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LOCALLY SMOOTH TORUS GROUP ACTIONS ON INTEGRAL COHOMOLOGY COMPLEX PROJECTIVE SPACES

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Let X be a HCP^n which admits a nontrivial smooth S^1 action. Petrie defined and studied a set functions $\delta_i(m)$ which are important in the study of local representations. In this paper, we extended Petrie's result to locally smooth torus group actions on integral cohomology complex projective spaces.

Introduction. Let X be a closed smooth manifold homotopically equivalent to CP^n which admits a nontrivial smooth S^1 action. An interesting problem is to study the structure of the representations of S^1 on the normal fibers to the various components of the fixed point set. Let the fixed point set X^{S^1} consist of k connected components X_i, $i = 1, 2, \cdots, k$. Let η be the equivariant Hopf bundle [3]. Choose $x_i \in X_i$ and define k integers a_i by $\eta|_{x_i}(t) = t^{a_i}$. Then Petrie [3] proved the following.

THEOREM 0.1. *The k integers a_i are distinct.*

Suppose further that $X_i = x_i$, isolated points. Let $TX|_{x_i}(t) = \sum_{j=1}^{n} t^{x_{ij}}$. For each integer m and each $i = 1, \cdots, k$, set

$$n_j(m) = \text{number of } j \neq i \text{ such that } m \text{ divides } a_j - a_i\,;$$
$$d_i(m) = \text{number of } j = 1, \cdots, n \text{ such that } m \text{ divides } x_{ij}\,,$$
$$\delta_i(m) = n_i(m) - d_i(m)\,.$$

Then Petrie [3] proved the following.

THEOREM 0.2. *$\delta_i(m) \geqq 0$ and $\delta_i(p^r) = 0$ if p is a prime.*

Although so far the actions are smooth, it is not difficult to see that the numbers a_i can be defined in the same way for an S^1 action on an integral cohomology complex projective space and the numbers x_{ij} are defined if the action is locally smooth [1]. It turns out that these results are also true for locally smooth S^1 actions on integral cohomology complex projective spaces. The main purpose of this paper is to extend these results to the category of locally smooth torus actions on integral cohomology complex projective spaces such that the fixed point sets do not necessarily consist of isolated points.

Our approach is different from [3] and is more elementary in the sense that we do not need equivariant K-theory and the Atiyah Singer Index Theorem. This paper is organized as follows. In §1, we study the torus actions on principal S^1-bunles in a very general setting and prove an extension of Theorem 0.1. In §2, we study the upper bound of the dimensions of certain invariant subspace. §3, we study the functions $\delta_1(H)$ and prove an extension of Theorem 0.2.

1. Torus actions on principal S^1 bundles. Let X be a paracompact space which supports a left T^s action. Let $\pi\colon P \to X$ be a principal S^1 bundle. The following result is due to Stewart and Su [3, p. 126].

THEOREM 1.1. *If $H^1(X; Z) = 0$, then a T^s action on X lifts to a T^s action on P which commutes with the principal S^1 action on P. If $(t, p) \to t \cdot p$ and $(t, p) \to t \circ p$ denote two liftings of T^s to P, then there is a homomorphism $\theta\colon T^s \to S^1$ such that*

$$t \circ p = t \cdot p \cdot \theta(t) . \qquad \square$$

Let $(t, p) \to tp$ be a fixed lifting. We define an equivariant complex line bundle η over X by letting $E(\eta) = P \times_{S^1} C$ and $t[p, z] = [tp, z]$ where $t \in T^s$, $z \in C$, $p \in P$. Suppose that the fixed point set X^{T^s} is the disjoint union of $k+1$ components X_i, $i = 0, i, \cdots, k$. For each i, choose $x_i \in X_i$ and define a character χ_i of T^s by $\eta|_{x_i}(t) = \chi_i(t)$. In this section we will study the general properties of these characters under the assumption that the fixed point set of any T^s action on P is connected (including the empty set as usual).

