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RINGS WHERE THE ANNIHILATORS OF α-CRITICAL MODULES ARE PRIME IDEALS

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For a ring R with Krull dimension α, we investigate the property that the annihilators of α-critical modules are prime ideals. If R satisfies the large condition then this property holds iff R/I_0 is semiprime, where I_0 is the maximal right ideal of Krull dimension $<\alpha$. The property holds in the following rings, (i) R is weakly ideal invariant, (ii) R satisfies the left AR property, or (iii) the prime ideals of R are right localizable. In addition, if R is a hereditary Noetherian α-primitive ring, then R is a prime ring.

1.1. Introduction. This paper will provide conditions on a ring R with Krull dimension α, which imply the property that the annihilator of any α-critical module is a prime ideal. In the terminology of [2], this property means the α-coprimitive ideals are prime.

In §2, using the procedures of [2] and [4], we find necessary and sufficient conditions for this property in rings which satisfy the large condition. In addition, for a ring R with Krull dimension this property is true under any one of the following conditions; (i) R is weakly ideal invariant (ii) R satisfies the left AR-condition, (iii) the prime ideals of R are right localizable. For right Noetherian ring, the conditions (i) and (iii) are shown to imply this property in [5]. For Noetherian AR-rings the same is true from [5] and [13]. We extend the results of K. Brown, T. H. Lenagan, and J. T. Stafford [5] for (i), (ii), and (iii) to rings with Krull dimension. The proofs are short and direct, utilizing the procedures of [2] and [4]. This should be helpful in the study of related problems.

One can show directly that if a right Noetherian ring R is smooth and the α-coprimitive ideals are prime, then R has a right Artinian, right classical quotient ring.

In §3, we shall investigate right hereditary α-primitive rings R. We show that the associated α-prime ideal P is a direct summand, and R/P is a right hereditary prime ring. This implies from [6] and [11], that if R is a hereditary Noetherian α-primitive ring, then R is a hereditary Noetherian prime ring of Krull dimension 0 or 1.

All rings will have identity, and all modules are right unitary modules. Ideal shall mean two-sided ideal, and a ring is Noetherian if it is both right and left Noetherian. The injective hull of a module M is denoted by $E(M)$, and $|M|$ denotes the Krull dimension of M.

If S is a subset of a module M over R, then ann $S = S^r = \{x \in R \mid sx = 0$ for all $s \in S\}$. The Socle of M is the sum of the critical submodules of M, and is denoted by Soc M.

2.1. What α-coprimitive ideals are prime. As in [2], an ideal D of R is called *α-coprimitive* if D is the annihilator of an α-critical module C, where $|C| = |R| = \alpha$. A ring R is *α-primitive* provided 0 is an α-coprimitive ideal. If I is an indecomposable injective module containing an α-critical module, then I is called an α-indecomposable injective module.

The following is known and the proof direct.

PROPOSITION 2.2. *If R is a semiprime ring, where $|R| = \alpha$, then every α-coprimitive ideal is prime.*

From §2 of [4], if $|R| = \alpha$, for every α-indecomposable module I, there is a unique minimal α-coprimitive ideal D, such that $D =$ ann Soc I, and if C is any α-critical module in I, then $D \subsetneqq$ ann $I \subseteqq P$, where $P =$ ass I. Thus we can write.

PROPOSITION 2.3. *If $|R| = \alpha$, then every α-coprimitive ideal is prime* iff *for every α-indecomposable injective module I, we have* ann (Soc I) *is prime ideal.*

Since there is but a finite number of isomorphic classes of α-indecomposable injective modules, then from 2.2 and 2.3 we have.

PROPOSITION 2.4. *If $|R| = \alpha$, and $M = I_1 \oplus \cdots \oplus I_n$, where I_i, for $i = 1, 2, \cdots, n$, is an α-indecomposable injective, and each isomorphic class of α-indecomposable injective modules is represented in this sum, then every α-coprimitive ideal is prime* iff ann (Soc M) *is a semiprime ideal of R.*

A ring R with Krull dimension α is said to satisfy the *large condition,* provided $|R/L| < |R|$, for all large right ideals L of R. A ring R is *α-smooth* or *α-homogeneous* if $|K| = |R| = \alpha$, for all nonzero right ideals K of R.

