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ISOMETRIES OF $C^{(n)}[0, 1]$

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By $C^{(n)}[0, 1]$ (henceforth denoted by $C^{(n)}$) we denote the Banach algebra of complex valued n times continuously differentiable functions on [0, 1] with norm given by

$$\|f\| = \sup_{x \in [0,1]} \left(\sum_{r=0}^{n} \left(\frac{|f^{(r)}(x)|}{r!} \right) \right) \quad \text{for} \quad f \in C^{(n)} .$$

By an isometry of $C^{(n)}$ we mean a norm-preserving linear map of $C^{(n)}$ onto itself.

The purpose of this article is to describe the isometries of $C^{(n)}$ for any positive integer n. More precisely, we show that any isometry of $C^{(n)}$ is induced by a point map of the interval [0, 1] onto itself.

The isometries of $C^{(1)}$ (with the same norm as above) are determined by M. Cambern [1]. N. V. Rao and A. K. Roy [2] have also determined the isometries of $C^{(1)}$ with norm of $f \in C^{(1)}$ given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and even for more general norms.
In the proof we shall follow the techniques of [1].

1. Let W denote the compact space $[0, 1] \times [-\pi, \pi]^n$. We prove the following propositions.

PROPOSITION 1.1. *Given $(x, \theta_1, \cdots, \theta_n) \in W$, then there exists $h \in C^{(n)}$ such that*

$$\sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} > \sum_{r=0}^{n} \frac{|h^{(r)}(y)|}{r!}$$

for $y \in [0, 1]$, $y \neq x$, with $|h(x)| = h(x) > 0$, $|h'(x)| = e^{i\theta_1}h'(x) > 0$, $|h''(x)| = e^{i\theta_2}h''(x) > 0, \cdots, |h^{(n)}(x)| = e^{i\theta_n}h^{(n)}(x) > 0$.

Proof. Let f_0 be the real valued, nonnegative continuous function on [0, 1] defined as follows

$$f_0(y) = \begin{cases} 0 & \cdots (y - x) \leqq -\dfrac{1}{2(n!)} \\ 1 + 2(n!)(y - x) & \cdots -\dfrac{1}{2(n!} < (y - x) \leqq 0 \\ 1 - 2(n!)(y - x) & \cdots 0 < (y - x) \leqq \dfrac{1}{2(n!)} \\ 0 & \cdots \dfrac{1}{2(n!)} < (y - x) . \end{cases}$$

For $1 \leqq r \leqq n$ define $f_r(y)$ as $f_r(y) = \int_x^y f_{r-1}(t)dt$. It can be easily verified that for $1 \leqq r \leqq n$, $f_r(y)$ is as follows:

$$f_r(y) = \begin{cases} -\sum_{j=1}^{r} \frac{1}{(j+1)!(2(n!))^j} \frac{(y-x)^{r-j}}{(r-j)!} \cdots (y-x) \leqq \frac{-1}{2(n!)} \\ \frac{(y-x)^r}{r!} + \frac{2(n!)(y-x)^{r+1}}{(r+1)!} \cdots -\frac{1}{2(n!)} < (y-x) \leqq 0 \\ \frac{(y-x)^r}{r!} - \frac{2(n!)(y-x)^{r+1}}{(r+1)!} \cdots 0 < (y-x) \leqq \frac{1}{2(n!)} \\ \sum_{j=1}^{r} \frac{(-1)^{j-1}}{(j+1)!(2(n!))^j} \frac{(y-x)^{r-j}}{(r-j)!} \cdots \frac{1}{2(n!)} < (y-x) . \end{cases}$$

Now let

$$g(y) = \frac{1}{(2n-1)!}\left[\sum_{j=1}^{(n-1)} e^{i(\theta_1-\theta_j)} \frac{(y-x)^j}{j!}\right] + e^{i(\theta_1-\theta_n)} f_n(y) .$$

Clearly, for $1 \leqq r \leqq n$, $f_n^{(r)} = f_{n-r}$. Therefore $g \in C^{(n)}$ and

$$g^{(r)}(y) = \frac{1}{(2n-1)!} \sum_{j=r}^{(n-1)} e^{i(\theta_1-\theta_j)} \frac{(y-x)^{j-r}}{(j-r)!} + e^{i(\theta_1-\theta_n)} f_{n-r}(y) \text{ for } 1 \leqq r \leqq n .$$

Thus

$$g(x) = 0,\ g^{(r)}(x) = \frac{1}{(2n-1)!} e^{i(\theta_1-\theta_r)} \text{ for } 1 \leqq r \leqq n-1 ,$$

and $g^{(n)}(x) = e^{i(\theta_1-\theta_n)}$. Therefore

$$\sum_{r=0}^{n} \frac{|g^{(r)}(x)|}{r!} = \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{1}{r!} + \frac{1}{n!} .$$

Now consider $\sum_{r=0}^{n} (|g^{(r)}(y)|/r!)$ for $y \in [0, 1]$ and $y \neq x$.

