PACIFIC JOURNAL OF MATHEMATICS
Vol. 93, No. 2, 1981

CONTINUOUS SELECTIONS AND REALCOMPACTNESS

I. BLUM AND S. SWAMINATHAN

The class of basically fixed, lowersemicontinuous carriers is defined, and the existence of continuous selections for members of this class is investigated. It is shown that, barring the existence of measurable cardinals, a completely regular Hausdorff space is realcompact iff every basically fixed, lowersemicontinuous carrier of infinite character from the space to the convex subsets of a locally convex space admits a selection. One application of this result is the proof that the union of a locally finite collection of realcompact cozero sets is realcompact, provided the union is of nonmeasurable cardinal.

1. Introduction. The well-known selection theorem of Michael [6, 3.2″] says that a T_1-space X is paracompact iff every lowersemicontinuous (l.s.c.) carrier from X to the family of closed convex subsets of a Banach space Y admits a selection. In this paper we examine the analogous question for realcompact completely regular Hausdorff spaces. We define the class of basically fixed carriers, and find conditions under which every basically fixed carrier on a realcompact space admits a selection. The main result is a characterization of realcompactness by a property of basically fixed carriers as well as by a selection property similar to Michael's [6, 3.2″ (b)]. Our results can easily be generalized to topologically complete spaces. As an application of the characterization of realcompactness we prove a sum theorem for realcompactness and examine its relation to known theory.

2. Preliminaries. In general, the terminology of Gillman-Jerison [5] is used. All spaces are assumed to be completely regular T_1-spaces. $C(X)$ denotes the ring of real valued continuous functions on X. A maximal ideal M of $C(X)$ is free (fixed) if the z-ultrafilter $Z(M)$ of X, consisting of zero sets of members of M, has empty (nonempty) intersection. M is real if $Z(M)$ has the countable intersection property; otherwise M is hyperreal.

An open cover of X is a cozero cover if its members are cozero sets. An open (cozero) cover $\mathscr{U}$ is maximal, if, for every open (cozero) set $V \notin \mathscr{U}$, the cover $\mathscr{U} \cup \{V\}$ has a finite subcover.

For the definitions of carrier, lower semicontinuity and selection, refer to [6]. As in [6], $\mathscr{F}(Y)$ denotes the collection of all closed, convex subsets of a linear topological space Y, and $\mathscr{K}(Y)$ the collection of all convex subsets of Y.

A partition of unity on a space X is a subset P of $C(X)$ such that (i) $f \geqq 0$ for every $f \in P$ and (ii) $\sum_{f \in P} f(x) = 1$ for every $x \in X$. A partition of unity P is called locally finite if the collection of cozero sets of functions in P form a locally finite cover of X. P is said to be subordinate to a cover $\mathscr{U}$ of X if the family of cozero sets of members of P refines $\mathscr{U}$.

A cardinal number m is called measurable if the discrete space of cardinal m admits a nontrivial $\{0, 1\}$-valued (countably additive) measure. Measurable cardinals, if any exist, are strongly inaccessible [5, 125]. The following theorem of DeMarco-Wilson [3] is a motivating factor for our work:

THEOREM DMW. *The following are equivalent for the maximal ideal M of $C(X)$:* (a) *M is hyperreal.* (b) *M contains a (locally finite) partition of unity of nonmeasurable cardinal.*

3. Fixed carriers. Let X and Y be topological spaces and let $\mathscr{S}$ a given family of subsets of X. A carrier $\phi: X \to 2^Y$ is called *$\mathscr{S}$-fixed* if, for every $S \in \mathscr{S}$, $\cap \{\phi(x): x \in S\} \neq \emptyset$.

The following simple properties of $\mathscr{S}$-fixed carriers are easily verified. Let $\phi: X \to 2^Y$ denote an $\mathscr{S}$-fixed carrier:

(a) If the carrier $\psi: X \to 2^Y$ satisfies $\psi(x) \supseteqq \phi(x)$ for every $x \in X$, then ψ is $\mathscr{S}$-fixed.