LEMMA 1.2. *For each i, there is a lifting of T^s to P such that the associated equivariant complex line bundle η_i satisfies*

$$\eta_i|_{x_j}(t) = \chi_j(t)\chi_i(t)^{-1} .$$

We will say that η_i is the normalization of η at x_i and the lifting is normalized at x_i.

Proof. Define a new lifting $(t, p) \to t \circ p$ by $t \circ p = tp\chi_i(t)$ and let η_i be the associated equivariant complex line bundle. Let $p \in \pi^{-1}(X_j)$ and $t \in T^s$, $z \in C$,

$$\begin{aligned}[p, \eta_i|_{x_j}(t)z] &= t \circ [p, z] = [t \circ p, z] = [tp\chi_i(t), z] = t[p\chi_i(t), z] \\ &= [p\chi_i(t), \chi_j(t)z] = [p, \chi_j(t)\chi_i(t)^{-1}z] .\end{aligned}$$

Hence $\eta_i|_{x_j}(t) = \chi_j(t)\chi_i(t)^{-1}$. $\square$

For the rest of this section, we will always assume that the lifting is normalized at x_0 and $\chi_0(t) = 1$.

LEMMA 1.3. *Let $H \subset T^s$ be a subgroup. Then $\pi^{-1}(X_j) \subset P^H$ if and only if $\chi_j(g) = 1$ for all $g \in H$.*

Proof. Suppose that $\chi_j(g) = 1$ for all $g \in H$. Let $p \in \pi^{-1}(X_j)$.

$$[gp, z] = g[p, z] = [p, \chi_j(g)z] = [p, z] .$$

Now it is clear that $gp = p$ for all $g \in H$ and consequently, $p \in P^H$.

Conversely, suppose that $\pi^{-1}(X_j) \subset P^H$. Let $p \in \pi^{-1}(X_j)$.

$$[p, z] = [gp, z] = g[p, z] = [p, \chi_j(g)z] .$$

Hence $\chi_j(g) = 1$ for all $g \in H$. □

THEOREM 1.4. *The $k + 1$ characters χ_i are distinct.*

Proof. If the $k + 1$ characters are not distinct, we may assume without loss of generality that $\chi_i = \chi_0$. By Lemma 1.3, we have $X_0 \cup X_i \subset \pi(P^{T^s})$. Since we have assumed that P^{T^s} is connected, $\pi(P^{T^s})$ is a connected subset of X^{T^s}. It follows that X_i and X_0 are in the same component of X^{T^s}. This is a contradiction. Hence $\chi_i \neq \chi_j$ for $i \neq j$. □

COROLLARY 1.5. $\pi^{-1}(X_0) = P^{T^s}$. □

PROPOSITION 1.6. *Let $H \subset T^s$ be a subgroup. If X_i and X_0 are contained in the same component of X^H, then $\chi_i(g) = 1$ for all $g \in H$. Conversely, if P^H is connected and $\chi_i(g) = 1$ for all $g \in H$, then X_i and X_0 are contained in the same component of X^H.*

Proof. If X_i and X_0 are contained in the same component of X^H, then, for $g \in H$,

$$\chi_i(g) = \eta|_{x_j}(g) = \eta|_{x_0}(g) = \chi_0(g) = 1 .$$

Conversely, if $\chi_i(g) = 1$ for all $g \in H$, then $\pi^{-1}(X_0 \cup X_i) \subset P^H$ by Lemma 1.3. Since by assumption, P^H is connected, X_0 and X_i are contained in the same component, namely $\pi(P^H)$, of X^H. □

2. The dimensions of invariant subspaces. For convenience, let $G = T^s$. Let $E_G \to B_G$ be the universal principal G-bundle. It is well-known that $H^*(B_G, Q) = Q[t_1, \cdots, t_s]$ where $\deg t_i = 2$. For the sake of simplicity, we will write $t = (t_1, \cdots, t_s)$. For a graded

$H^*(B_G, Q)$-module M, we denote by $M[t^{-1}]$ for the localization of M with respect to the multiplicative system generated by t. For a G-space A, let $A_G = A \times_G E_G$ which the associated bundle over B_G with fiber A. We will need the following result in [2].

