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ON THE TOPOLOGY OF DIRECT LIMITS OF ANR'S

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Let $\{(X_n, a_n)\}$ be a sequence of pointed, locally compact, finite-dimensional, nondegenerate, connected ANR's. It is shown that the dirct limit of the system

$$X_1 \longrightarrow X_1 \times \{a_2\} \subset X_1 \times X_2 \longrightarrow X_1 \times X_2 \times \{a_3\} \subset X_1 \times X_2 \times X_3 \longrightarrow \cdots$$

is homeomorphic to an open subset of $R^\infty = \varinjlim R^n$, R the reals. As a consequence, if $f: X \to Y$ is a homotopy equivalence between ANR's as above then $\varinjlim f^n: \varinjlim X^n \to \varinjlim Y^n$ is homotopic to a homeomorphism.

A. Introduction. Infinite countable products of complete AR's have been shown to be in most cases homeomorphic to either the Hilbert cube or a Hilbert space: by combined results of Anderson [1], West [9] and Edwards [2] the product $\prod X_i$ is homeomorphic to $\prod_i [0, 1]_i$ provided all the X_i are compact and nondegerate; similarly, any product of countably many noncompact AR's of the same weight is, topologically, a Hilbert space (see [8]). The latter result can be used to show that if (X_i, a_i) are pointed, finite-dimensional, σ-compact AR's then the space

(i) $\sum (X_i, a_i) = \{(x_i) \in \prod X_i : x_i = a_i \text{ for almost all } i\}$

is, in the product topology, homeomorphic to the incomplete linear subspace l_2^f consisting of all eventually zero sequences in l_2, the Hilbert space.

In this note we show that, under the additional assumption that the X_i's are locally compact, the space (i) considered in the *direct limit topology* is homeomorphic to another familiar topological space, namely R^∞, the direct limit of finite products of R, the real line. More generally, we have the following:

THEOREM. *Let $\{(X_n, a_n)\}$ be a sequence of pointed, locally compact, finite-dimensional, connected* ANR*'s having more than one point. Then the direct limit of the system*

$$X_1 \longrightarrow X_1 \times \{a_2\} \subset X_1 \times X_2 \longrightarrow X_1 \times X_2 \times \{a_3\} \subset X_1 \times X_2 \times X_3 \longrightarrow \cdots$$

is homeomorphic to an open subset of R^∞.

For results concerning the topological properties of R^∞ we refer the reader to [4] and [5]. It is shown there that the R^∞-manifolds posess many of the properties of l_2-manifolds; in particular, if f is a homotopy equivalence between R^∞-manifolds then f is homotopic to a homeomorphism. Combined with a result of Hansen, Theorem 6.2 of [3], this gives the following.

COROLLARY. *If $f: X \to Y$ is a homotopy equivalence betwee locally compact, finite-dimensional, connected* ANR*'s having more than one point, then* $\varinjlim f^n: \varinjlim X^n \to \varinjlim Y^n$ *is homotopic to a homeomorphism.*

Despite the above-mentioned similarity of R^∞ and l_2 manifolds no intrinsic characterization of R^∞-manifolds corresponding to the characterizations of l_2 and Q-manifolds (see [8]) is known. The motivation of this paper was to show that the direct limit operation leads naturally to such manifolds (see also the Proposition in § C). Earlier, it was shown by Henderson [6] that taking products of R^∞ with locally compact, finite-dimensional ANR's yields open subsets of R^∞. Our result generalizes Henderson's. However, while Henderson's technique involved the linear structure of R^∞ (and has since been applied to studying factors of other linear topological spaces) our proofs involve merely the construction of embeddings from finite-dimensional compacta into products of ANR's.

B. Notation and lemmas. In this section all spaces are separable and metric. If d_i is the metric on X_i, $i \leqq n$, we take max $\{d_i(x_i, y_i): i \leqq n\}$ as the metric on $X_1 \times \cdots \times X_n$. By I and I^k was denote [0, 1] and the k-fold product of [0, 1], respectively. If $k = 0$, I^k is the singleton.