(b) If $\mathscr{T}$ is a collection of subsets of X such that $\mathscr{T} \subset \mathscr{S}$, then ϕ is $\mathscr{T}$-fixed.

In the sequel, we shall be considering $\mathscr{S}$-fixed carriers for the following families $\mathscr{S}$:

$\mathscr{A} = \{A \subset X$: A is a cozero set in X, $\bar{A}$ is compact, and $X - A$ is not compact$\}$;

$\mathscr{B} = \{B \subset X$: B is a realcompact cozero set in X and $X - B$ is not compact$\}$;

$\mathscr{C} = \{C \subset X$: C is a topologically complete cozero set in X and $X - C$ is not compact$\}$.

Clearly $\mathscr{A} \subset \mathscr{B} \subset \mathscr{C}$. $\mathscr{B}$-fixed carriers will be referred to as *basically fixed* carriers because of their basic relationship to realcompact spaces.

The classes of $\mathscr{S}$-fixed carriers for $\mathscr{S} \in \{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$ are nonempty since the trivial carrier $\theta(x) = Y$ $(x \in X)$, belongs to each class. For this carrier, as well as for any carrier θ which satisfies $\cap \{\theta(x): x \in X\} \neq \emptyset$, the selection problem is trivially solved by a constant function.

Even the discrete space N of positive integers admits a nontrivial basically fixed carrier.

EXAMPLE. A lowersemicontinuous, $\mathscr{C}$-fixed (hence $\mathscr{B}$- and $\mathscr{A}$-fixed) carrier to the closed convex subsets of a Banach space need not admit a constant selection.

Proof. Let p satisfy $1 \leqq p \leqq \infty$; and let $Y = \ell_p(R)$. Let $\theta: N \to \mathscr{F}(Y)$ be defined as follows: For $n \in N$, $\theta(n) = \{x = (x_i) \in \ell_p: x_1 = 1, x_{n+1} = 2x_n, x_i \in R \text{ if } i \neq 1, n+1\}$. $\theta(n)$ is evidently closed and convex for each $n \in N$, and if some $x \in \cap\{\theta(n): n \in N\}$, then x is a strictly increasing sequence and thus cannot belong to ℓ_p. Thus no constant function can be a selection for θ. To see that θ is $\mathscr{C}$-fixed, note that if U is any proper subset of N, and $k \in N - U$, the sequence x^* defined by $x_i = 2^{i-1}(i = 1, \cdots, k)$ and $x_i = 0(i > k)$ is an element of $\theta(n)$ for each $n \neq k$. Since N is discrete, lowersemicontinuity is obvious.

While this carrier admits nonconstant selections, results below show that this need not always be true. Theorem 1 shows that in one sense the above example is the best possible: No nontrivial $\mathscr{A}$-fixed (hence $\mathscr{B}$-, or $\mathscr{C}$-fixed) carriers exist from a discrete space of uncountable cardinal to the closed convex subsets of a separable Banach space.

Note further that the carrier θ above is actually $\mathscr{S}$-fixed for the family of all proper subsets of N. By modifying the definition of $\theta(x)$ to include only strictly positive sequences, we obtain a carrier to the convex subsets of Y which is $\mathscr{A}$-fixed, but not $\mathscr{A}'$-fixed for any family $\mathscr{A}'$ which contains a subset with compact complement.

4. Realcompact spaces. Recall that a topological space X is realcompact if every free maximal ideal of $C(X)$ is hyperreal. We wish to find conditions which ensure selection theorems for basically fixed carriers defined on realcompact spaces. The following theorem is a simple result concerning constant selections for such carriers on a class of realcompact spaces.