PROPOSITION 2.5. *Let R be an α-smooth ring with Krull dimension α. Then R satisfies the large condition and every α-coprimitive ideal is prime* iff *R is a semiprime ring.*

Proof. If R satisfies the large condition, then every α-indecomposable injective module embeds in $E(R)$. Hence, since R is smooth, then $R \cong I_1 \oplus \cdots \oplus I_n$, where each I_i is an α-indecomposable injective module, and all the isomorphic classes are represented. From Corollary 2.6 of [2, p. 61], we have $\operatorname{Soc} I_i = I_i$ for all i. Hence $\operatorname{Soc} E(R) = E(R)$. Now if every α-coprimitive ideal is prime, then from 2.4, we have $0 = \operatorname{ann} R = \operatorname{ann} E(R)$ is a semiprime ideal.

Since semiprime rings with Krull dimension all satisfy the large condition the converse is true.

THEOREM 2.6. *Let R be a right Noetherian ring with Krull dimension α, then R satisfies the large condition and the α-coprimitive ideals are prime* iff *I_0 is a closed semiprime ideal, where I_0 is the maximal right ideal of Krull dimension $< \alpha$.*

Proof. If D is an α-coprimitive ideal of R, then since R/D is α-smooth, it follows that $D \supseteq I_0$, which is an ideal of R. Thus the α-coprimitive ideals of R/I_0 are just of the form D/I_0, where D is an α-coprimitive ideal of R.

From Proposition 3.4 of [3], we know that R satisfies the large condition iff I_0 is closed and R/I_0 satisfies the large condition. Since R/I_0 is smooth, the result follows from 2.5.

EXAMPLE 2.7. Let Z denote the integers and Z_p the integers modulo a prime element p. If

$$R_1 = \begin{bmatrix} Z & Z_p \\ 0 & Z_p \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} Z & Z_p[x] \\ 0 & Z_p[x] \end{bmatrix},$$

then R_1 and R_2 satisfy the large condition, and R_1 satisfies the conditions of 2.6. However, R_2 is smooth, but certainly not semiprime. The ideal $\begin{bmatrix} (p) & 0 \\ 0 & 0 \end{bmatrix}$ is coprimitive, and not prime.

If R is a ring with Krull dimension, an ideal T of R is said to be *weakly ideal invariant* provided $|T/IT| < |R/T|$ for every right ideal I of R with $|R/I| < |R/T|$. If every ideal of R is weakly ideal invariant, then R is said to be *weakly ideal invariant.*

The proof of the following is direct.

LEMMA 2.8. *If A, B, and C are right ideals of a ring with Krull dimension α and if $|A/B| = \alpha$ and $|R/C| < \alpha$, then $|A \cap C/B \cap C| = \alpha$.*

THEOREM 2.9. *Let R be a ring where $|R| = \alpha$. If N, the prime* *radical of R, is weakly ideal invariant, then the α-coprimitive ideals are prime.*

Proof. Let $N = P_1 \cap \cdots \cap P_m \cap P_{m+1} \cap \cdots \cap P_n$, where the P_i are minimal prime ideals, and $|R/P_i| = \alpha$ for $i = 1, 2, \cdots, m$, and $|R/P_i| < \alpha$ for $i = m+1, \cdots, n$.

As in [4], let D_i be the unique minimal α-coprimitive ideal contained in P_i for $i = 1, \cdots, m$, and $H_i = (D_i : P_i) = \{x \in R \mid xP_i \subseteqq D_i\}$ for $i = 1, 2, \cdots, m$. Then H_i/D_i is large in R/D_i, since P_i/D_i is the assassinator for every uniform right ideal of R/D_i. From [2], we have that R/D_i satisfies the large condition. Hence $|R/H_i| < \alpha$. From Corollary 1.3 of [9], for rings will Krull dimension, if $W = H_1 \cdots H_m P_{m+1} \cdots P_n$, then $|R/W| < \alpha$. Now $WN \subset D_1 \cap \cdots \cap D_m \cap P_{m+1} \cap \cdots \cap P_n$. If $D_i \neq P_i$ for some i, then $|P_1 \cap \cdots \cap P_m / D_1 \cap \cdots \cap D_m| = \alpha$ since $R/D_1 \cap \cdots \cap D_m$ is α-smooth, and $P_1 \cap \cdots \cap P_m$ is not contained in $D_1 \cap \cdots \cap D_m$. To show this last statement, suppose $P_1 \cap \cdots \cap P_m = D_1 \cap \cdots \cap D_m$. Then $P_i(P_1 \cdots P_{i-1} P_{i+1} \cdots P_m) \subseteqq D_i$. However, D_i is P_i primary, which means $P_1 \cdots P_{i-1} P_{i+1} \cdots P_m \subseteqq P_i$, and $P_j \subseteqq P_i$ for $i \neq j$, which is a contradiction.

Thus, by Lemma 2.8, we have $|N/WN| = \alpha$. However $|R/W| < \alpha$, which contradicts the assumption that N is weak ideal invariant.

