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TOPOLOGIES ON THE RING OF INTEGERS OF A GLOBAL FIELD

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A characterization of all nondiscrete, locally bounded ring topologies on the ring of integers of an algebraic number field and all those on the ring of integers of an algebraic function field for which the field of constants is bounded is given. As a consequence of these results we obtain Mahler's classic description of the seminorms on the ring of integers of an algebraic number field.

1. Introduction and basic definitions. Let R be a ring and let T be a ring topology on R, that is, T is a topology on R making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R. A subset A of R is *bounded* for T if given any neighborhood U of zero, there is a neighborhood V of zero such that $AV \subseteqq U$ and $VA \subseteqq U$. T is a *locally bounded topology* on R if there exists a bounded neighborhood of zero for T.

We recall that a *seminorm* $||\cdot\cdot||$ on a ring R is function from R to the nonnegative reals satisfying $||x|| = 0$ if $x = 0$, $||x - y|| \leqq ||x|| + ||y||$ and $||xy|| \leqq ||x||\ ||y||$ for all x and y in R. A seminorm on R is a *norm* on R if $||x|| = 0$ implies $x = 0$.

If $||\cdot\cdot||$ is a seminorm on R, for each $\varepsilon > 0$ define $B_\varepsilon = \{r \in R: ||r|| < \varepsilon\}$. Then $\{B_\varepsilon: \varepsilon > 0\}$ is a fundamental system of neighborhoods of zero for a locally bounded topology $T_{||\cdot\cdot||}$ on R. Two seminorms on R are *equivalent* if they define the same topology.

An *algebraic number field* is a finite extension of the rational field Q. An *algebraic function field* is a finite extension of the field $F(x)$ of rational functions over the field F. A *global field* is either an algebraic number field or an algebraic function field.

Weber gave a complete description of the locally bounded topologies on the rational integers Z [12] and Mahler characterized the normed topologies on the ring of integers of an algebraic number field [7]. A description of the locally bounded topologies on the ring of integers of $Q(\sqrt{-d})$ where d is a positive, squarefree integer was obtained by Wieslaw [14]. In this paper we use Weber's description of the locally bounded topologies on global fields ([11, Theorem 3.3] and [13, Theorem 4.4]) to characterize all the locally bounded topologies on the ring of integers of an algebraic number field and also those on the integral closure of $F[x]$ in K for which the subfield F is bounded where K is an algebraic function field. The results of Weber [12], Mahler [7], and Wieslaw [14] are a consequence of this characterization.

II. The ring of integers of a global field. If K is an algebraic number field, let R be the integral closure of the rational integers Z in K, P the set of nonzero prime ideals of R, and let P_∞ be a set of archimedean absolute values on K such that each archimedean absolute value on K is equivalent to exactly one member of P_∞. Then R is a Dedekind domain properly contained in K [5, Theorem 6.1, p. 23], K is the quotient field of R [10, Theorem I, p. 74] and P_∞ is a finite set [5, Proposition 1.4, p. 81]. Denote $P \cup P_\infty$ by P'.

If K is a finite extension of $F(x)$, let Z be $F[x]$, R the integral closure of Z in K, P the set of nonzero prime ideals of R, v_∞ the valuation on $F(x)$ defined by, $v_\infty(f/g) = \deg g - \deg f$, $\{v_1, \cdots, v_n\}$ a complete set of extensions of v_∞ to K [1, Definition 3, p. 140; and Theorem 1, p. 143], and let $P_\infty = \{|\cdot\cdot|_i : i \in [1, n]\}$ where for each $i \in [1, n]$, $|y|_i = 2^{-v_i(y)}$ for all y in K. Then R is a Dedekind domain properly contained in K and K is the quotient field of R [5, Theorem 6.1, p. 23; 10, Theorem I, p. 74]. As before, denote $P \cup P_\infty$ by P'.

If K is a global field, R is called the *ring of integers* of K. Each $p \in P$ defines a p-adic valuation v_p on K (and hence on R) and furthermore, for each a in $K \backslash \{0\}$, $v_p(a) = 0$ for all but finitely many p in P [2, p. 25]. Let T_p denote the topology on R associated with the valuation v_p. Then $\{p^m : m \geqq 0\}$ is a fundamental system of neighborhoods of zero for T_p. For each $p \in P$ and each $m \geqq 0$, let $T'_p m$ be the topology on R for which $\{p^m\}$ is a fundamental system of neighborhoods of zero. We note that T'_{p^m} is the topology on R defined by the seminorm $||\cdot\cdot||_{p^m}$ where

$$||r||_{p^m} = \begin{cases} 0 \text{ if } r \in p^m \\ 1 \text{ if } r \notin p^m \end{cases}.$$

Finally, for each $|\cdot\cdot| \in P_\infty$, let $T_{|\cdot\cdot|}$ be the corresponding absolute value topology on R.