Case 1. Let $(y - x) \leqq (-1/2(n!))$.

$$(1) \quad \sum_{r=0}^{n} \frac{|g^{(r)}(y)|}{r!} \leqq \frac{1}{(2n-1)!} \sum_{j=1}^{(n-1)} \frac{|y-x|^j}{j!} + \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{1}{r!} \times \left\{ \sum_{j=r}^{(n-1)} \frac{|y-x|^{j-r}}{(j-r)!} \right\} + \sum_{r=0}^{n} \frac{1}{r!} \left\{ \sum_{j=1}^{(n-r)} \frac{|y-x|^{(n-r-j)}}{(j+1)!\,(2(n!))^j (n-r-j)!} \right\} .$$

For $n = 1, 2$, it can be easily verified that right hand side of (1) is less than $\sum_{r=0}^{n} (|g^{(r)}(x)|/r!)$. When $n \geqq 3$, denoting $(n!/(n-j)!j!)$ by C_j^n, (1) gives

$$\sum_{r=0}^{n}\frac{|g^{(r)}(y)|}{r!} \leqq \frac{1}{(2n-1)!}\sum_{j=1}^{(n-1)}\frac{1}{j!}+\frac{1}{(2n-1)!}\sum_{r=1}^{(n-1)}\frac{(n-1)}{r!}$$
$$+\frac{1}{2(n!)}\sum_{r=0}^{(n-1)}\frac{1}{r!}\sum_{j=1}^{(n-r)}\left\{\frac{1}{j(j+1)(n-r-1)!}C_{j-1}^{n-r-1}\frac{1}{(2(n!))^{j-1}}\right\}.$$

Now

$$\frac{1}{(2n-1)!}\sum_{r=1}^{(n-1)}\frac{(n-1)}{r!} \leqq \frac{(n-1)}{(2n-1)!}\sum_{r=1}^{(n-1)}\frac{1}{2^{r-1}} < \frac{2(n-1)}{(2n-1)!} < \frac{1}{4(n!)} \quad \text{for all} \quad n \geqq 3 .$$

Thus we have

$$\text{(2)} \qquad \text{for} \quad n \geqq 3, \frac{1}{(2n-1)!}\sum_{r=1}^{(n-1)}\frac{(n-1)}{r!} < \frac{1}{4(n!)} .$$

Also

$$\frac{1}{2(n!)}\sum_{r=0}^{(n-1)}\frac{1}{r!}\sum_{j=1}^{(n-r)}\left\{\frac{1}{j(j+1)(n-r-1)!}C_{j-1}^{n-r-1}\frac{1}{(2(n!))^{j-1}}\right\}$$
$$\leqq \frac{1}{2(n!)}\sum_{r=0}^{(n-1)}\frac{1}{r!}\frac{1}{2(n-r-1)!}\left\{\sum_{j=1}^{(n-r)}C_{j-1}^{n-r-1}\frac{1}{(2(n!))^{j-1}}\right\}$$
$$=\frac{1}{(2(n!))}\cdot\frac{1}{2(n-1)!}\sum_{r=0}^{(n-1)}C_r^{n-1}\left(1+\frac{1}{2(n!)}\right)^{(n-1-r)}$$
$$=\frac{\left\{\left(1+\frac{1}{2(n!)}\right)+1\right\}^{n-1}}{2(2(n!))(n-1)!} \leqq \frac{\left(\frac{9}{4}\right)^{n-1}}{2(2(n!))(n-1)!} < \frac{81}{64}\cdot\frac{1}{2(n!)} .$$

Thus

$$\text{(3)} \qquad \frac{1}{2(n!)}\sum_{r=0}^{(n-1)}\frac{1}{r!}\sum_{j=1}^{(n-r)}\left\{\frac{1}{j(j+1)(n-r-1)!}\cdot C_{j-1}^{n-r-1}\cdot\frac{1}{(2(n!))^{j-1}}\right\} < \frac{3}{4}\cdot\frac{1}{n!} .$$

By (2) and (3) it follows immediately that for all $y \in [0, 1]$ and $y \neq x$

$$\sum_{r=0}^{n}\frac{|g^{(r)}(y)|}{r!} < \sum_{r=0}^{n}\frac{|g^{(r)}(x)|}{r!} .$$

Case 2. Let $-(1/2(n!)) < (y-x) < 0$

$$\sum_{r=0}^{n}\frac{|g^{(r)}(y)|}{r!} \leqq \frac{1}{(2n-1)!}\sum_{j=1}^{(n-1)}\frac{|y-x|^j}{j!}+\frac{1}{(2n-1)!}\sum_{r=1}^{(n-1)}\frac{1}{r!}\left\{\sum_{j=r}^{(n-1)}\frac{|y-x|^{j-r}}{(j-r)!}\right\}$$
$$+\sum_{r=0}^{n}\frac{1}{r!}\left|\frac{(y-x)^{n-r}}{(n-r)!}+\frac{2(n!)(y-x)^{n-r+1}}{(n-r+1)!}\right|$$