THEOREM 1. *Let X be a realcompact space in which every countable subset is disjoint from some noncompact closed subset of X, and let B be a separable Banach space. Then any basically fixed carrier θ to the closed subsets of B has a constant selection.*

Proof. Let $\Upsilon = \{\theta(x): x \in X\}$. Υ is a family of zero sets of B. To see that Υ has the finite intersection property, let $\{x_n\}_{n=1}^k$ be any finite subset of X. By hypothesis, there is a closed, noncompact subset F of X disjoint from $\{x_n\}_{n=1}^k$, and by [5; 3D1], a zero set Z disjoint from $\{x_n\}_{n=1}^k$ containing F. It follows that $X - Z$ is a realcompact cozero subset of X with noncompact complement, and because θ is basically fixed we have

$$\cap \{\theta(x_n): n \leqq k\} \supseteqq \cap \{\theta(x): x \in X - Z\} \neq \varnothing .$$

Let Υ' be the z-filter generated by Υ. It suffices to show that Υ' is fixed, since in that case Υ is *a fortiori* fixed, and for any $p \in \cap \Upsilon$, the constant function $f(x) = p$ $(x \in X)$ is a selection for θ. Since separable metric spaces are Lindelöf, by [5; 8H5] it suffices to show that the z-filter Υ' on B has the countable intersection property. To this end, let $\{Z_n\}_{n=1}^{\infty}$ be any countable subfamily of Υ'. Since Υ' is generated by Υ, for each $n \in N$, there is a finite subset $\{x_{i,n}\}_{i=1}^{k(n)}$ of X such that

$$Z_n \supseteqq \cap \{\theta(x_{i,n}): 1 \leqq i \leqq k(n)\} .$$

By assumption, the countable set $\{x_{n,i}: 1 \leqq i \leqq k(n), n \in N\}$ is disjoint from some noncompact closed set F, and, as above, we can conclude the existence of a realcompact cozero set C with noncompact complement which contains $\{x_{n,i}: 1 \leqq i \leqq k(n), n \in N\}$. Since θ is basically fixed, we have

$$\cap \{Z_n: n \in N\} \supseteqq \cap \{\theta(x_{i,n}): 1 \leqq i \leqq k(n): n \in N\} \supseteqq \cap \{\theta(x): x \in C\} \neq \varnothing .$$

This completes the proof.

Let X be a space, Y a locally convex space and V a symmetric, convex neighborhood of 0 in Y. Let $\phi: X \to \mathscr{K}(Y)$ be an l.s.c. carrier and, for $y \in Y$, $U_y = \{x \in X: y \in \phi(x) + V\}$. Then the collection $\mathscr{U}(V) = \{U_y: y \in Y\}$ is an open cover of X. ϕ is said to be *of infinite character* iff there exists V such that $\mathscr{U}(V)$ has no finite subcover. If ϕ is not of infinite character, it is *of finite character*.

THEOREM 2. *Let X be realcompact, Y a locally convex space and $\phi: X \to \mathscr{K}(Y)$ an* l.s.c. *basically fixed carrier. Then ϕ is of finite character.*

Proof. If X is compact, the conclusion is obvious. If X is not compact, then it cannot be pseudocompact either. In such a case, by [5, 1G4] there exist two disjoint zero sets in X neither of which is compact. Let C_1 and C_2 denote the complementary cozero sets. Then $X = C_1 \cup C_2$. Since ϕ is basically fixed, and C_i is realcompact, there exist $y_i \in \cap \{\phi(x): x \in C_i\}$, $i = 1, 2$. For each symmetric convex neighborhood of 0 in Y, we then have the sets $\{U_{y_i}: i = 1, 2\}$ form a cover of X.

This condition is, however, not strong enough to guarantee the existence of a selection for ϕ. The carriers defined in [6, Examples 6.2 and 6.3] are of finite character, but do not admit selections. We do have the following result, analogous to [6, 4.1].

THEOREM 3. *Let X be realcompact, Y a locally convex space, $\phi: X \to \mathscr{K}(Y)$ an* l.s.c., *basically fixed carrier, and V a convex zero neighborhood in Y. Then there exists a continuous function $f: X \to Y$ such that $f(x) \in \phi(x) + V$ for each $x \in X$.*

Proof. By Theorem 2, ϕ is of finite character. Thus there is a finite subset $T \subset Y$ such that $X = \cup \{U_t: t \in T\}$. Consider the collection $\mathscr{W} = \{W \subset X: W$ is a cozero set and there exists $t \in T$ such that $W \subset U_t\}$.