Henceforth, let K be a global field and define Z, R, P, P_∞ and P' as above. We denote topologies on R by T or T' and those on K by $\hat{T}$.

LEMMA 1. *Let T be a ring topology on R and let $|\cdot\cdot| \in P_\infty$. If $\{y \in R : |y| < M\}$ is a T-neighborhood of zero for some $M > 0$, then $T \supseteq T_{|\cdot\cdot|}$.*

Proof. Let $\varepsilon > 0$. Then there exists a c in R such that $|c| > M/\varepsilon$. Indeed, if R is the ring of integers of an algebraic number field, let c be a positive integer such that $|c| > M/\varepsilon$ and if R is the ring of integers of an algebraic function field, let $c = x^n$ where n is a positive integer such that $|x^n| > M/\varepsilon$. As $y \to yc$ is continuous at zero, there exists a T-neighborhood U of zero such that $Uc \subseteq \{y \in R: |y| < M\}$. Then for all u in U, $|u|\ |c| < M$ and hence $|u| < \varepsilon$. Consequently, $\{y \in R: |y| < \varepsilon\}$ is a T-neighborhood of zero and so $T \supseteq T_{|\cdot\cdot|}$.

We recall that a subset I of a field F is an *almost order* of F if (1) $0, 1 \in I$, (2) $-I \subseteq I$, (3) $II \subseteq I$, (4) there exists a nonzero element h in I such that $h(I + I) \subseteq I$, and (5) for each x in F^*, there exist y and z in I^* such that $x = yz^{-1}$. If T is a nondiscrete, locally bounded ring topology on F, then there is an almost order I of F which is a T-bounded neighborhood of zero. Conversely, if I is an almost order of F, then there exists a unique nondiscrete, locally bounded ring topology T_I on F for which I is a bounded neighborhood of zero. Furthermore, T_I is Hausdorff if and only if $I \neq F$ [6, Theorems 5 and 6].

We note also that if U is a bounded neighborhood of zero for a topology T on a field F, then $\{aU: a \neq 0\}$ is a fundamental system of neighborhoods of zero for T [3, Exercise 20b, p. 120]. Consequently, if U and V are two T-bounded neighborhoods of zero, then there exist nonzero elements a and b such that $aU \subseteq V$ and $bV \subseteq U$.

LEMMA 2. *If T is a nondiscrete, locally bounded topology on R, then there exists a T-bounded neighborhood H of zero such that H is a proper almost order of K. Furthermore, if a subset A of R is T-bounded, then A is a T_H-bounded subset of K as well.*

Proof. Let V be a T-bounded neighborhood of zero such that $1 \in V$ and $-V \subseteq V$. Define H by, $H = \{r \in R: rV \subseteq V\}$. H clearly satisfies properties $1 - 3$ in the definition of an almost order. Since V is a T-bounded neighborhood of zero, H is a T-neighborhood of zero and as $1 \in V$, $H \subseteq V$. H is therefore T-bounded as well. Consequently, $H + H$ is T-bounded and so as T is nondiscrete, there exists a nonzero element h in H such that $h(H + H) \subseteq H$. H is a proper subset of K since $H \subseteq R$. So it suffices to show that if d is a nonzero element of K, then there exist nonzero elements y and z in H such that $d = yz^{-1}$. Let s and t be nonzero elements of R such that $d = st^{-1}$. As the mappings $w \to sw$ and $w \to tw$ from R to R are continuous at zero, there exists a nonzero element u in H such that su and tu are in H. Then $d = yz^{-1}$ where $y = su$ and $z = tu$.

To prove the final assertion of the lemma, we note that if A is T-bounded, then there exists a nonzero element h in H such that

$hA \subseteqq H$ as T is nondiscrete. Hence A is bounded for T_H as well.