$$
\begin{aligned}
&= \frac{1}{(2n-1)!}\sum_{j=1}^{(n-1)}\frac{(-1)^j(y-x)^j}{j!} + \frac{1}{(2n-1)!}\sum_{r=1}^{n-1}\frac{1}{r!} \\
&\quad + \frac{1}{(2n-1)!}\sum_{r=1}^{(n-2)}\frac{1}{r!}\left\{\sum_{j=r+1}^{(n-1)}\frac{(-1)^{j-r}(y-x)^{j-r}}{(j-r)!}\right\} \\
&\quad + \sum_{r=0}^{n}\frac{(-1)^{n-r}}{r!}\left\{\frac{(y-x)^{n-r}}{(n-r)!} + \frac{2(n!)(y-x)^{n-r+1}}{(n-r+1)!}\right\} \\
&= \frac{1}{(2n-1)!}\sum_{r=1}^{(n-1)}\frac{1}{r!} + \frac{1}{n!} + \sum_{s=1}^{(n-1)}(y-x)^s\left\{\frac{(-1)^s}{s!(2n-1)!}\right. \\
&\quad \left. + \frac{(-1)^s}{s!(n-s)!} + \frac{2(n!)(-1)^{s-1}}{s!(n-s+1)!} + \sum_{r=1}^{n-1-s}\frac{(-1)^s}{s!r!}\right\} \\
&\quad + (y-x)^n\left\{\frac{(-1)^n}{n!} + \frac{2(n!)(-1)^{n-1}}{n!}\right\} + \frac{(-1)^n(y-x)^{n+1}}{(n+1)!} \\
&= \sum_{r=0}^{n}\frac{|g^{(r)}(x)|}{r!} + \sum_{s=1}^{(n-1)}\frac{(-1)^s(y-x)^s}{s!}\left\{\frac{1}{(2n-1)!}\right. \\
&\qquad \left. + \sum_{r=1}^{(n-s)}\frac{1}{r!} - \frac{2(n!)}{(n-s+1)!}\right\} \\
&\qquad + \frac{(-1)^n(y-x)^n}{n!}\{1-2(n!)\} + \frac{(-1)^n(y-x)^{n+1}}{(n+1)!} \\
&< \sum_{r=0}^{n}\frac{|g^{(r)}(x)|}{r!}
\end{aligned}
$$

since all the other terms are negative. Verification in cases when $0 < (y-x) \leqq (1/2(n!))$ and $(1/2(n!)) < (y-x)$ is similar. From this it follows that the function $h \in C^{(n)}$ defined by $h(y) = 1 + e^{-i\theta_1}g(y)$ has the desired properties.

PROPOSITION 1.2. *For any* $f \in C^{(n)}$

$$
\sum_{j=1}^{n}(-1)^{j-1}C_{j-1}^{n}(f^{n-j+1})^{(k)}(x)(f(x))^{j-1} = \begin{cases} 0 & \text{if } 1 \leqq k < n \\ n!(f'(x))^n & \text{if } k = n \end{cases}
$$

where $(f^{n-j+1})^{(k)}(x)$ *means the* kth *derivative of* f^{n-j+1} *at* x.

Proof. We prove this proposition by induction on n. For $n = 1$ it is obvious. Let it be true for $n = r$. Then we have

$$
\sum_{j=1}^{r}(-1)^{j-1}C_{j-1}^{r}(f^{r-j+1})^{(k)}(x)(f(x))^{j-1} = 0\,, \qquad \text{for } 1 \leqq k < r\,,
$$

and

$$
\sum_{j=1}^{r}(-1)^{j-1}C_{j-1}^{r}(f^{r-j+1})^{(r)}(x)(f(x))^{j-1} = r!(f'(x))^r\,.
$$

Now let $n = r + 1$ and $k = r + 1$.

Since $(f^{r-j+2})'(x) = (r-j+2)\ (f^{r-j+1})(x)f'(x)$

$$\begin{aligned}
&\sum_{j=1}^{(r+1)}(-1)^{j-1}C_{j-1}^{r+1}(f^{r-j+2})^{(r+1)}(x)(f(x))^{j-1} \\
&\quad = \sum_{j=1}^{r+1}(-1)^{j-1}C_{j-1}^{r+1}(f(x))^{j-1}\left\{(r-j+2)\sum_{s=0}^{r}C_s^r(f^{r-j+1})^{(r-s)}(x)(f')^{(s)}(x)\right\} \\
&\quad = \sum_{j=1}^{r+1}(-1)^{j-1}(r+1)C_{j-1}^{r}(f(x))^{j-1}(f^{r-j+1})^{(r)}(x)f'(x) \\
&\qquad + \sum_{j=1}^{r+1}(-1)^{j-1}(r+1)C_{j-1}^{r}(f(x))^{j-1}\left\{\sum_{s=1}^{r}C_s^r(f^{r-j+1})^{(r-s)}(x)(f')^{(s)}(x)\right\} \\
&\quad = (r+1)\left\{\sum_{j=1}^{r}(-1)^{j-1}C_{j-1}^{r}(f(x))^{j-1}(f^{r-j+1})^{(r)}(x)\right\}f'(x) \\
&\qquad + (r+1)\sum_{j=1}^{r+1}(-1)^{j-1}(f(x))^{j-1}C_{j-1}^{r}\left\{\sum_{s=1}^{r}c_s^r(f^{r-j+1})^{(r-s)}(x)(f')^{(s)}(x)\right\} \\
&\quad = (r+1)!(f'(x))^{r+1} + (r+1)\sum_{s=1}^{r-1}C_s^r(f')^{(s)}(x) \\
&\qquad \times\left\{\sum_{j=1}^{r}(-1)^{j}C_{j-1}^{r}(f^{r-j+1})^{(r-s)}(x)(f(x))^{j-1}\right\} \\
&\qquad + (r+1)\sum_{j=1}^{r+1}(-1)^{j-1}C_{j-1}^{r}(f(x))^{r}(f'(x))^{(r)} \\
&\quad = (r+1)!(f'(x))^{r+1}\ .
\end{aligned}$$