Case 1. $\mathscr{W}$ has a finite subcover. Then each U_t is a cozero set of X. For each $t \in T$, let $f_t \in C(X)$ satisfy $0 \leqq f_t \leqq 1$, $U_t = \operatorname{coz} f_t$; and define $g_t \in C(X)$ by $g_t(x) = f_t(x)/\sum \{f_s: s \in T\}$. Then $\{g_t: t \in T\}$ is a finite partition of unity on X, and for each $t \in T$, $\operatorname{coz} g_t = U_t$. Then $f: X \to Y$, defined by $f(x) = \sum \{tg_t(x): t \in T\}$ satisfies the required properties.

Case 2. $\mathscr{W}$ has no finite subcover. Let $\mathscr{W}'$ be a maximal cozero cover of X which contains $\mathscr{W}$. Then by [4, 1.4], $\mathscr{W}'$ has a countable subcover $\{W_n\}_{n=1}^{\infty}$. For each $n \in N$, let $f_n \in C(X)$ satisfy $0 \leqq f_n \leqq 1$, $W_n = \operatorname{coz} f_n$, and define $g_n \in C(X)$ by $g_n(x) = 2^n f_n(x)/\sum_{i=1}^{\infty} 2^{-i} f_i(x)$. Then $\{g_n\}_{n=1}^{\infty}$ is a partition of unity on X, and $W_n = \operatorname{coz} g_n$. By [4, 1(c)] there exists a locally finite partition of unity $\{g'_n\}_{n=1}^{\infty}$ on X such that $\operatorname{coz} g'_n \subseteqq \operatorname{coz} g_n = W_n$. We can conclude that $\operatorname{coz} g_n$ has noncompact complement, since otherwise W_n would have compact complement and could not belong to a maximal cover. By the $\mathscr{B}$-fixedness of ϕ, we can now find, for each $n \in N$, $y_n \in Y$ such that $y_n \in \cap \{\phi(x): x \in \operatorname{coz} g'_n\}$. Since $\{g'_n: n \in N\}$ forms a locally finite partition of unity and $\phi(x)$ is convex for each $x \in X$, the function $f: X \to Y$ defined by $f(x) = \sum_{n=1}^{\infty} g'_n(x) \cdot y_n$ satisfies the required properties. (If Case 2 occurs, we can actually conclude that f is a selection for ϕ.)

The next result gives a sufficient condition for the existence of a selection.

THEOREM 4. *Let X be normal and countably paracompact. If X is also realcompact, then every lowersemicontinuous basically fixed carrier ϕ from X to the closed convex subsets of a Banach space admits a selection.* (*It will be shown below that the converse is also true if X is a space of nonmeasurable cardinal.*)

Proof. Let $\{V_n\}_{n=1}^{\infty}$ be a base for the convex, symmetric neighborhoods of the origin in Y, such that $V_{n+1} \subset (1/2)^n V_n$. Since X is realcompact, it follows from Theorem 2 that θ is of finite character. For each $n \in N$, there exists a finite subset $T_n \subseteqq Y$ such that the cover

$\mathscr{U}(V_n)$ (defined above) has $\{U_t: t \in T_n\}$ as a finite subcover. Let $T = \bigcup_{n=1}^{\infty} T_n$. The linear subspace Y' of Y generated by T is separable. Define the carrier $\psi: X \to \mathscr{F}(Y')$ by $\psi(x) = \phi(x) \cap Y'$. Then ψ is l.s.c., $\psi(x) \neq \phi$ for each $x \in X$, and, by [6, 3.1], ψ will have a selection, which will be a selection for ϕ also.

An entirely analogous proof establishes the following:

THEOREM 5. *Let X be a perfectly normal space. If X is realcompact, then every* l.s.c. *carrier to the convex subsets of a Banach space admits a selection. (By Theorem 9 below, the converse is also true if X has nonmeasurable cardinal.)*

5. **Necessary conditions.** In this section we show, that in contrast to Theorem 2, a nonrealcompact space always admits basically fixed carriers of infinite character.