A map (= continuous function) $g: X \to Y$ is said to be *approximable* by elements of the family $\mathscr{F}$ of maps $X \to Y$ if for any admissible metric d for Y there is an $f \in \mathscr{F}$ such that $d(f, g) < 1$. (If X is compact this coincides with the concept of being in the closure of $\mathscr{F}$ in the compact—open topology.)

We say that $A \subset X$ is a Z^k-set, $k \geqq 0$, if any map $I^k \to X$ can be approximated by maps whose images are disjoint from A. A map whose image is a Z^k-set will be called a Z^k-map.

We shall consider spaces X having the following property, sometimes called the disjoint k-cube property.

$(*)_k$ Any map $I^k \times \{1, 2\} \to X$ is approximable by maps sending $I^k \times \{1\}$ and $I^k \times \{2\}$ to disjoint sets.

The following generalizes the fact that R^{2k+1} has property $(*)_k$.

LEMMA 1. *If $X_1, X_2, \cdots, X_{2k+1}$ are locally contractible spaces with no isolated points then $X_1 \times \cdots \times X_{2k+1}$ has the property $(*)_k$.*

For a proof see [8].

LEMMA 2. *If X is complete and satisfies $(*)_k$ then any map $I^k \to X$ is approximable by Z^k-maps.*

The proof is the same as that of Remark 3 of [7].

LEMMA 3. *Let X be an* ANR *satisfying $(*)_k$, let A and B be disjoint compacta of dimension $\leqq k$, and let X_0 be a closed Z^k-set in X. Then any map $A \cup B \to X$ is approximable by maps $g: A \cup B \to X$ satisfying $g(A) \cap g(B) = \varnothing$ and $g(A \cup B) \cap X_0 = \varnothing$.*

Proof. Since X is an ANR each map $A \cup B \to X$ can be approximated by compositions of the form $A \cup B \to K \to X$, where K is a polyhedron of dimension $\leqq \dim(A \cup B)$. Thus, we may assume that A and B are compact polyhedra, and the result follows from the fact that in this case A and B are finite unions of cells of dimension $\leqq k$. (Details are left to the reader; cf. the proof of the next result.)

PROPOSITION 4. *Let A and X be locally compact spaces, let A_0 be a closed subset of A and let $f: A \to X$ be a map such that $f(A_0)$ is a closed Z^k-set. If $\dim A \leqq k$ and X is an* ANR *satisfying $(*)_k$, then f is approximable by Z^k-maps $g: A \to X$ such that $g|A_0 = f|A_0$, $g(A \backslash A_0) \cap g(A_0) = \varnothing$, and $g|(A \backslash A_0)$ is one-to-one.*

Proof. A proof is given in [7] for the case $k = \infty$ and $A_0 = \varnothing$. The proof of the general case is similar; we include it for completeness.

Fix a metric d_0 for X. Let $d \geqq d_0$ be a complete metric for X and let $\{A_i\}_{i \in N}$ be a family of compact subsets of $A \backslash A_0$ such that for any pair x and y of distinct points of $A \backslash A_0$ there are $i, j \in N$ with $x \in A_i$, $y \in A_j$, and $A_i \cap A_j = \varnothing$. Let $\{f_i\}_{i \in N}$ be a dence subset of $C(I^k, X)$ consisting of Z^k-maps such that $f_i(I^k) \cap f(A_0) = \varnothing$ (see Lemma 2). With $F = \{g \in C(A, X): g|A_0 = f|A_0\}$ it follows from Lemma 3 and [7, Lemma C] that for each $i, j \in N$ with $A_i \cap A_j = \varnothing$, the set

$$G_{i,j,l} = \{g \in F: g(A_i) \cap g(A_j) = \varnothing \text{ and } g(A_i \cup A_j) \cap [f_l(I^k) \cup f(A_0)] = \varnothing\}$$

is dense and open in F. (We equip F with the sup metric $\hat{d}$ induced by d.) Since $(F, \hat{d})$ is complete it follows that $G = \cap \{G_{i,j,l}: A_i \cap A_j = \varnothing,\ l \in N\}$ is dense in F. This completes the proof since $f \in F$ and any $g \in G$ satisfies the desired conditions.