Now let $n = r+1$ and $k < (r+1)$. Then

$$\begin{aligned}
&\sum_{j=1}^{r+1}(-1)^{j-1}C_{j-1}^{r+1}(f^{r-j+2})^{(k)}(x)(f(x))^{j-1} \\
&\quad = \sum_{j=1}^{r+1}(-1)^{j-1}C_{j-1}^{r+1}(r-j+2)(f(x))^{j-1}\left\{\sum_{s=0}^{k-1}C_s^{k-1}(f^{r-j+1})^{k-1-s}(x)(f')^{(s)}(x)\right\} \\
&\quad = (r+1)\sum_{s=0}^{k-2}C_s^{k-1}(f')^{(s)}(x)\left\{\sum_{j=1}^{r}(-1)^{j-1}C_{j-1}^{r}(f(x))^{j-1}(f^{r-j+1})^{(k-1-s)}(x)\right\} \\
&\qquad + (r+1)\sum_{j=1}^{r+1}(-1)^{j-1}C_{j-1}^{r}(f(x))^{k-1}(f')^{(k-1)}(x) \\
&\quad = 0\ .
\end{aligned}$$

Hence the proposition follows by mathematical induction.

2. If X is any compact Hausdorff space, we will denote by $C(X)$ the Banach algebra of continuous complex functions defined on X with norm $\|\ \|_\infty$ determined by $\|g\|_\infty = \sup_{x\in X}|g(x)|$ for $g \in C(X)$.

Given $f \in C^{(n)}$, we define $\tilde{f} \in C(W)$ by

$$\tilde{f}(x, \theta_1, \cdots, \theta_n) = f(x) + e^{i\theta_1}f'(x) + \frac{e^{i\theta_2}}{2!}f''(x) + \cdots + \frac{e^{i\theta_n}}{n!}f^{(n)}(x)\ , \qquad (x, \theta_1, \cdots, \theta_n) \in W\ .$$

The following lemma is then obvious.

LEMMA 2.1. *The mapping* $f \to \tilde{f}$ *establishes a linear and norm-preserving correspondence between* $C^{(n)}$ *and the closed subspace* S *of* $C(W)$, $S = \{\tilde{f}: f \in C^{(n)}\}$.

Next given $(x, \theta_1, \cdots, \theta_n) \in W$, we define a continuous linear functional $L(x, \theta_1, \cdots, \theta_n)$ on $C^{(n)}$ by

$$L_{(x,\theta_1,\cdots,\theta_n)}(f) = \tilde{f}(x, \theta_1, \cdots, \theta_n) , \quad f \in C^{(n)} .$$

In view of Proposition 1.1 the proof of the following lemma is analogous to the proof of Lemma 1.2 in [1].

LEMMA 2.2. *An element of* $C^{(n)*}$ *is an extreme point of the unit ball* U^* *of* $C^{(n)*}$ *if and only if* f^* *is of the form* $e^{i\eta}L_{(x,\theta_1,\cdots,\theta_n)}$ *for some* $\eta \in [-\pi, \pi]$, $(x, \theta_1, \cdots, \theta_n) \in W$.

We now suppose that T is an isometry of $C^{(n)}$. The adjoint T^* is then an isometry of $C^{(n)*}$, and thus carries extreme points of U^* onto itself.

LEMMA 2.3. *The image by* T *of the constant function* 1 *of* $C^{(n)}$ *is a constant function* $e^{i\lambda}$, $\lambda \in [-\pi, \pi]$.

Proof. For each extreme point $e^{i\eta}L_{(x,\theta_1,\cdots,\theta_n)}$ of U^*,

$$|(e^{i\eta}L_{(x,\theta_1,\cdots,\theta_n)})(1)| = 1 .$$

Thus for each extreme point $|T^*(e^{i\eta}L_{(x,\theta_1,\cdots,\theta_n)})(1)| = 1$. Therefore, $|L_{(x,\theta_1,\cdots,\theta_n)}(T(1))| = 1$. Thus for a fixed x, $|(T(1))(x) + e^{i\theta_1}(T(1))'(x) + \cdots + (e^{i\theta_n}/n!)(T(1))^{(n)}(x)| = 1$ for all $(\theta_1, \cdots, \theta_n) \in [-\pi, \pi]^n$. Choosing $\theta_1, \theta_2, \cdots, \theta_n$, so that

$$\arg((T(1))(x)) = \arg(e^{i\theta_1}(T(1))'(x)) = \cdots = \arg\left(\frac{e^{i\theta_n}}{n!}(T(1))^{(n)}(x)\right)$$

we get

$$|(T(1))(x)| + |(T(1))'(x)| + \cdots + \frac{|(T(1))^{(n)}(x)|}{n!} = 1 .$$

Again by choosing $\theta_1, \cdots, \theta_n$, so that

$$\arg((T(1))(x)) = \pi + \arg(e^{i\theta_1}(T(1))'(x)) = \cdots = \pi + \arg(e^{i\theta_n}(T(1))^{(n)}(x))$$

we get

$$\left| |(T(1))(x)| - \left\{ |(T(1))'(x)| + \cdots + \frac{|(T(1))^{(n)}(x)|}{n!} \right\} \right| = 1 .$$

