

NEW CONDITIONS FOR SUBNORMALITY

TAVAN T. TRENT

The purpose of this paper is to establish some new characterizations of subnormality. One of these characterizations is interesting, in that the conditions are applied to "one vector at a time". This type of criterion is applied to show that verifying subnormality can be reduced to considering the restrictions of the operator to its cyclic invariant subspaces.

Denote the bounded linear operators on a separable Hilbert space H by $B(H)$. An operator $A \in B(H)$ is called subnormal if there exists an operator $N \in B(H \oplus H)$ so that N is a normal operator, $H \oplus 0$ is invariant for N and the restriction of N to $H \oplus 0$ equals A [8]. Some previous intrinsic characterizations of subnormality can be found in [2], [7], [8]. Also a summary of these results appears in [5].

An operator $T \in B(H)$ is called hyponormal if $T^*T - TT^* \geq 0$. It is easy to see that T is hyponormal if and only if $\|Tx\| \geq \|T^*x\|$ for all x in H . By a theorem of Douglas [6], this is equivalent to the existence of an operator $K \in B(H)$ satisfying $\|K\| \leq 1$ and $T^* = KT$. This fact was explicitly brought to the author's attention in [3].

Now the subnormal operators comprise a subset of the hyponormal ones. Thus the question arises as to whether the contraction operator K relating T^* and T , as above, has properties which enable one to tell whether T is not only hyponormal, but subnormal as well. The following example shows that this is not the case. Let K , T , and S denote Toeplitz operators with symbols \bar{z}^2 , $\bar{z} + z^3$, and z , respectively. (Here z stands for the identity function on the boundary of the unit disc.) Then $T^* = KT$ and $S^* = KS$, but S is subnormal and T is not [cf. 1]. The example for T comes from [4].

However if S is subnormal then so is S^n for $n = 0, 1, 2, \dots$. Hence for $n = 0, 1, 2, \dots$ there exist contractions $K_n \in B(H)$ with $S^{*n} = K_n S^n$. Also it is known that there are hyponormal operators T , which are not subnormal, with T^n hyponormal for $n = 0, 1, \dots$ [13]. One might ask for conditions on the K_n guaranteeing that if $T^{*n} = K_n T^n$, $n = 0, 1, \dots$, then T is subnormal. The following theorem provides these conditions.

THEOREM 1. *Let $T \in B(H)$. The following conditions on T are equivalent.*

- (a) T is subnormal.

(b) *There exists a unitary operator $U \in B(H \oplus H)$ such that for $n = 0, 1, \dots$ $T^{*n} = P_H U^n T^n$, where P_H is the orthogonal projection of $H \oplus H$ onto $H \oplus 0$.*

(c) *For $n = 0, 1, \dots$*

$$T^{*n} = \left[\int_{\partial D} e^{int} dQ(t) \right] T^n$$

where Q is a positive operator measure (denoted by POM) defined on the boundary of the unit disc, ∂D .

(d) *There exists a sequence of operators $K_n \in B(H)$ satisfying $T^{*n} = K_n T^n$ for $n = 0, 1, \dots$. Moreover if we define*

$$L_n = \begin{cases} K_n & n \geq 0 \\ K_n^* & n < 0 \end{cases}$$

then for any finite set $\{x_0, x_1, \dots, x_n\}$ contained in H ,

$$\sum_{j, k \geq 0}^n \langle L_{j-k} x_j, x_k \rangle \geq 0.$$

(e) *There exists a sequence of operators $K_n \in B(H)$ satisfying $T^{*n} = K_n T^n$ for $n = 0, 1, \dots$. Moreover if we define*

$$L_n = \begin{cases} K_n & n \geq 0 \\ K_n^* & n < 0 \end{cases}$$

then for each $x \in H$ and each $n = 0, 1, \dots$ the matrix $[\langle L_{j-k} x, x \rangle]_{j, k \geq 0}^n$ is positive definite.

Proof. (a) \Rightarrow (b). Let N be a normal extension of T acting on $H \oplus H$. Since the kernel of N reduces N we may write $N = N_1 \oplus 0$ acting on $(\ker N)^\perp \oplus \ker N$, where N_1 is normal and one-to-one. By normality, N_1 is also densely ranged, so if $N_1 = U_1 |N_1|$ is a polar decomposition of N_1 , then U_1 is unitary. Let V_1 be any unitary operator in $B(\ker N)$ and define $U = U_1 \oplus V_1$. Thus $N = U |N|$ where $U \in B(H \oplus H)$ is unitary. By normality U_1 commutes with N_1 , thus U commutes with N . Computing

$$N^{*n} = (U |N|)^{*n} = U^{*n} |N|^n = U^{*2n} (U |N|)^n = [U^{*2}]^n N^n.$$

Projecting onto $H \oplus 0$ we see that (b) holds.

