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PROJECTIVE REPRESENTATIONS OF FINITE GROUPS IN CYCLOTOMIC FIELDS

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It is known that every complex projective representation of a finite group is realizable in a cyclotomic field $Q(\xi_k)$. This note is concerned with the problem of finding the minimal k with this property and gives a partical answer in this direction.

1. Let G be a finite group and let $P: G \to PGL(n, C)$ be a complex projective representation of G. A representative of P is a mapping $T: G \to GL(n, C)$ such that for all $x \in G$ we have $\pi(T(x)) = P(x)$ where π denotes the canonical homomorphism $GL(n, C) \to GL(n, C)/C^* \cong PGL(n, C)$. P is called irreducible if in the vector space $V \cong C^n$ there is no proper subspace W such that $T(x)W \subset W$ for all $x \in G$. P is called realizable in a subfield K of C if there is a function $\alpha: G \to C^*$ and some matrix $A \in GL(n, C)$ such that the coefficients of all the matrices $\alpha(x)AT(x)A^{-1}$, $x \in G$, belong to K. The representative T of P is said to be realizable in a subfield K of C if for some matrix $A \in GL(n, C)$ the coefficients of all the matrices $AT(x)A^{-1}$, $x \in G$, belong to K. A subfield K of C is called a (*weak*) *projective splitting field* for G if every (irreducible) projective representation of G is realizable in K. For any $k \in N$ denote by ξ_k a primitive kth root of unity in C. In [6] it is shown that the cyclotomic field $Q(\xi_k)$, $k = |G|$, is a projective splitting field for G, and it was asked in the same paper (p. 191) whether $|G|$ could be replaced by the exponent $\exp G$ of G. We shall point out an example which gives a negative answer to this question, and then we shall prove the following theorem.

THEOREM. *The cyclotomic field* $Q(\xi_m)$, $m = \exp G' \exp M(G)$, *is a weak projective splitting field for* G.

Here G' denotes the commutator subgroup and $M(G)$ the multiplicator of G. Note that $\exp G \exp M(G)$ divides $|G|$, comp. [1].

For basic concepts in the theory of projective representations the reader may consult [3], V, § 23–25.

2. Let $T: G \to GL(n, C)$ be a representative of the complex projective representation $P: G \to PGL(n, C)$. Then for all $x, y \in G$ we have $T(x)T(y) = f(x, y)T(xy)$ for some $f(x, y) \in C^*$, and the

mapping $f: G \times G \to C^*$ is a central factor system (2-cocycle). If P is realizable in a subfield K of C we may choose T in such a way that all values of f belong to K. Now let $H^2(G, E^*)$ be the second cohomology group of the finite group G with respect to the multiplicative group E^* of a field E, E^* regarded as a trivial G-module. ($H^2(G, C^*) = : M(G)$ is also called "*multiplicator*".) Any subfield $E \subset C$ contains a primitive eth root of unity, e denoting the exponent of the image of the canonical homomorphism $H^2(G, E^*) \to H^2(G, C^*)$ induced by the embedding $E^* \subset C^*$, comp. [4], (2.1). If $K \subset C$ is a (weak) projective splitting field for G, then, by the above argument, the homomorphism $H^2(G, K^*) \to H^2(G, C^*)$ is surjective, hence K contains a primitive $\exp M(G)$th root of unity.

In [2], p. 88, an example of a finite nilpotent group of exponent 4 with multiplicator of exponent 8 is given. Therefore $Q(\sqrt{-1})$ is not a projective splitting field for this group. So the answer to the above mentioned question of Reynolds is negative.

The following example (due to H. Pahlings) shows that in general the cyclotomic field $Q(\xi_k)$, $k = \exp M(G)$, is not a (weak) projective splitting field for G. Let $D = \langle a, b \mid a^8 = a^2 = 1, bab = a^7 \rangle$ be the dihedral group of order 16. It has an irreducible projective representation P with representative T such that

$$T(a) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad T(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$\varepsilon =$ primitive root of unity of order 16. The factor system f, which corresponds to T, has order 2. It is known that $\exp M(D) = 2$, comp. [3], V, (25.6). But there is no function $\alpha: D \to C^*$ such that $\det(\alpha(a)T(a)) = \alpha(a)^2$ and $\operatorname{trace}(\alpha(a)T(a)) = \alpha(a)(2 + \sqrt{2})^{1/2}$ both belong to Q.