We begin with the following useful lemmas. Lemma 6 summarizes Michael's technique of constructing an l.s.c. carrier to the closed convex subsets of a Banach space from an open cover of X.

LEMMA 6. *Let $\mathscr{U}$ be an open cover of a space X and $Y = \ell_1(\mathscr{U})$, the Banach space of all functions $y: \mathscr{U} \to R$ such that $||y|| = \sum_{U \in \mathscr{U}} |y(U)| < \infty$. Let $C = \{y \in Y: ||y|| = 1$ and $y(U) \geqq 0$ for every $U \in \mathscr{U}\}$ and $\phi(x) = \{y \in C \mid y(U) = 0$ for all $U \in \mathscr{U}$ such that $x \notin U\}$. Then ϕ is* l.s.c. *and $\phi(x) \in \mathscr{F}(Y)$. Further, if there is a continuous selection for ϕ, then there exists a locally finite partition of unity subordinate to $\mathscr{U}$.*

For proof, we refer to [6, p. 369].

Recall that, by [5, 8.4], the free real maximal ideals of $C(X)$ may be indexed by the points of $\upsilon X - X$, where υX denotes the Hewitt realcompactification of X.

LEMMA 7. $\bigcap \{M^p: p \in \upsilon X - X\} = \{f \in C(X): \operatorname{coz}(f)$ *is realcompact*$\}$.

Proof. If $\operatorname{coz}(f)$ is realcompact, then $\operatorname{coz}(f) \cup \operatorname{cl}_{\beta X} Z(f)$ is realcompact subset of βX containing X. Hence $\upsilon X \subseteqq \operatorname{coz}(f) \cup \operatorname{cl}_{\beta X} Z(f)$ and $\upsilon X - X \subseteqq \operatorname{cl}_{\beta X} Z(f)$. Thus, for every $p \in \upsilon X - X$, $f \in M^p$.

To prove the converse let $f \in \bigcap \{M^p: p \in \upsilon X - X\}$, and assume for a contradiction that $\operatorname{coz}_X(f)$ is not realcompact. If f^υ denotes the continuous extension of f to υX, then $\operatorname{coz}_{\upsilon X} f^\upsilon = \upsilon(\operatorname{coz}_X f)$, [1, 5.2]. Since $\operatorname{coz}_X f$ is not realcompact, it follows that there is $p \in (\operatorname{coz}_{\upsilon X} f^\upsilon) - \operatorname{coz}_X f$. Thus $p \in \upsilon X - X \subseteqq \operatorname{cl}_{\beta X} Z_X(f) = \operatorname{cl}_{\beta X} Z_{\upsilon X}(f^\upsilon)$. (The first inclusion follows from the assumption, the second by [3, 8.8].) This is impossible since $\text{coz}_{vX} f^v \cap z_{vX}(f^v) = \emptyset$.

THEOREM 8. *If X is not realcompact, then there exists a Banach space Y, and a basically fixed* l.s.c. *carrier $\phi: X \to \mathscr{F}(Y)$, which is of infinite character. Furthermore, if ϕ admits a selection, then X has measurable cardinal.*

Proof. Suppose X is not realcompact. Let M be a free real maximal ideal. Consider the open cover $\mathscr{W}$ of X defined by $\mathscr{W} = \{\text{coz}(f): f \in M\}$. Let $Y = \ell_1(\mathscr{W})$ and $\phi: X \to 2^Y$ be defined as in Lemma 6. Then ϕ is l.s.c. We show that ϕ is basically fixed. Let $A \in \mathscr{B}$ and $A = \text{coz}(h)$. Then, by Lemma 7, $h \in M^{vX-X} \subseteqq M$. Thus $A \in \mathscr{W}$. Define $y_0: \mathscr{W} \to R$ as follows: for $W \in \mathscr{W}$,

$$y_0(W) = \begin{cases} 0 & \text{if} \quad W \neq A \\ 1 & \text{if} \quad W = A\,. \end{cases}$$

Then $y_0 \in \cap \{\phi(x): x \in A\}$, i.e., ϕ is basically fixed.