Thus either

$$\left\{|(T(1))(x)| = 1 \quad \text{and} \quad |(T(1))'(x)| + \cdots + \frac{|(T(1))^{(n)}(x)|}{n!} = 0\right\}$$

or

$$\left\{|(T(1))(x)| = 0 \quad \text{and} \quad |(T(1))'(x)| + \cdots + \frac{|(T(1))^{(n)}(x)|}{n!} = 1\right\} . \tag{4}$$

Therefore, for any $x \in [0, 1]$, $|(T(1))(x)| = 1$ or $|(T(1))(x)| = 0$. But since $|T(1)|$ is a continuous function on $[0, 1]$ we have

$$|(T(1))(x)| \equiv 0 \quad \text{or} \quad |(T(1))(x)| \equiv 1 .$$

Now $|(T(1))(x)| \equiv 0$ implies that $(T(1))(x) \equiv (T(1))'(x) \equiv (T(1))''(x) \equiv \cdots \equiv (T(1))^{(n)}(x) \equiv 0$ which contradicts (4).

Hence $|(T(1))(x)| \equiv 1$ from which it follows that $(T(1))'(x) \equiv 0$ and hence

$$T(1) \equiv e^{i\lambda} \quad \text{for some fixed} \quad \lambda \in [-\pi, \pi] .$$

We denote $T^*(L_{(x,\theta_1,\cdots,\theta_n)})$ by

$$e^{i\lambda(x,\theta_1,\cdots,\theta_n)} L_{(y(x\ \theta_1\ \cdots\ \theta_n),\psi_1(x\ \theta_1\ \cdots\ \theta_n),\cdots,\psi_n(x\ \theta_1\ \cdots\ \theta_n))} .$$

The above Lemma 2.3, shows that $\lambda(x, \theta_1, \cdots, \theta_n) \equiv \lambda$ for all $(\theta_1, \cdots, \theta_n) \in [-\pi, \pi]$. For

$$(T^*(L_{(x,\theta_1,\cdots,\theta_n)}))(1) = e^{i\lambda(x,\theta_1,\cdots,\theta_n)} L_{(y(x,\theta_1,\cdots,\theta_n),\psi_1(x,\theta_1,\cdots,\theta_n)\cdots\psi_n(x,\theta_1,\cdots,\theta_n))}(1) ,$$

so that $L_{(x,\theta_1,\cdots,\theta_n)}(T(1)) = e^{i\lambda(x,\theta_1,\cdots,\theta_n)}$ and thus $L_{(x,\theta_1,\cdots,\theta_n)}(e^{i\lambda}) = e^{i\lambda(x,\theta_1,\cdots,\theta_n)}$. Hence $\lambda(x, \theta_1, \cdots, \theta_n) \equiv \lambda$.

LEMMA 2.4. *If $x \in [0, 1]$, then for all $(\theta_1, \cdots, \theta_n) \in [-\pi, \pi]^n$,*

$$y_{(x,\theta_1,\cdots,\theta_n)} = y_{(x,0,\cdots,0)} .$$

Proof. For fixed $x \in [0, 1]$, we consider the map $\rho: [-\pi, \pi]^n \to [0, 1]$ given by

$$\rho(\theta_1, \theta_2, \cdots, \theta_n) = y_{(x,\theta_1,\cdots,\theta_n)} .$$

It is easy to verify that this mapping is continuous. Hence the image of $[-\pi, \pi]^n$ in $[0, 1]$ is a connected subset of $[0, 1]$. It is, in fact, a singleton. For otherwise we could find g in $C^{(n)}$ such that $g \equiv g' \equiv \cdots \equiv g^{(n)} \equiv 0$ on an open subinterval $I \subset \rho([-\pi, \pi]^n)$ while for some $y_{(x,\varphi_1,\cdots,\varphi_n)} \notin I$,

$$\Big| g(y_{(x,\varphi_1,\cdots,\varphi_n)}) + e^{i\psi_1(x\ \varphi_1,\cdots,\varphi_n)} g'(y_{(x,\varphi_1,\cdots,\varphi_n)})$$

$$+ e^{i\psi_2(x,\varphi_1,\cdots,\varphi_n)} \cdot \frac{1}{2!} g''(y_{(x,\varphi_1,\cdots,\varphi_n)}) + \cdots$$
$$+ e^{i\psi(n-1)(x,\varphi_1,\cdots,\varphi_n)} \cdot \frac{1}{(n-1)!} g^{(n-1)}(y_{(x,\varphi_1,\cdots,\varphi_n)}) \Big|$$
$$< \left| \frac{1}{n!} g^{(n)}(y_{(x,\varphi_1,\cdots,\varphi_n)}) \right| .$$