3. Now we shall prove the theorem stated in § 1. A key role is played by Clifford's theory for projective representations, comp. [5], § 2.

Let P be an irreducible complex projective representation of the finite group G and let T be a representative of P such that the values of the central factor system $f: G \times G \to C^*$ which is determined by the relation $T(x)T(y) = f(x, y)T(xy)$, $x, y \in G$, belong to the group W of $\exp M(G)$th roots of unity in C. Let χ be the character of T, i.e., $\chi(x) = \operatorname{trace}(T(x))$ for all $x \in G$. The restriction $T \mid G'$ decomposes as a direct sum of representatives of irreducible projective representations of G'. Let S be one of them and let ϕ be its character. S is realizable in the cyclotomic field $Q(\xi_m)$, $m =$

$\exp G' \exp M(G)$. This is easily seen as follows. Consider the group $G'(f) := \{(a, h) \mid a \in W, \ h \in G'\}$, the multiplication rule given by $(a_1, h_1)(a_2, h_2) = (a_1 a_2 f(h_1, h_2), h_1 h_2)$. Then lift S to a linear representation $\tilde{S}$ of $G'(f)$ by defining $\tilde{S}((a, h)) := aS(h)$. It is well known (comp. e.g., [3], V, (19.11)) that $\tilde{S}$ is realizable in the cyclotomic field $K = \boldsymbol{Q}(\xi_k)$, $k = \exp G'(f)$. Therefore this is also true for S. But $\exp G'(f)$ divides $\exp G' \exp M(G)$. Now let I be the inertia group of ϕ, i.e.,

$$I = \{x \in G \mid (f(x, h) f(xh, x^{-1}) / f(x, x^{-1})) \phi(xhx^{-1}) = \phi(h) \text{ for all } h \in G'\} .$$

By Clifford's theory there is a representative R of a projective representation of I such that T is induced by R, and the restriction $R \mid G'$ decomposes as a direct sum of some copies of S. Furthermore, for every $x \in I$ there is an invertible matrix $A'(x)$ with coefficients in K such that $A'(x)S(h)A'(x)^{-1} = S^x(h)$ for all $h \in G'$. Schur's lemma yields a central factor system $t' \colon I \times I \to K^*$ such that $A'(x)A'(y) = t'(x, y)A'(xy)$ for all $x, y \in I$. Choose a set $\{u_\delta\}$ of right coset representatives of G' in I and define

$$A(hu_\delta) := f(h, u_\delta)^{-1} S(h) A'(u_\delta), \quad A(h) := S(h)$$

for all $h \in G'$. Then A is a representative of a projective representation of I which is realizable in K. Denote by $t \colon I \times I \to K^*$ the central factor system which belongs to A. The central factor system $ft^{-1} \colon I \times I \to K^*$ satisfies $ft^{-1}(h_1 x_1, h_2 x_2) = ft^{-1}(x_1, x_2)$ for all $x_1, x_2 \in I$, $h_1, h_2 \in G'$, comp. [6], § 2. Hence it may be regarded as a factor system of $F := I/G'$.

There is a representative U of an irreducible complex projective representation of F such that $R = A \otimes \inf_F^I(U)$. Since F is abelian there is a subgroup $F_0 \leqq F$ such that U is induced by some function $\lambda \colon F_0 \to \boldsymbol{C}^*$ which satisfies $\lambda(a)\lambda(b) = ft^{-1}(a, b)\lambda(ab)$. λ defines an element $\bar{\lambda} \in \operatorname{Hom}(F_0, \boldsymbol{C}^*/K^*)$. We can find an element $\bar{\alpha} \in \operatorname{Hom}(G/G', \boldsymbol{C}^*/K^*)$ such that $\bar{\alpha} \mid F_0 = \bar{\lambda}^{-1}$ and all values of the trivial factor system $\delta\alpha$ belong to K^*. Define $T' := \alpha \otimes T$, $f' := f\delta\alpha$. By Frobenius reciprocity (comp. [4], (1.12))T' is induced by $A \otimes \inf(\alpha \mid F \otimes U)$ and $\alpha \mid F \otimes U$ is induced by $\alpha \mid F_0 \otimes \lambda$. All values of $\alpha \mid F_0 \otimes \lambda$ belong to K. Hence $\alpha \mid F \otimes U$ is realizable in K and then also T' is realizable in K. This shows that P is realizable in K.

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