To show that ϕ is of infinite character, let $V = \{y: \|y\| < 1\}$ and assume that there exists a finite subset $\{y_1, y_2, \cdots, y_k\}$ of Y such that $\cup \{U_{y_i}: i = 1, \cdots, k\} = X$. The collection $\mathscr{W}'$ of all those $W \in \mathscr{W}$ for which there exists an $i \in 1, \cdots, k$ such that $y_i(W) > 0$ is countable. Since M is a real ideal, we have $X \neq \cup \{W: W \in \mathscr{W}'\}$. Let x be an element of X which is not in the union. Then every $y \in \phi(x)$ satisfies $y(W) = 0$ for every $W \in \mathscr{W}'$. Hence $\|y - y_i\| \geqq \|y_i\| = 1 > 0$ for every $i \leqq k$. It follows that $x \notin \cup \{U_{y_i}: i = 1, \cdots, k\}$, a contradiction. Thus ϕ is an l.s.c. basically fixed carrier of infinite character. To prove the second statement, assume ϕ has a selection. Then by Lemma 6, we have a partition of unity P subordinated to $\mathscr{W}$, i.e., $P \subseteqq M^P$. Theorem DMW yields that P must be measurable. Since measurable cardinals are strongly inaccessible, $|X|$ is measurable, a contradiction.

We can summarize the above results in the following theorem:

MAIN THEOREM 9. *Let X be a completely regular space of nonmeasurable cardinal. The following are equivalent*:

(a) *X is realcompact.*

(b) *Every basically fixed* l.s.c. *carrier from X to the convex subsets of a locally convex space is of finite character.*

(c) *Every basically fixed,* l.s.c. *carrier of infinite character from X to the convex subsets of a locally convex space admits a selection.*

Proof. (a) $\to$ (b) Theorem 2.
(b) $\to$ (c) Trivial.
(c) $\to$ (a) Theorem 8.

REMARK. The restriction of nonmeasurable cardinal is not needed for (a) $\to$ (b) $\to$ (c). However, this restriction is necessary for (c) $\to$ (a): Any discrete space of measurable cardinal is not realcompact [5; 12.2], but is paracompact, and Michael's Theorem [6; 3.2″] applies.

6. Application to unions of realcompact spaces. Several conditions which imply that a union of realcompact spaces is also realcompact have been investigated. Some of these are given in [1] and [2]. The example of S. Mrowka cited in [4, §4] shows that the union of two closed realcompact subspaces need not be realcompact. Open subsets of a realcompact space need not be realcompact, hence it follows that an arbitrary union of realcompact cozero sets need not be realcompact. However, using the condition of Theorem 8 we prove:

THEOREM 10. *Let $\{C_\alpha: \alpha \in \Lambda\}$ be a locally finite collection of realcompact cozero subspaces of X, let $C = \cup\{C_\alpha: \alpha \in \Lambda\}$, and assume $|C|$ is a nonmeasurable cardinal. Then C is realcompact.*

Proof. If any C_α has compact complement, C may be expressed as a finite union of realcompact cozero sets, which is realcompact by Lemma 7. Otherwise, each C_α has noncompact complement. Let $\phi: C \to \mathscr{K}(Y)$ be any l.s.c. basically fixed carrier from C to the convex subsets of a locally convex space Y. By Theorem 8 it suffices to show that ϕ admits a selection.