For each nonempty subset S of P', define $O(S)$ by,

$$O(S) = \{y \in K\colon v_p(y) \geqq 0 \text{ for all } p \in S \cap P$$

and

$$|y| \leqq 1 \text{ for all } |\cdot\cdot| \in S \cap P_\infty\} .$$

We note that $O(P_\infty) \cap Z$ is the set $\{0, \pm 1\}$ if K is an algebraic number field. Therefore, if T is a locally bounded topology on the ring of integers of an algebraic number field, then $O(P_\infty) \cap Z$ is bounded for T. We note further that if K is an algebraic function field, then $O(P_\infty) \cap Z$ is the field F of constants.

Weber proved that if $\hat{T}$ is any Hausdorff, nondiscrete, locally bounded topology on a global field K for which $O(P_\infty) \cap Z$ is bounded, then there exists a nonempty, proper subset S of P' such that $\hat{T}$ is the topology defined by the almost order $O(S)$ [11, Theorem 3.3 and 13, Theorem 4.4]. (S must be a proper subset of P'. Indeed, if K is an algebraic function field, then as $O(P')$ is contained in the set of elements of K which are algebraic over F [4, Corollary, p. 12], $O(P')$ fails to satisfy property 5 in the definition of an almost order. If K is an algebraic number field, then $O(P')$ is finite [9, Theorem 33: 4, p. 69] and so once again, property 5 does not hold.) We use this result and the following lemma due to Seth Warner.

LEMMA 3. *If A is a Hausdorff topological ring with identity* 1 *that has a bounded subfield D containing* 1 *such that the left D-vector space A is finite dimensional, then A is discrete.*

Proof. Let $n = \dim_D A$. First, A is bounded, since if $\{e_1, e_2, \cdots, e_n\}$ is a basis of A, then $A = De_1 + \cdots + De_n$. We next show that any neighborhood U of zero contains an open left ideal. Indeed, let V be a neighborhood of zero such that $V + \cdots + V$ (n times) $\subseteqq U$. As A is bounded, there exists a neighborhood W of zero such that $AW \subseteqq V$. Let $\{a_1, \cdots, a_m\}$ be a maximal, linearly independent subset of W. Then $m \leqq n$ and $W \subseteqq Da_1 + \cdots + Da_m$, for if $a \in W \backslash (Da_1 + \cdots + Da_m)$ then $\{a_1, \cdots, a_m, a\}$ is a linearly independent subset of W. Thus $W \subseteqq Da_1 + \cdots + Da_m \subseteqq Aa_1 + \cdots + Aa_m \subseteqq V + \cdots + V$($m$ times) $\subseteqq U$ as $m \leqq n$. So $Aa_1 + \cdots + Aa_m$ is the desired open left ideal contained in U.

As A is finite dimensional over D and therefore artinian, it has a minimal open ideal J which is clearly not only minimal but the smallest. Consequently by the preceding, $J = (0)$ as A is Hausdorff. So A is discrete.

THEOREM 1. *Let T be a nondiscrete, locally bounded topology on the ring R of integers of a global field K for which $O(P_\infty) \cap Z$ is bounded. Then there exists a proper subset S of P_∞, a subset P_1 of P and nonnegative integers $n(p)$ for each $p \in P\backslash P_1$ such that*

$$T = \sup(\sup_{p \in P_1} T_p, \ \sup_{p \in P\backslash P_1} T'_{p^{n(p)}}, \ \sup_{|\cdot\cdot| \in S} T_{|\cdot\cdot|}) .$$

In particular, if K is an algebraic number field or if K is a finite dimensional extension of the field $F_q(x)$ of rational functions over a finite field F_q having q elements, then every nondiscrete, locally bounded topology on R can be described as above.

Proof. Let H be a T-bounded neighborhood of zero such that H is a proper almost order of K. Let $\hat{T}$ be the unique Hausdorff, nondiscrete, locally bounded topology on K for which H is a bounded neighborhood of zero. (We note that $\hat{T}|_R \supseteq T$.) By Lemma 2, $O(P_\infty) \cap Z$ is $\hat{T}$-bounded and hence by [11, Theorem 3.3; 13, Theorem 4.4], there exists a proper subset S' of P' such that $\hat{T}$ is the topology associated with the almost order $O(S')$. So there exists a nonzero element a in K such that $aO(S') \subseteq H \subseteq R$. If $O(S') \nsubseteq R$, let $y \in O(S')\backslash R$, let $p \in P$ be such that $v_p(y) < 0$ and let $m > 0$ be such that $v_p(a) + mv_p(y) < 0$. Then $ay^m \in aO(S') \subseteq R$ but $v_p(ay^m) < 0$. Contradiction! So $O(S') \subseteq R = O(P)$ and hence $O(P \cup S') = O(S')$. Therefore, $P \cup S'$ is a proper subset of P' as $O(S')$ is an almost order of K and so by replacing S' with $P \cup S'$, we may assume that S' is a nonempty, proper subset of P' and $P \subseteq S'$. Consequently, the set S defined by, $S = S' \cap P_\infty$, is a proper subset of P_∞.