(b) \Rightarrow (c). By the spectral theorem $U^n = \int_{\partial D} e^{int} dE(t)$, n an integer, for a projection valued measure E defined on ∂D . Hence for $n = 1, 2, \dots$

$$\begin{aligned} T^{*n} &= P_H U^n P_H T^n \\ &= \left[\int_{\partial D} e^{int} dQ(t) \right] T^n \end{aligned}$$

where $Q(t) = P_H E(t) P_H$ is a POM on ∂D .

(c) \Rightarrow (d). By hypothesis we may choose

$$K_n = \int_{\partial D} e^{int} dQ(t) \quad \text{for } n = 0, 1, \dots .$$

Then

$$K_n^* = \int_{\partial D} e^{-int} dQ^*(t) = \int_{\partial D} e^{-int} dQ(t) .$$

Hence

$$L_n = \int_{\partial D} e^{int} dQ(t) \quad \text{for all integers } n .$$

Take any finite subset $\{x_0, \dots, x_M\}$ of H . Let $\{\Delta_p\}_{p=1}^n$ be any partition of ∂D and choose $e^{ikt_p} \in \Delta_p$.

Then for any fixed p

$$\sum_{j,k \geq 0}^M e^{ijt_p} e^{-ikt_p} \langle Q(\Delta_p)x_j, x_k \rangle = \left\langle Q(\Delta_p) \sum_{j=0}^M e^{ijt_p} x_j, \sum_{k=0}^M e^{ikt_p} x_k \right\rangle \geq 0 .$$

Summing over p and interchanging the orders of summation, we get

$$\sum_{j,k \geq 0}^M \sum_{p=1}^n e^{i(j-k)t_p} \langle Q(\Delta_p)x_j, x_k \rangle \geq 0 .$$

The innermost sum is a Riemann sum for $\int_{\partial D} e^{i(j-k)t} d\langle Q(t)x_j, x_k \rangle$.

We may conclude that

$$\sum_{j,k \geq 0}^M \langle L_{j-k}x_j, x_k \rangle = \sum_{j,k \geq 0}^M \int_{\partial D} e^{i(j-k)t} d\langle Q(t)x_j, x_k \rangle \geq 0 .$$

(d) \Rightarrow (e). For any x in H and any finite subset $\{t_0, \dots, t_M\}$ of complex numbers, denote $t_j x$ by x_j and apply (d).

(e) \Rightarrow (d). By a result of Herglotz (see [9], p. 125), the hypotheses say that $\{\langle L_n x, x \rangle\}_{n=-\infty}^{\infty}$ is a trigonometric moment sequence for a positive Borel measure μ_x on ∂D , whose total variation is $\langle L_0 x, x \rangle = \|x\|^2$. Thus

$$(1) \quad \langle L_n x, x \rangle = \int_{\partial D} e^{int} d\mu_x(t) \quad \text{for } n = 0, 1, \dots .$$

Fix x in H . For each Borel set $\Delta \subset \partial D$ define the positive form $Q(\Delta)$ by

$$\langle Q(\Delta)x, x \rangle = \int_{\Delta} 1 d\mu_x(t) .$$

Extend this form to a bilinear form on H by polarization. The bilinear form is bounded since the positive form is. So the positive operator $Q(\Delta)$ is defined and $Q(\Delta)$ is in $B(H)$.

By polarization and (1) we have

$$\langle L_n x, y \rangle = \int_{\partial D} e^{int} d\langle Q(t)x, y \rangle$$

for x, y in H . Thus

$$\langle T^{*n}x, y \rangle = \langle L_n T^n x, y \rangle = \int_{\partial D} e^{int} d\langle Q(t)T^n x, y \rangle .$$

Thus (c) holds and so (d) must.

(d) \Rightarrow (a). Let $\{x_0, \dots, x_n\}$ be any finite subset of H . By (d)

$$\sum_{j, k \geq 0}^n \langle L_{j-k} T^j x_j, T^k x_k \rangle \geq 0 .$$

Now if $k - j \geq 0$,

$$\begin{aligned} \langle L_{j-k} T^j x_j, T^k x_k \rangle &= \langle T^j x_j, K_{k-j} T^{k-j} T^j x_k \rangle \\ &= \langle T^j x_j, T^{*k-j} T^j x_k \rangle \\ &= \langle T^k x_j, T^j x_k \rangle . \end{aligned}$$

A similar result follows when $k - j < 0$, thus

$$0 \leq \sum_{j, k \geq 0}^n \langle L_{j-k} T^j x_j, T^k x_k \rangle = \sum_{j, k \geq 0}^n \langle T^k x_j, T^j x_k \rangle .$$

It follows from the Bram-Halmos criterion that T is subnormal [2].