PROPOSITION 2.1. *Let A be a G-space. Then*
(i) *The inclusion $j: A^G \to A$ induces an isomorphism*

$$j^*: H^*(A_G, Q)[t^{-1}] \longrightarrow H^*(A_G^G, Q)[t^{-1}] .$$

(ii) *the top class in $H^*(A, Q)$ does not die in $H^*(A_G, Q)[t^{-1}]$ if A is orientable and $A^G \neq \emptyset$.*

Let X be a closed topological manifold of dimension $2n$ which is also a rational cohomology complex projective space and which admits a G action. Write the fixed point set $X^G = X_0 \cup \cdots \cup X_k$ as a disjoint union of its components. By a theorem of Bredon [1, p. 378], each X_j is a rational cohomology complex projective space and the inclusion $X_j \to X$ induces isomorphisms $j^*: H^i(X, Q) \to H^i(X_j, Q)$ for $i \leqq \dim X_j$. Let $Y \subset X$ be an oriented invariant submanifold and let $C = \{i \mid X_i \subset Y\}$. In this section, we will study the dimension of Y under the assumption that $Y^G = \bigcup_{i \in C} X_i$ where $C \neq \emptyset$. We would like to point out that the most interesting case is that Y is a component of the fixed point set of a subgroup of G.

THEOREM 2.2. *With the above assumptions.*

$$\dim Y \leqq 2\Big(\sum_{i \in C} \frac{1}{2} \dim X_i + 1\Big) .$$

Proof. By Proposition 2.1, $H^*(Y_G, Q)[t^{-1}](H^*(X_G, Q)[t^{-1}]$, respectively) is isomorphic to $H^*(Y_G^G, Q)[t^{-1}](H^*(X_G^G, Q)[t^{-1}]$, respectively). Since $Y_G^G = Y^G \times B_G$ and $X_G^G = X^G \times B_G$.

$$H^*(X_G^G, Q)[t^{-1}] = \bigoplus_{i=0}^{k} H^*(X_i) \otimes H^*(B_G)[t^{-1}]$$

and

$$H^*(Y_G^G, Q)[t^{-1}] = \bigoplus_{i \in C} H^*(X_i) \otimes H^*(B_G)[t^{-1}] .$$

This implies the rank of $H^*(Y_G, G)[t^{-1}]$ over $Q[t, t^{-1}]$ is equal to $\sum_{i \in C} (1/2 \dim X_i + 1)$. Let $\alpha \in H^2(H_G, Q)$ represent the generator of $H^2(X, Q)$ and let α_Y be its image in $H^2(Y_G, Q)$. Since $H^*(X_G, Q)[t^{-1}]$ is a free algebra over $Q[t, t^{-1}]$ with basis $1, \alpha, \cdots, \alpha^n$, $H^*(Y_G, Q)[t^{-1}]$ is also a free algebra over $Q[t, t^{-1}]$ with basis $1, \alpha_Y, \cdots, \alpha_Y^k$ where k

is easily seen to be $\sum_{i\in C}(1/2 \dim X_i + 1)$. Since the top classes in $H^*(Y, Q)$ does not die in $H^*(Y_G, Q)[t^{-1}]$, we have

$$\dim Y \leqq 2\Big(\sum_{i\in C}\frac{1}{2}\dim X_i + 1\Big)\,. \qquad \square$$

COROLLARY 2.3. *With the above assumptions,*

$$\dim Y \leqq 2(rkH^*(Y, Q) - 1)$$

and the equality holds if and only if Y is a rational cohomology complex projective space.