Since ϕ is basically fixed, and each $C_\alpha \in \mathscr{B}$, we have, for each $\alpha \in \Lambda$, $y_\alpha \in \cap\{\phi(x): x \in C_\alpha\}$. Since $\{C_\alpha: \alpha \in \Lambda\}$ is locally finite, there is a subordinate locally finite partition of unity $\{g_\alpha\}_{\alpha\in\Lambda}$ (Argue as in the proof of Theorem 3.) Let $f(x) = \Sigma\{g_\alpha(x)\cdot y_\alpha: \alpha \in \Lambda\}$ for each $x \in C$. Then f is continuous, since $\{g_\alpha: \alpha \in \Lambda\}$ is a locally finite partition of unity, and for every $x \in X$, $g_\alpha(x) > 0$ implies $x \in C_\alpha$ and consequently $y_\alpha \in \phi(x)$. Thus $f(x) \in \phi(x)$, being a convex combination of elements of $\phi(x)$ and f is a selection for ϕ. This completes the proof.

A covering $\mathscr{U}$ of X is said to be *normal*, if there exists a continuous pseudometric on X for which the 1-balls refine $\mathscr{U}$. Since a locally finite cozero covering is normal, the above result may also be derived from the following:

THEOREM (*Blair*, [2, 2.1]). *Let $\mathscr{U}$ be a normal cover of X of nonmeasurable cardinal, with $U \subseteq X$ for each $U \in \mathscr{U}$. Then $\upsilon X = \upsilon\{\upsilon U: U \in \mathscr{U}\}$.*

Blair and Hager show that this result implies the Katětov-Shirota theorem that a complete, nonmeasurable uniform space is realcompact. We now show that Theorem 10 is equivalent to a slightly weakened form of Blair's result, which also implies the Katětov-Shirota theorem. The following characterization is given in [2, 2.4].

LEMMA (*Blair and Hager*). *Let* $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ *be a cover of* X. $\mathscr{U}$ *is normal if, and only if, there is a locally finite cover* $\mathscr{V} = \{V_\alpha : \alpha \in \Lambda\}$ *of* X, *consisting of cozero sets, with* V_α *and* $X - U_\alpha$ *completely separated for each* $\alpha \in \Lambda$.

LEMMA 11. *Let* $|\upsilon X|$ *be a nonmeasurable cardinal. Let* Λ *be an index set nonmeasurable cardinal. A cover* $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ *of* X *is normal if, and only if,* $\upsilon\mathscr{U} = \{\upsilon U_\alpha : \alpha \in \Lambda\}$ *is a normal cover of* υX.

Proof. Let $\mathscr{U}$ be a normal covering of X. By applying the previous lemma twice it is possible to obtain cozero coverings $\mathscr{V} = \{V_\alpha : \alpha \in \Lambda\}$ and $\mathscr{W} = \{W_\alpha : \alpha \in \Lambda\}$ of X such that for each $\alpha \in \Lambda$, V_α is completely separated from $X - U_\alpha$ and W_α is completely separated from $X - V_\alpha$. Now, by [1, 5.2], $\upsilon\mathscr{W} = \{\upsilon W_\alpha : \alpha \in \Lambda\}$ is a family of cozero sets of υX. To see that $\upsilon\mathscr{W}$ is locally finite, it suffices to show that any neighborhood N in υX such that $N \cap X$ is disjoint from V_α, cannot meet υW_α. To this end, let Z_1 and Z_2 be a complete separation of W_α and $X - V_\alpha$ in X. Then $\mathrm{cl}_{\upsilon X} Z_1 \cap \mathrm{cl}_{\upsilon X} Z_2 = \emptyset$ and since W_α is a cozero set, $\upsilon W_\alpha \subseteqq \mathrm{cl}_{\upsilon X} Z_1$, while $N \subseteqq \mathrm{cl}_{\upsilon X} N \cap X \subseteqq \mathrm{cl}_{\upsilon X} X - V_\alpha \subseteqq \mathrm{cl}_{\upsilon X} Z_2$.