For instance, one may take

$$g(y) = \begin{cases} 0 & y \leqq y_1 \\ (y - y_1)^{(n+1)} & y > y_1 \end{cases}$$

where y_1 is least upper bound of I and $y_{(x,\varphi_1,\cdots,\varphi_n)}$ sufficiently near to y_1. Thus for an infinite number of $(\theta_1, \theta_2, \cdots, \theta_n) \in [-\pi, \pi]^n$ with $y_{(x,\theta_1,\cdots,\theta_n)} \in I$,

$$\begin{aligned} L_{(x,\theta_1,\cdots,\theta_n)}(T(g)) &= T^* L_{(x,\theta_1,\cdots,\theta_n)}(g) \\ &= e^{i\lambda} L_{(y(x,\theta_1,\cdots,\theta_n), \psi_1(x,\theta_1,\cdots,\theta_n), \cdots, \psi_n(x,\theta_1,\cdots,\theta_n))}(g) \\ &= 0 \end{aligned}$$

while

$$\begin{aligned} &L_{(x,\varphi_1,\cdots,\varphi_n)}(T(g)) \\ &\quad = e^{i\lambda} L_{(y(x,\varphi_1,\cdots,\varphi_n), \psi_1(x,\varphi_1,\cdots,\varphi_n), \cdots, \psi_n(x,\varphi_1,\cdots,\varphi_n))}(g) \neq 0 . \end{aligned}$$

Since ρ is continuous, $\rho^{-1}(I)$ is open in $[-\pi, \pi]^n$ and therefore for each $i = 1, 2, \cdots, n$ there exist an infinite number of θ_i's such that

$$(5) \qquad L_{(x,\theta_1,\cdots,\theta_n)}(T(g)) = 0 \quad \text{while} \quad L_{(x,\varphi_1,\cdots,\varphi_n)}(T(g)) \neq 0 .$$

Therefore $(T(g))(x) + e^{i\theta_1}(T(g))'(x) + \cdots + (e^{i\theta_n}/n!)(T(g))^{(n)}(x) = 0$.

For any j with $1 \leqq j \leqq n$, by keeping θ_i constant for $i \neq j$ and varying θ_j we can see that $(T(g))^{(j)}(x) = 0$. Thus $L_{(x,\varphi_1,\cdots,\varphi_n)}(T(g)) = 0$ which contradicts (5).

Hence $y_{(x,\theta_1,\cdots,\theta_n)} = y_{(x,0,\cdots,0)}$ for all $(\theta_1, \cdots, \theta_n) \in [-\pi, \pi]^n$.

Finally, we define a point map τ of $[0, 1]$ to $[0, 1]$ by

$$\tau(x) = y_{(x,0,\cdots,0)} .$$

Consideration of $(T^{-1})^*$ shows that τ is onto, and, applying Lemma 2.4, one-one.

THEOREM 2.5. *Let T be an isometry of $C^{(n)}$. Then, for $f \in C^{(n)}$,*

$$(T(f))(x) = e^{i\lambda} f(\tau(x))$$

with $e^{i\lambda} = T(1)$. Moreover, τ is one of the two functions $F, 1 - F$ where F is the identity mapping of $[0, 1]$ onto itself.

Proof. Given $x \in [0, 1]$ and $\theta \in [-\pi, \pi]$, consider the function g of the Proposition 1.1 constructed for $(x, \theta, \cdots, \theta)$. Clearly, g does not depend on θ; $g(x) = 0$; $g'(x), g''(x), \cdots, g^{(n)}(x)$ are positive reals and $\sum_{r=1}^{n} (g^{(r)}(x)/r!) > \sum_{r=0}^{n} (|g^{(r)}(y)|/r!)$ for all $y \in [0, 1]$, $y \neq x$. Therefore,

$$\begin{aligned} \|g\| &= g'(x) + \frac{1}{2!}g''(x) + \cdots + \frac{1}{n!}g^{(n)}(x) \\ &= e^{-i\theta}L_{(x,\theta,\cdots,\theta)}(g) \\ &= e^{-i\theta}T^*L_{(x,\theta,\cdots,\theta)}(T^{-1}(g)) \\ &= e^{i(\lambda-\theta)}L_{(\tau(x),\psi_1(x\ \theta,\cdots,\theta),\cdots,\psi_n(x,\theta,\cdots,\theta))}(T^{-1}(g)) . \end{aligned}$$

Thus we have for all $\theta \in [-\pi, \pi]$

$$\begin{aligned} \|g\| = e^{i(\lambda-\theta)}[(T^{-1}(g))(\tau(x)) &+ e^{i\psi_1(x\ \theta,\cdots,\theta)}(T^{-1}(g))'(\tau(x)) \\ &+ \cdots + \frac{1}{n!}e^{i\psi_n(x\ \theta,\cdots\ \theta)}(T^{-1}(g))^{(n)}(\tau(x))] . \end{aligned} \tag{6}$$

Since

$$\begin{aligned} \|g\| &= \|T^{-1}(g)\| \\ &= \sup_{y \in [0,1]} \sum_{r=0}^{n} \frac{|(T^{-1}(g))^{(r)}(y)|}{r!} , \end{aligned}$$

by (6) we have

$$\|g\| = |(T^{-1}(g))(\tau(x))| + |(T^{-1}(g))'(\tau(x))| + \cdots + \frac{1}{n!}|(T^{-1}(g))^{(n)}(\tau(x))| .$$

Again since g is independent of θ,

$$(T^{-1}(g))(\tau(x)), (T^{-1}(g))'(\tau(x)), \cdots, (T^{-1}(g))^{(n)}(\tau(x))$$

are independent of θ but

$$A(\theta) = \left\{e^{i\psi_1(x\ \theta,\cdots,\theta)}(T^{-1}(g))'(\tau(x)) + \cdots + \frac{1}{n!}e^{i\psi_n(x,\theta,\cdots,\theta)}(T^{-1}(g))^{(n)}(\tau(x))\right\}$$

depends on θ for otherwise (6) cannot be true. In other words, $A(\theta)$ is not constant. Now by (6) $A(\theta)$ must be on a circle with center as $\{-(T^{-1}(g))(\tau(x))\}$ and radius equal to $\|g\|$.