Proof. Since $rkH^*(Y^G, Q) \leqq rkH^*(Y, Q)$ [see [1, p. 163]), and $rkH^*(Y^G, Q) = 1/2\sum_{i\in C}(\dim X_i + 1)$, the inequality follows immediately. If the equality holds then $rkH^*(Y, Q) = rkH^*(Y^G, Q)$ and $H^{odd}(Y, Q) = 0$. This implies that Y is totally nonhomologous to zero in Y_G. Let $a = j^*\alpha_Y$ where $j\colon Y \to Y_G$ is the inclusion. It follows that $H^*(Y, Q)$ is generated by a. This implies that Y is a rational cohomology complex projective space. The converse is obvious. $\square$

REMRAK. Petrie's examples in [4] provide exotic S^1 actions on CP^n such that the fixed point set component Y of $Z_{pq} \subset S^1$ has property that $\dim Y < 2(rkH^*(Y, Q) - 1)$.

3. The functions $\delta_i(H)$. Let G be a compact Lie Groups and let V be a enclidian space on which H operates orthogenally. We say that a G action on a paracompact space X is locally smooth if for each orbit Y of type G/H there is a linear tube

$$\varphi\colon G \times_H V \longrightarrow X$$

about Y in X, i.e., φ is an equivariant embedding onto an open neighborhood of Y in X. Note that X must be a topological manifold and the components of the fixed point set are topological submanifolds. If x is a fixed point, then the action in a neighborhood of x is equivalent to an orthogonal action. See [1] for details.

Let X be a closed integral cohomology complex projective space of dimension $2n - 2$ and let X admit a locally smooth T^s action. Then X is the orbit space of a free S^1 action on a closed integral cohomology sphere $\sum$ and the T^s action on X lifts to a T^s action on $\sum$ which commutes with the free S^1 action. Let η be the equivariant complex line bundle over X associated to a fixed lifting. Then η is the equivariant Hopf bundle defined in [4]. Let the fixed point set

$$X^{T^s} = X_0 \cup X_1 \cup \cdots \cup X_k$$

be the disjoint union its components X_i. Each X_i is a closed integral cohomology complex projective space of dimension $2n_i - 2$ and $\sum_{i=0}^{k} n_i = n$. Choose $x_i \in X_i$. Define $k+1$ characters χ_i as in §1. Since $\sum$ is an integral cohomology sphere, $\sum^{T^s}$ is connected. By Theorem 1.4, the $k+1$ characters χ_i are distinct. For each i, let $\rho_i(t) = \sum_{j=1}^{n-n_i} \lambda_{ij}(t)$ be the representation in the normal fiber at x_i to X_i.

The following definition is essentially due to Petrie [3].

DEFINITION 3.1. *For each subgroup $H \subset T^s$ and each $i = 0, 1, \cdots, k$, set $\alpha_i(H) = \sum_{j \in A_i} n_j$ where $A_i = \{j \neq i \mid \chi_j(g) = \chi_i(g)$ for all $g \in H\}$, $\beta_i(H) = card\, B_i$ where $B_i = \{j \mid \lambda_{ij}(g) = 1$ for all $g \in H\}$. Let $\delta_i(H) = \alpha_i(H) - \beta_i(H)$.* □

It is easy to see that the function $\delta_i(H)$ are independent of the choice of the lifting. In this section, we will study the properties of $\delta_i(H)$.

LEMMA 3.2. *Let p be a prime and let $H \subset T^s$ be a subgroup of order p^a. Then X^H has no p-torsions.*

Proof. Choose a prime $q \neq p$ such that X^H has no q-torsions. Let $K \subset T^s$ be a cyclic subgroup of order q^b where b is so large that $X^K = X^{T^s}$. It follows from the results of [1],

$$\begin{aligned} rkH^*(X; Z) &= rk_pH^*(X; Z_p) = rk_pH^*(X^H; Z_p) \geqq rkH^*(X^H; Z) \\ &= rk_qH^*(X^H; Z_q) \geqq rk_qH^*((X^H)^K; Z_q) = rk_qH^*(X^{T^s}; Z_q) \\ &= rkH^*(X^{T^s}; Z) = rkH^*(X; Z)\ . \end{aligned}$$