By Theorem 10, it follows that $\cup\{\upsilon W_\alpha : \alpha \in \Lambda\}$ is realcompact, and since this set obviously contains X, $\upsilon\mathscr{W}$ is a locally finite cozero cover of υX. Moreover, for each $\alpha \in \Lambda$, υW_α is completely separated from $\upsilon X - \upsilon U_\alpha$: Let Z_1 and Z_2 be a complete separation of $X - V_\alpha$ and W_α. Since V_α is a cozero subset of U_α we have $\upsilon X - \upsilon U_\alpha \subseteqq \upsilon X - \upsilon V_\alpha = \mathrm{cl}_{\upsilon X} X - V_\alpha \subseteqq \mathrm{cl}_{\upsilon X} Z_1$, where the first containment follows from [1, 3.5], the second by [5, 8.8] and [1, 5.2]. Also $\upsilon W_\alpha \subseteqq \mathrm{cl}_{\upsilon X} Z_2$, so $\mathrm{cl}_{\upsilon X} Z_1$, and $\mathrm{cl}_{\upsilon X} Z_2$ are a complete separation of $\upsilon X - \upsilon U_\alpha$ and υW_α. Thus $\upsilon\mathscr{U}$ is a normal cover of υX. The converse is obvious.

THEOREM 12. *Let* $\mathscr{U}$ *be a normal cover by sets of of nonmeasurable cardinal, with* $|\mathscr{U}|$ *nonmeasurable. If* $\upsilon U \subset \upsilon X$ *for each* $U \in \mathscr{U}$ *then* $\upsilon X = \cup\{\upsilon U : U \in \mathscr{U}\}$.

Proof. By Lemma 11, $\upsilon\mathscr{U}$ is a normal cover of υX. Thus there is a locally finite cozero cover $\mathscr{V}$ of υX. Since the cardinalities involved are nonmeasurable, $\upsilon\{V : V \in \mathscr{V}\}$ is a realcompact subset of

$\cup\{\upsilon U: U \in \mathscr{U}\}$, and the former union contains X. Thus we have shown

$$\upsilon X \subseteq \cup\{V: V \in \mathscr{U}\} \subseteq \cup\{\upsilon U: U \in \mathscr{U}\} \subseteq \upsilon X$$

where the last inclusion follows from the hypotheses $\upsilon U \subseteq \upsilon X$, and the proof is complete.

REMARKS. (i) The extra restriction in Theorem 12 on the cardinality of the members of $\upsilon\mathscr{U}$ is satisfied in any space of nonmeasurable cardinal; we may infer, using the Blair-Hager procedure [2, 2.3], the Katetov-Shirota Theorem from Theorem 12.

(ii) Substitution of the phrases "topologically complete" for "realcompact, and "$\mathscr{C}$-fixed" for "$\mathscr{B}$-fixed" (basically fixed) in the theorems of §§3 and 4 above, will leave all the theorems valid. Similarly it may be shown that the paracompactness of a completely regular T_1-space X is equivalent to the property that every $\mathscr{A}$-fixed, l.s.c. carrier from X to the closed convex subsets of a Banach space admits a selection.

(iii) The following question remains open: If ϕ is a carrier of finite character to the closed convex subsets of a Banach space Y, must ϕ admit a selection in general, or even when X is realcompact?

Finally, the authors wish to thank the referee for his remarks and suggestions.

REFERENCES

1. R. L. Blair, *On υ-embedded sets in topological spaces*, Lecture Notes, 378, Springer-Verlag, 1974.
2. R. L. Blair and M. Hager, *Notes on the Hewitt realcompactification of a product*, General Topology and Applications, **5** (1975), 1-8.
3. G. DeMarco and R. G. Wilson, *Realcompactness and partitions of unity*, Proc. Amer. Math. Soc., **30** (1971), 189-194.
4. N. Dykes, *Generalizations of realcompact spaces*, Pacific J. Math., **33** (1970), 571-581.
5. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, N.Y., 1960.
6. E. Michael, *Continuous selections, I*, Ann. of Math., **63** (1956), 361-382.
7. K. Sundaresan, *Spaces of continuous functions into a Banach space*, Studia Math., **48** (1973), 15-22.

Received October 10, 1979 and in revised form May 16, 1980. Research supported by NSERC Grants A 3577 and A 5615.

MOUNT SAINT VINCENT UNIVERSITY
AND
DALHOUSIE UNIVERSITY
HALIFAX, N.S.