On the other hand $A(\theta)$ must be on or within the circle with center as origin and radius equal to $\rho = \sum_{r=1}^{n} (|(T^{-1}(g))^{(r)}(x)|/r!) = \|g\| - |(T^{-1}(g))(\tau(x))|$. This implies that $(T^{-1}(g))(\tau(x)) = 0$ for otherwise $A(\theta)$ has to be a constant (see Figure 2.1) which is false.

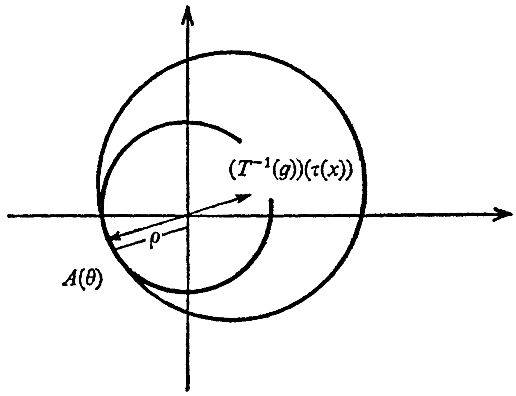

FIGURE 2.1.

Therefore, we have

$$\arg e^{i\psi_{1(x,\theta,\cdots,\theta)}}(T^{-1}(g))'(\tau(x)) = \arg \cdot \frac{1}{2!} e^{i\psi_{2(x,\theta,\cdots,\theta)}}(T^{-1}(g))''(\tau(x)) = \cdots$$
$$= \arg \cdot \frac{1}{n!} e^{i\psi_{n(x,\theta,\cdots,\theta)}}(T^{-1}(g))^{(n)}(\tau(x)) .$$

Thus for all $\theta \in [-\pi, \pi]$, $1 \leqq k \leqq n$, $1 \leqq j \leqq n$

$$\psi_{k(x,\theta,\cdots,\theta)} - \psi_{j(x,\theta,\cdots,\theta)} = \psi_{k(x,0,\cdots,0)} - \psi_{j(x,0,\cdots,0)} .$$

Also by (6)

$$||g|| = e^{i(\lambda-\theta)}\left[\sum_{k=1}^{n} \frac{1}{k!} e^{i\psi_{k(x,\theta,\cdots,\theta)}}(T^{-1}(g))^{(k)}(\tau(x))\right]$$
$$= e^{i(\lambda-\theta+\psi_{j(x,\theta,\cdots,\theta)})}\left[\sum_{k=1}^{n} \frac{1}{k!} e^{i(\psi_{k(x,\theta,\cdots,\theta)}-\psi_{j(x,\theta,\cdots,\theta)})}(T^{-1}(g))^{(k)}(\tau(x))\right]$$
$$= e^{i(\lambda-\theta+\psi_{j(x,\theta,\cdots,\theta)})}\left[\sum_{k=1}^{n} \frac{1}{k!} e^{i(\psi_{k(x,0,\cdots,0)}-\psi_{j(x,0,\cdots,0)})}(T^{-1}(g))^{(k)}(\tau(x))\right] .$$

Since the left hand side is independent of θ, we have

$$\lambda - \theta + \psi_{j(x,\theta,\cdots,\theta)} = \lambda + \psi_{j(x,0,\cdots,0)}) .$$

Hence for all $\theta \in [-\pi, \pi]$, $1 \leqq j \leqq n$

$$\psi_{j(x,\theta,\cdots,\theta)} = \psi_{j(x,0,\cdots,0)} + \theta .$$

Now let f be any element of $C^{(n)}$ such that $f(x) = 0$ then for all $\theta \in [-\pi, \pi]$

$$f'(x) + \frac{1}{2!} f''(x) + \cdots + \frac{1}{n!} f^{(n)}(x)$$

$$
\begin{aligned}
&= e^{-i\theta}L_{(x,\theta,\cdots,\theta)}(f)\\
&= e^{-i\theta}T^*L_{(x,\theta,\cdots,\theta)}(T^{-1}(f))\\
&= e^{i(\lambda-\theta)}L_{(\tau(x),\psi_1(x,\theta,\cdots,\theta),\cdots\psi_n(x,\theta,\cdots,\theta))}(T^{-1}(f))\\
&= e^{i(\lambda-\theta)}\left[(T^{-1}(f))(\tau(x)) + \sum_{k=1}^{n}\frac{1}{k!}e^{i\psi_k(x,\theta,\cdots,\theta)}(T^{-1}(f))^{(k)}(\tau(x))\right]\\
&= e^{i\lambda}\left[e^{-i\theta}(T^{-1}(f))(\tau(x)) + \sum_{k=1}^{n}\frac{1}{k!}e^{i\psi_k(x,0,\cdots,0)}(T^{-1}(f))^{(k)}(\tau(x))\right]
\end{aligned}
$$