It follows that $rk_pH^*(X^H; Z_p) = rkH^*(X^H; Z)$ and X^H has no p-torsions. □

PROPOSITION 3.3. *Let $H \subset T^s$ be a subgroup. Then X^H is orientable.*

Proof. Let $K \subset H$ be a subgroup of order 2^a such that H/K has no element of even order. By Lemma 3.2, X^K has no 2-torsions. Hence X^K is orientable. It has been shown in [1] that the fixed point set of a locally smooth action of a finite group of odd order or a torus group on an orientable manifold is always orientable. It follows that X^H is orientable. □

THEOREM 3.4. *$\delta_i(H) \geqq 0$ for any subgroup $H \subset T^s$ and $\delta_i(H) = 0$ if H is a toral subgroup or a subgroup of order p^a where p is a prime.*

Proof. We may assume without loss of generality that $i = 0$. Let Y be a component of X^H which contains X_0. Let $C = \{j | X_j \subset Y\}$. It is obvious that $Y^{T^s} = \bigcup_{j \in C} X_j$. By Proposition 1.6, $C \subseteqq A_0 \cup \{0\}$.

It is clear that

$$\frac{1}{2}\dim Y = (n_0 - 1) + \beta_0(H) .$$

By Theorem 2.2, we have

$$\frac{1}{2}\dim Y \leqq \sum_{j \in C} n_j - 1 \leqq \sum_{j \in A_0} n_j + n_0 - 1 = \alpha_0(H) + n_0 - 1 .$$

Hence $\delta_0(H) = \alpha_0(H) - \beta_0(H) = \alpha_0(H) + n_0 - 1 - 1/2 \dim Y \geqq 0$.

If H is a subgroup of order p^a, it follows from P. A. Smith Theorem that $\sum^H$ is connected. Hence $C = A_0 \cup \{0\}$ by Proposition 1.6. By Lemma 3.2, Y is a mod p cohomology complex projective space without p-torsions. Hence Y is a rational cohomology complex projective spaces. It follows easily from Corollary 2.3 $\delta_0(H) = 0$. Similarly, we can prove the case that H is a toral subgroup. $\square$

COROLLARY 3.5. *If $\delta_i(H)=0$ for $i=0, \cdots, k$, then X^H is a disjoint union of rational cohomology complex projective spaces for any subgroup $H \subset T^s$ and $\sum^H$ is connected for any normalized lifting.*

We also include the following properties for future application.

PROPOSITION 3. *Let $H \subset T^s$ be a subgroup. If X_i and X_j are contained in the same component of X^H, then $\delta_i(H) = \delta_j(H)$.*

Proof. Let Y be the component of X^H which contains X_i and X_j. It is clear that $\alpha_i(H) + n_i = \alpha_j(H) + n_j$ and $1/2 \dim Y = \beta_i(H) + n_i - 1 = \beta_j(H) + n_j - 1$. It follows that $\delta_i(H) = \delta_j(H)$. $\square$

PROPOSITION 3. *Let $H \subset T^s$ be a subgroup. If $\chi_i(H) = \chi_j(H) = 1$ and $\delta_j(H) = 0$, then $\delta_i(H) = 0$.*

Proof. It is clear from the proof of Theorem 3.4 that $\delta_j(H) = 0$ implies $\sum^H$ is connected where lifting is normalized at x_j. By Proposition 1.6, X_i and X_j are contained in the same component of X^H. It follows from Proposition 3.5 that $\delta_i(H) = \delta_j(H) = 0$. $\square$

REFERENCES

1. G. E. Bredon, *Introduction to Compact Transformation groups,* Academic Press, New York, 1972.

2. G. E. Bredon, *Fixed point sets of actions on Poincare duality spaces,* Topology, **12** (1973), 159-176.
3. T. Petrie, *Smooth S^1 actions on homotopy complex projective spaces and related topics,* Bull. Amer. Math. Soc., **78** (1972), 105-153.
4. ———, *A setting for smooth S^1 actions with applications to real algebraic actions on* $P(C^{4n})$, Topology, **13** (1974), 363-374.

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