COROLLARY 5. *If in Lemma 4 it is additionally assumed that*

f is proper and $f|A_0$ is an embedding, then the approximations $g: A \to X$ can be taken to be closed Z^k-embeddings.

Proof. Use the facts that a map sufficiently close to a proper map of locally compact spaces is itself proper and that one-to-one proper maps are closed embeddings.

REMARK. If $A_0 = \emptyset$ and $X = R^{2k+1}$ then the above corollary reduces to the classical Menger-Nöbeling embedding theorem.

LEMMA 6. *Let $X_1, \cdots, X_k$ be nondegenerate, connected* ANR's. *Then the singletons are Z^k-sets in $X_1 \times \cdots \times X_{k+1}$. Accordingly, $X_0 \times \{b\}$ is a Z^k-set in $X_0 \times X_1 \times \cdots \times X_{k+1}$, for any space X_0 and any point $b \in X_1 \times \cdots \times X_{k+1}$.*

Proof. (By induction on k.) Let $b = (b_1, \cdots, b_{k+1}) \in X_1 \times \cdots \times X_{k+1}$, $f = (f_1, \cdots, f_{k+1}): I^k \to X_1 \times \cdots \times X_{k+1}$ and $\varepsilon > 0$ be given. Let $\mathscr{T}$ be a triangulation of I^k so fine that for each simplex $\sigma \in \mathscr{T}$, $f_{k+1}(\sigma)$ is contractible in X_{k+1} within a set of diameter less than ε. Let $\mathscr{T}^{k-1}$ be the $(k-1)$-skeleton of $\mathscr{T}$. By the induction hypothesis and [7, Lemma C] we may assume without loss of generality that $(f_1, \cdots, f_k)(|T^{k-1}|)$ misses $(b_1, \cdots, b_{k-1})$; cf. proof of 4.

Now, using the ε-contractions of $f_{k+1}(\sigma)$, we may alter f on k-dimensional simplices, modulo their boundaries, so that the resulting map $g: I^k \to X_1 \times \cdots \times X_{k+1}$ is within ε of f and satisfies (a) $g_i(I^k) = f_i(I^k)$, $i \leqq k$, and (b) for each $\sigma \in \mathscr{T} \backslash \mathscr{T}^{k-1}$ there is a point $p_\sigma \in X_{k+1}$ with

$$g(\sigma) \subset [(f_1, \cdots, f_k)(\partial\sigma) \times X_{k+1}] \cup [(f_1, \cdots, f_k)(\sigma) \times \{p_\sigma\}] .$$

Since X_{k+1} has no isolated points, all the p_σ's can clearly be chosen distinct from b_{k+1}. Thus, g is an ε-approximation to f whose image misses b.

Finally, we need the following.

LEMMA 7. *Let A be a locally compact space and let A_0 be a closed subset of A. Then any proper map $f: A_0 \to [0, 1)$ has a continuous extension $\bar{f}: A \to [0, 1)$ which is also proper.*

Proof. Let $A \cup \{\infty\}$ be the one point compactification of A, and extend f to $g: A_0 \cup \{\infty\} \to [0, 1]$ by defining $g(\infty) = 1$. Letting $\bar{g}: A \cup \{\infty\} \to [0, 1]$ be an extension of g we may take $\bar{f}(a) = h(a)\bar{g}(a)$, where $h: A \cup \{\infty\} \to [0, 1]$ is a map with $h^{-1}(1) = A_0 \cup \{\infty\}$.