For invertible operators, (d) is essentially Embry's condition [7]. In both cases "polar coordinates" are used. For Embry's the measures are supported on $[0, 1]$ (radial) and in our case the relevant support set is ∂D (angular).

Condition (e) of Theorem 1 gives a criterion for subnormality which involves looking at only one vector of H at a time. A similar related result is due to Lambert [11]. As a consequence of this type of criterion, we have the following corollaries.

Fix $S \in B(H)$. Denote the closed linear span of $\{S^k x: k = 0, 1, \dots\}$ by H_x .

COROLLARY 1. *S is subnormal if and only if for every x in a dense linear manifold of H the restriction of S to H_x is subnormal.*

Proof. The necessity of the condition is trivial. Let \mathcal{D} denote the dense linear manifold of H given in the hypotheses and let $S|_{H_x}$ denote the restriction of S to H_x . If $S|_{H_x}$ is subnormal for all x in \mathcal{D} , then so is $(\lambda - S)|_{H_x}$. For all large λ H_x is invariant for $(\lambda - S)^{-1}$.

Thus without loss of generality we assume that S is invertible and that

$$(S|_{H_x})^{-1} = S^{-1}|_{H_x} .$$

Fix x in \mathcal{D} . Since $S|_{H_x}$ is subnormal there exist contractions B_n in $B(H_x)$ such that

$$P_{H_x} S^{*n}|_{H_x} = B_n S^n|_{H_x} \text{ for } n = 0, 1, \dots .$$

So

$$P_{H_x} S^{*n} S^{-n}|_{H_x} = B_n .$$

By (e) of Theorem 1 applied to x and B_n we see that

$$[\langle S^{*j-k} S^{k-j} x, x \rangle]_{j,k \geq 0}^n \text{ is positive definite}$$

for $n = 0, 1, \dots$. But $S^{*n} = (S^{*n} S^{-n}) S^n$, so (e) of Theorem 1 now shows that S is subnormal.

COROLLARY 2. *N is normal if for each x in a dense linear manifold of H . We have $N|_{H_x}$ is normal.*

Proof. Let \mathcal{D} denote the dense manifold. For x in \mathcal{D} , $N|_{H_x}$ normal implies that

$$\|Nx\| = \|P_H N^* x\| \leq \|N^* x\| .$$

But Corollary 2 says that N is subnormal, thus hyponormal. Hence for x in \mathcal{D} .

$$\|Nx\| = \|N^* x\| .$$

Since N and N^* are continuous, the proof is complete.

An observation due to R. L. Moore might be of interest. Let U be the unilateral shift of infinite multiplicity. In contrast to the result of Corollary 2, by model theory every cyclic normal operator of norm less than one can be obtained as the restriction of U^* to a (cyclic) invariant subspace.

Using Corollary 1 and the fact that cyclic subnormal operators correspond to compactly supported Borel measures [2], it might be possible to find "functional" criteria for classes of subnormal operators. As a modest example motivated by a function used in [10], we have the following result.

THEOREM 2. *Let $S \in B(H)$ and $\|S\| \leq 1$. S is an isometry if and only if for each x in H the function*

$$\phi_x(z) = (1 - |z|^2) \|(1 - \bar{z}S)^{-1} x\|^2 \text{ is harmonic in the unit disc.}$$

The proof follows by an easy computation.

We wish to thank R. L. Moore for his example and Warren Wogen for his suggestion that Theorem 1 which was originally proved for invertible operators should hold in general.

REFERENCES

1. B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J., **43** (1976), 597-604.
2. J. Bram, *Subnormal operators*, Duke Math. J., (1955), 75-94.
3. K. Clancey, *Seminormal operators*, lecture notes, University of Georgia, 1978.
4. K. Clancey and B. Morrel, *On the essential spectra of some Toeplitz operators*, Proc. Amer. Math. Soc., **17** (1966), 367-379.
5. J. Conway, *Lecture notes on subnormal operators*, University of Indiana, 1978.
6. R. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc., **17** (1966), 413-415.
7. M. Embry, *Generalization of the Halmos-Bram criterion for subnormality*, Acta Scien. Math., (Szeged) **35** (1973), 61-64.
8. P. Halmos, *Normal dilations and extensions of operators*, Summa Bras. Math., (1950), 125-134.
9. ———, *A Hilbert Space Problem Book*, Van Nostrand, New York, 1967.
10. T. L. Kriete and T. Trent, *Growth near the boundary in $H^2(u)$ spaces*, Proc. Amer. Math. Soc., **62** (1977), 83-88.
11. A. Lambert, *Subnormality and weighted shifts*, J. London Math. Soc., **14** (1976), 476-480.
12. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.
13. J. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc., **117** (1965), 709-718.

Received January 2, 1980.

UNIVERSITY OF ALABAMA
UNIVERSITY, AL 35486