It is not known whether the converse of this theorem is true for rings with Krull dimension. However, Theorem 2.5 of [5] establishes this theorem and its converse for right Noetherian rings.

If I is an ideal of ring R, and $C(I) = \{c \in R \mid c + I$ is regular in $R/I\}$, then I is said to be *right localizable* provided $C(I)$ is a right Ore set.

THEOREM 2.10. *Let R be an α-primitive ring with unique α-prime ideal P, then P is right localizable* iff $P = 0$.

Proof. If $P = 0$, the result follows from [7]. Suppose now $P \neq 0$, and P is right localizable. Then $[0: C(P)] = \{x \in R \mid xa = 0$ for some $a \in C(P)\}$ is an ideal of R, and $[0: C(P)] \subseteqq P$.

Suppose $[0: C(P)] = P$. Now P^l, the left annihilator of P, is a large right ideal of R, hence $P^l \cap P \neq 0$. There exists $a \in C(P)$, and $0 \neq x \in P^l \cap P$, so that $x(aR + P) = 0$. However, $|R/aR + P| < \alpha$, which implies $|xR| < \alpha$. This is not possible since R is α-smooth. Hence $[0: C(P)] \subsetneqq P$.

Since P^l is not contained in P, then $P^l \cap C(P) \neq 0$. Let $a \in P^l \cap C(P)$, and $x \in P$, but $x \notin [0: C(P)]$. By the Ore condition, there exist, $d \in R$, $b \in C(P)$ such that $ad = xb$. Thus $d \in P$, and since $a \in P^l$, then $as = 0$. This implies $xb = 0$ and $x \in [0: C(P)]$, a contradiction.

Note that if P is right localizable in R, then P/K is right localizable in R/K for any ideal K contained in P. Thus

COROLLARY 2.11. *If R is a ring with $|R| = \alpha$, and if every α-prime P is right localizable, then the α-coprimitive ideals are prime.*

For right Noetherian rings this result follows from 2.5 and 3.1 of [5].

An ideal I of a ring R is said to satisfy the *left* AR *property* provided for every left ideal E of R, there exists a positive integer n such that $E \cap I^n \subseteq IE$. A ring R satisfies the left AR *property* if every ideal of R satisfies this property. The right AR property is defined in a similar fashion.

THEOREM 2.12. *If R is an α-primitive ring with unique α-prime ideal P, then $P = 0$ iff P^l satisfies the left* AR *property.*

Proof. If $P = 0$, the result is trivial. Suppose P^l satisfies the left AR property. Since P^l is large, and $Z(R) = 0$, then $(P^l)^n$ is large for all positive integer n. Thus, if $P \neq 0$, then $0 \neq P \cap (P^l)^n \subseteq P^l P = 0$, a contradiction.

Note that if I satisfies the left AR-condition in R, then I/K satisfies this condition in R/K for all ideals K contained in I.

COROLLARY 2.13. *If a ring R with Krull dimension satisfies the left* AR *property, then the α-coprimitive ideals are prime.*

If R is a Noetherian ring with both the right and left AR-condition, the result follows from 3.4 of [5], which is a consequence of 3.4 of [13].

PROPOSITION 2.14. *Let R be an α-coprimitive ring with unique α-prime P, then P is nilpotent* iff *P satisfies the right* AR-*condition.*

Proof. Now $P^l \cap P^n \subseteq P^l P = 0$ for some positive integer n. Since P^l is large, we have $P^n = 0$.

For an example of this type of ring see 4.3 of [4].

3.1. Right hereditary α-primitive rings. Currently, we have no example of a Noetherian α-primitive ring R, which is not prime. If $|{}_R M| = |M_R|$ for all (R, R) modules M, one easily shows R is prime. Thus an example would likely depend on finding a Noetherian ring, whose right Krull dimension is not equal to its left Krull dimension.

We show here that a hereditary Noetherian α-primitive ring is prime. We begin with an investigation of right hereditary α-primitive rings.

PROPOSITION 3.2. *Let R be a right hereditary α-primitive ring with faithful α-critical module C. Then*

(1) *If K is a right of R, then K^r is a direct summand of R.*

(2) *The* $\operatorname{ass} C = P$ *is a direct summand of R, and R/P is a right hereditary ring.*

(3) *$P^r = 0$, and R is right Noetherian.*

Proof. Since R is smooth, then for a right ideal K of R, we have $K^r = x_1^r \cap \cdots \cap x_n^r$, for $x_1, x_2, \cdots, x_n \in K$. Thus R/K imbeds in $R/x_1^r \oplus \cdots \oplus R/x_n^r$, and by Proposition 7 of [10, p. 85], we have that R/K is projective. Hence K is a direct summand of R.