C. Proof of the theorem. The theorem follows immediately from Lemmas 1 and 6 and the following.

PROPOSITION. *Let* $\{X_k; i_k\}$ *be a direct system of closed embeddings* $i_k: X_k \to X_{k+1}$ *of locally compact, finite-dimensional* ANR's. *Assume that for any positive integers* k, p *there is an integer* $l > k$ *such that* X_l *has property* $(*)_p$ *and* $i_{l-1} \circ \cdots \circ i_k(X_k)$ *is a* Z^p*-set in* X_l. *Then,* $\varinjlim \{X_k; i_k\}$ *is homeomorphic to an open subset of* R^∞.

Proof. Let $d_k = \dim X_k$. Passing to a subsequence, if necessary, we may assume that

(a) $i_k(X_k)$ is a Z^{3d_k}-set in X_{k+1} and

(b) X_{k+1} has property $(*)_{3d_k}$ and, hence, $d_{k+1} \geqq 3d_k$.

Let $J = [-1, \infty)$ and let $j_k: J^{3d_{k-1}} \to J^{3d_{k-1}} \times (0, 0, \cdots, 0) \subset J^{3d_k}$ be the natural inclusion. We shall inductively construct manifolds with boundary M_k in $(-1, \infty)^{3d_{k-1}}$ and closed embeddings $f_k: M_k \to X_k$ and $g_k: X_k \to M_{k+1}$ such that, for each k,

(c) M_{k+1} is a neighborhood of $j_k(M_k)$, and

(d) the following diagram commutes.

$$\begin{array}{ccc} X_k & \xrightarrow{i_k} & X_{k+1} \\ \uparrow f_k & \searrow g_k & \uparrow f_{k+1} \\ M_k & \xrightarrow{j_k} & M_{k+1} \end{array} .$$

Assume $\{(M_k, f_k, g_k)\}$ have been constructed. It is then clear that bot $\varinjlim \{X_k; i_k\}$ and $\varinjlim \{M_k; j_k\}$ are homeomorphic to the direct limit of the system $M_0 \xrightarrow{f_0} X_0 \xrightarrow{g_0} M_1 \xrightarrow{f_1} X_1 \xrightarrow{g_1} M_2 \xrightarrow{f_2} \cdots$. Also, it is clear that $\varinjlim \{M_k; j_k\}$ is homeomorphic to $\varinjlim \{\operatorname{Int} M_k; j_k\}$ which is open in $\varinjlim \{(-1, \infty)^{3d_{k-1}}; j_k\} \cong R^\infty$. Thus, $\varinjlim \{X_k; i_k\}$ is homeomorphic to an open subset of R^∞.

We now give the construction of the embeddings f_k and g_k. Assuming, without loss of generality, that $X_0 = R^0$, the singleton, we take for f_0 the identity. Having established f_k consider the closed embedding $j_k f_k^{-1}: im(f_k) \to J^{3d_k}$. By Lemma 7 we can extend $j_k f^{-1}$ to a proper map $X_k \to J^{3d_k}$ which we may then, by 1 and 6, alter so as to get a closed embedding $g_k: X_k \to J^{3d_k}$ coinciding with $j_k f_k^{-1}$ on $im(f_k)$. Clearly, we may adjust g_k so that in addition $im(g_k) \subset (-1, \infty)^{3d_k}$.

The set $im(g_k)$ being a closed ANR subset of $(-1, \infty)^{3d_k}$, there is a manifold with boundary M_{k+1} contained in $(-1, \infty)^{3d_k}$ which is topologically closed in J^{3d_k}, contains a neighborhood of $im(g_k)$ in $(-1, \infty)^{3d_k}$, and which properly retracts to $im(g_k)$. Then $i_k g_k^{-1}: im(g_k) \to$

X_{k+1} extends to a proper map $M_{k+1} \to X_{k+1}$ which we again may alter modulo $im(g_k)$ to get a closed embedding $f_{k+1}: M_{k+1} \to X_{k+1}$ coinciding with $i_k g_k^{-1}$ on $im(g_k)$. This completes the inductive step and the proof of the proposition.

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