Let $|\cdot\cdot| \in S$. As H is a $\hat{T}$-bounded neighborhood of zero, there exists a nonzero element b in K such that $bH \subseteq O(S') \subseteq O(\{|\cdot\cdot|\})$. So for each $h \in H$, $|h| \leqq 1/|b|$. Consequently, $\{y \in R: |y| < 2/|b|\}$ is a T-neighborhood of zero and so by Lemma 1, $T \supseteq T_{|\cdot\cdot|}$.

For each $p \in P$, let $n(p) = \sup\{n \geqq 0: p^n$ is a T-neighborhood of zero$\}$. Let $P_1 = \{p \in P: n(p) = \infty\}$. We note that for all $p \in P\backslash P_1$, p^n is T-open if and only if $0 \leqq n \leqq n(p)$ and for all $p \in P_1$, p^n is T-open for $n \geqq 0$. Clearly, $T \supseteq \sup(\sup_{p \in P_1} T_p, \sup_{p \in P\backslash P_1} T'_{p^{n(p)}}, \sup_{|\cdot\cdot| \in S} T_{|\cdot\cdot|})$. So it suffices to prove that T is weaker than the supremum topology.

Let U be any T-closed neighborhood of zero. Then U is a $\hat{T}$-neighborhood of zero as well. So there exists a nonzero element a in R such that $aO(S') \subseteq U$. Let $p_1, \cdots, p_m \in P$ be such that $v_{p_i}(a) > 0$ for $i \in [1, m]$ and $v_p(a) = 0$ for $p \in P\backslash\{p_i: 1 \leqq i \leqq m\}$ [2, p. 25]. Consider the set $V \cap C_1$ where $V = \bigcap_{i=1}^m p_i^{v_{p_i}(a)}$ and $C_1 = \{y \in R: |y| < |a|$ for all $|\cdot\cdot| \in S\}$. If $y \in V \cap C_1$, then $a^{-1}y \in O(S')$ and so $y \in aO(S') \subseteq U$. Therefore, $V \cap C_1 \subseteq U$. Let $\bar{V}$ be the T-closure of V

and define W by, $W = \bar{V} \cap C_2$ where $C_2 = \{y \in R: |y| < |a|/2$ for all $|\cdot\cdot| \in S\}$.

We next show that $W \subseteqq U$. Let $w \in W$ and let W' be any T-neighborhood of zero. As $T_{|\cdot\cdot|} \subseteqq T$ for all $|\cdot\cdot| \in S$, we may assume that $w' \in C_2$ for all $w' \in W'$. Since $w \in \bar{V}$, there exists $w' \in W'$ such that $w + w' \in V$. Furthermore, $|w + w'| \leqq |w| + |w'| < |a|/2 + |a|/2 = |a|$ for all $|\cdot\cdot| \in S$ and so $w + w' \in V \cap C_1$. Therefore, w is in the T-closure of $V \cap C_1$. But $V \cap C_1 \subseteqq U$, a T-closed set. Hence $w \in U$ and consequently $W \subseteqq U$.

To complete the proof of the theorem, it suffices to show that W is open for the supremum topology. As V is an ideal of R, $\bar{V}$ is also an ideal of R [3, Proposition 5, p. 77] containing V. So $\bar{V} = \bigcap_{i=1}^{m} p_i^{n_i}$ where $n_i \leqq v_{p_i}(a)$ for $i = 1, 2, \cdots, m$ [2, p. 26]. The canonical epimorphism ϕ from R to $R/\bar{V}$ then defines a Hausdorff ring topology on $R/\bar{V}$ [3, Proposition 18, p. 25]. If K is an algebraic number field, $R/\bar{V}$ is a finite ring [9, Theorem 33: 2, p. 67] and the topology on $R/\bar{V}$ is therefore discrete. If K is an algebraic function field, as ϕ is open and continuous, $\phi(F)$ is bounded. Furthermore, $R/\bar{V}$ is finite dimensional over $\phi(F)$ [4, proof of Theorem, p. 23]. Therefore, by Lemma 3, $R/\bar{V}$ is discrete. So in both cases, $\bar{V}$ is open for T. Thus for $1 \leqq i \leqq m$, $p_i^{n_i}$ is T-open and so $n_i \leqq n(p_i)$. Hence W is open for the supremum topology.