so that $(T^{-1}(f))(\tau(x)) = 0$. For an arbitrary $f \in C^{(n)}$, define $g(y) = f(y) - f(x)$, $y \in [0, 1]$ then $g(x) = 0$ and so

$$
\begin{aligned}
0 = (T^{-1}(g))(\tau(x)) &= (T^{-1}(f))(\tau(x)) - f(x)(T^{-1}(1))(\tau(x))\\
&= (T^{-1}(f))(\tau(x)) - e^{-i\lambda}f(x) .
\end{aligned}
$$

Thus, replacing f by $T(f)$, it follows that for all $x \in [0, 1]$ and $f \in C^{(n)}$,

$$(T(f))(x) = e^{i\lambda}f(\tau(x)) .$$

Now if, for $0 \leqq r \leqq n-1$, F_r is the mapping of $[0, 1]$ onto itself given by $F_r(x) = x^{r+1}$ (where F_0 is the identity map F), we have

$$(T(F_r))(x) = e^{i\lambda}(\tau(x))^{r+1} = e^{i\lambda}(\tau^{r+1})(x) , \quad 0 \leqq r \leqq n-1 .$$

Therefore $(T(F_r))(x) = (T(F_{r-1}))(x)\cdot\tau(x)$. Now

$$
\begin{aligned}
\sum_{k=0}^{n}\frac{1}{k!}(T(F_r))^{(k)}(x) &= L_{(x,0,\cdots,0)}(T(F_r))\\
&= T^*L_{(x,0,\cdots,0)}(F_r)\\
&= e^{i\lambda}L_{(\tau(x),\psi_1(x,0,\cdots,0),\cdots,\psi_n(x,0,\cdots,0))}(F_r)\\
&= e^{i\lambda}\left[F_r(\tau(x)) + \sum_{j=1}^{n}\frac{1}{j!}e^{i\psi_j(x,0\cdots,0)}F_r^{(j)}(\tau(x))\right]\\
&= e^{i\lambda}\left[(\tau(x))^{r+1} + \sum_{j=1}^{r+1}e^{i\psi_j(x,0,\cdots,0)}C_j^{r+1}(\tau(x))^{r+1-j}\right] .
\end{aligned}
$$

Thus for $0 \leqq r \leqq n-1$

$$\sum_{k=1}^{n}\frac{1}{k!}(T(F_r))^{(k)}(x) = e^{i\lambda}\sum_{j=1}^{r+1}e^{i\psi_j(x\ 0,\cdots,0)}C_j^{r+1}(\tau(x))^{r+1-j} . \tag{7}$$

Taking $r = 0$ in (7), we get

$$\sum_{k=1}^{n}\frac{1}{k!}(T(F))^{(k)}(x) = e^{i(\lambda+\psi_1(x,0,\cdots,0))} .$$

Taking $r = 1$, we get

$$\sum_{r=1}^{n} \frac{1}{k!} (T(F_1))^{(k)}(x) = C_1^2(\tau(x)) e^{i(\lambda+\psi_1(x,0,\cdots,0))} + e^{i(\lambda+\psi_2(x,0,\cdots,0))} .$$

Hence

$$e^{i(\lambda+\psi_2(x,0,\cdots,0))}$$
$$= \sum_{k=1}^{n} \frac{1}{k!}(T(F_1))^{(k)}(x) - C_1^2(\tau(x)) \sum_{k=1}^{n} \frac{1}{k!}(T(F))^{(k)}(x) .$$

Thus by successive iterations we get for $1 \leqq r \leqq n$

$$e^{i(\lambda+\psi_r(x,0,\cdots,0))} = \sum_{k=1}^{n} \frac{1}{k!} \left\{ \sum_{j=1}^{r} (-1)^{j-1} C_{j-1}^{r} (T(F_{r-j}))^{(k)}(x)(\tau(x))^{j-1} \right\}$$
$$= e^{i\lambda} \sum_{k=1}^{n} \frac{1}{k!} \left\{ \sum_{j=1}^{r} (-1)^{j-1} C_{j-1}^{r} (\tau^{r-j+1})^{(k)}(x)(\tau(x))^{j-1} \right\} .$$

Therefore,

$$e^{i\psi_n(x,0,\cdots,0)} = \sum_{k=1}^{n} \frac{1}{k!} \left\{ \sum_{j=1}^{n} (-1)^{j-1} C_{j-1}^{n} (\tau^{n-j+1})^{(k)}(x)(\tau(x))^{j-1} \right\} .$$

Applying Proposition 1.2 to the function τ which clearly belongs to $C^{(n)}$ we get

$$e^{i\psi_n(x,0,\cdots,0)} = \{\tau'(x)\}^n .$$

Thus $\tau'(x)$ is an nth root of a complex number of absolute value one.

But since $\tau'(x)$ is real valued and continuous we have $\tau'(x) \equiv 1$ or $\tau'(x) \equiv -1$ and, therefore, $\tau(x) \equiv F$ or $1 - F$.

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