If C_0 is a compressible right ideal of R, then $C_0^r = P$ from [2]. Thus (2) follows from (1).

Since R is P primary, then $P^r \subseteqq P$. Thus $(P^r)^2 = 0$. If $P^r \neq 0$, then (1) implies that P^r contains a nonzero idempotent, which is impossible. The ring R is right Noetherian by Corollary 5.20 of [8, p. 149].

Since P is a direct summand of R, then $R = eR \oplus P$, where $(1 - e) R = P$. We can write R as a formal triangular matrix ring.

$$R \cong \begin{pmatrix} (1-e)R(1-e) & (1-e)Re \\ 0 & eR \end{pmatrix},$$

where

$$P = \begin{pmatrix} (1-e)R(1-e) & (1-e)Re \\ 0 & 0 \end{pmatrix},$$

and

$$R/P \cong \begin{pmatrix} 0 & 0 \\ 0 & eR \end{pmatrix}$$

is a right hereditary, right Noetherian prime ring, and $(1-e)R(1-e) = (1-e)P(1-e) \cong \operatorname{Hom}_R(P, P)$ is a right hereditary ring. Theorem 4.7 of [8, p. 111] provides a characterization of triangular matrix rings of this type.

If $P \neq 0$, these rings do not satisfy the left or the right AR-condition. Thus if R is a right hereditary α-primitive ring which satisfies the right AR-condition, then R is a prime ring.

THEOREM 3.3. *If R is a Noetherian right hereditary α-primitive ring, then R is a hereditary Noetherian prime ring of Krull dimension* 0 *or* 1.

Proof. We have R/P is a right hereditary prime ring, and by Theorem 3 of [12], then R/P is a hereditary Noetherian prime ring. Consequently, by 3.52 of [6, p. 310], then $|R/P| = 0$ or 1. Since $|R| = |R/P|$, then $|R| = 0$ or 1. If $|R| = 0$, then the faithful critical module C is simple. If $|R| = 1$, the result follows from Lemma 3.5 of [11].

From 2.3, we have

COROLLARY 3.4. *Let R is a Noetherian ring of Krull dimension α and I denote an α-indecomposable injective module. If $R/$*ann Soc *I is right hereditary for all I, then for every α-coprimitive ideal D we have R/D is a hereditary prime ring. If $\alpha = |R|$, then $\alpha = 0$ or* 1.

The upper triangular matrices over $F[x]$, where F is a field, is an example for this corollary.

PROPOSITION 3.5. *Let R be a right Noetherian α-primitive ring with faithful projective α-critical module C. Then R is right hereditary* iff *C is hereditary. In this case, R is a direct sum of critical right ideal, at least one of which is faithful.*

Proof. If C is projective, and R is right hereditary certainly C is hereditary. If C is hereditary, then as in [2], there exists $x_1, x_2, \cdots, x_n \in C$, such that $x_1^r \cap \cdots \cap x_n^r = 0$. As in 3.2, then R is right hereditary, and is a direct sum of critical right ideals.

If C is projective, then C embeds in direct sum of copies of the right hereditary ring R. Again by Proposition 7 of [10, p. 85], since C is critical, then C embeds in R. If $R = \sum_{i=1}^{n} C_i$, where C_i is a critical right ideal for each i, then $C_iC \neq 0$ for some i. Hence there exist a monomorphism of $C \to C_i$. Thus C_i is faithful, and the proof is complete.

COROLLARY 3.6. *Let R be a Noetherian ring of Krull dimension α, where all the α-indecomposable injective modules I are semi-hereditary. If D is the* ann Soc *I, then R/D is a hereditary prime ring, and the α-coprimitive ideals are prime.*

The α-primitive rings which are the direct sum of critical right ideals, at least one of which is faithful, is described in [1].

EXAMPLE 6.6. Let F be a field, and $F[x]$ the polynomial ring in x over F. Let

$$R_1 = \begin{vmatrix} F & F & F[x] \\ 0 & F & F[x] \\ 0 & 0 & F[x] \end{vmatrix}, \quad \text{and} \quad R_2 = \begin{vmatrix} F & 0 & F[x] \\ 0 & F & F[x] \\ 0 & 0 & F[x] \end{vmatrix}.$$

Then R_1 and R_2 have faithful a critical module $C = \begin{vmatrix} F & F & F[x] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$. Now R_1 and R_2 are right hereditary, by Theorem 4.7 of [8, p. 111]. Now C is hereditary over R_1, but C does not embed in R_2. Hence C is not projective over R_2.

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