COROLLARY 1. *If P_∞ has exactly one element, then the following statements are equivalent.*

1°. *T is a nondiscrete ring topology on R for which the T-open ideals form a fundamental system of neighborhoods of zero.*

2°. *T is a nondiscrete ring topology on R for which R is bounded.*

3°. *T is a nondiscrete, locally bounded topology on R for which $O(P_\infty) \cap Z$ is bounded.*

Proof. Clearly 1° implies 2° and 2° implies 3°. To prove that 3° implies 1° we need only notice that as the set S defined in Theorem 1 is a proper subset of P_∞, $S = \varnothing$.

COROLLARY 2 [12, Theorem 1.5; 14, Theorem 1]. *If R is Z, $F_q[x]$, or the integral closure of Z in $Q(\sqrt{-d})$ where d is a positive, squarefree integer, then statements* 1°, 2°, *and* 3° *are equivalent.*

Proof. Corollary 2 follows from Corollary 1 and the observation that P_∞ has exactly one element. (The proof for $Q(\sqrt{-d})$ is the same as the proof of Corollary 3 to Theorem 3 of [8].)

THEOREM 2. *Let K be a global field and let R be the ring of integers of K. If T is a nondiscrete ring topology on R, then the following statements are equivalent.*

1°. *There exists a proper subset S of P_∞, a finite subset P_1 of P, a finite subset P_2 of $P\backslash P_1$ and positive integers $n(p)$ for $p \in P_2$ such that $T = \sup(\sup_{p\in P_1} T_p, \sup_{p\in P_2} T'_{p^{n(p)}}, \sup_{|\cdot\cdot|\in S} T_{|\cdot\cdot|})$.*

2°. *T is defined by a seminorm which is bounded on $O(P_\infty) \cap Z$.*

3°. *T is a locally bounded ring topology on R for which $O(P_\infty) \cap Z$ is bounded and there exists a nonzero element c in R such that $c^n \to 0$ in T.*

Proof. We first show that 1° implies 2°. If $S \cup P_1 \cup P_2 = \emptyset$, then T is defined by the seminorm $||\cdot\cdot||$ where $||r|| = 0$ for all r in R. If $S \cup P_1 \cup P_2 \neq \emptyset$, then T is defined by the seminorm

$$\sup(\sup_{p\in P_1} |\cdot\cdot|_p, \sup_{p\in P_2} ||\cdot\cdot||_{p^{n(p)}}, \sup_{|\cdot\cdot|\in S} |\cdot\cdot|)$$

where for each $p \in P_1$, $|r|_p = 2^{-v_p(r)}$ for all r in R.

Since T is nondiscrete, if T is defined by a seminorm $||\cdot\cdot||$, then there exists a nonzero element c in R such that $||c|| < 1$. Therefore, $c^n \to 0$ in T and so 2° implies 3°.

To prove that 3° implies 1°, first notice that if $c^m \in p$ for some $m > 0$, then $v_p(c) > 0$. Define $n(p)$, P_1 and S as in Theorem 1. Then $\{p \in P: n(p) > 0\} \subseteqq \{p \in P: v_p(c) > 0\}$, a finite set [2, p. 25]. So P_1 is finite. Clearly $T = \sup(\sup_{p\in P_1} T_p, \sup_{p\in P_2} T'_{p^{n(p)}}, \sup_{|\cdot\cdot|\in S} T_{|\cdot\cdot|})$ where $P_2 = \{p \in P\backslash P_1: n(p) > 0\}$.

COROLLARY [7, p. 328]. *If $||\cdot\cdot||$ is a seminorm on the ring of integers R of a global K for which $O(P_\infty) \cap Z$ is bounded and if $||r|| \neq 0$ for some r in R, then $||\cdot\cdot||$ is equivalent to the supremum of finitely many p-adic absolute values, finitely many seminorms $||\cdot\cdot||_{p^n}$ defined by the nonzero ideals p^n of R, and finitely many absolute values from P_∞.*

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