

HOMOLOGY COBORDISMS OF 3-MANIFOLDS, KNOT CONCORDANCES, AND PRIME KNOTS

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There is a close relation between link concordances and homology cobordisms of 3-manifolds. Using this relationship we will prove that every closed, orientable 3-manifold is homology cobordant to an irreducible 3-manifold. Essentially the same construction will be used to prove that every knot in S^3 is concordant to a prime knot.

Kirby and Lickorish [3] have previously obtained the second result using a different construction. This answers problem 13 in Gordon [2].

Results of Schubert [6] show that the knots constructed are prime. In the second half of this paper we will generalize his results to characterize prime "generalized" cable knots. If K_1 is a knot in $S^1 \times B^2$ and K_2 is a knot in S^3 , we can form the K_1 cable of K_2 by mapping $S^1 \times B^2$ into S^3 , with $S^1 \times \{0\}$ going to K_2 . We will show that, with an obvious restriction, the K_1 cable of K_2 is prime if and only if K_1 is prime in $S^1 \times B^2$. The final section gives a large set of examples of prime knots in $S^1 \times B^2$.

I would like to thank Rob Kirby for his helpful suggestions and conversations.

1. Definitions and conventions. In all that follows manifolds and maps will be smooth and orientable. Intersections of submanifolds and of submanifolds with the boundary of a manifold will always be assumed to be transverse.

A *knot* K in S^3 is an embedded S^1 in S^3 , a *link* of n components, L , is a collection on n disjoint knots. By $N(K)$ we indicate a tubular neighborhood of K in S^3 . The *meridian* of a knot K is any embedded S^1 in $\partial N(K)$ which bounds a B^2 in $N(K)$, and which is nontrivial in $H_1(\partial N(K); \mathbb{Z})$. A *longitude* of K is an embedded S^1 in $\partial N(K)$ which is nontrivial in $H_1(\partial N(K); \mathbb{Z})$ but represents 0 in $H_1(S^3 - \text{int}(N(K)); \mathbb{Z})$.

Two links, L_1 and L_2 , each of n components are called *concordant* if there is an embedding \bar{L} of n disjoint copies of $S^1 \times I$ into $S^3 \times I$, with $\bar{L}(n(S^1 \times I)) \cap S^3 \times \{0\} = L_1$ and $\bar{L}(n(S^1 \times I)) \cap S^3 \times \{1\} = L_2$. Two 3-manifolds, M_1 and M_2 , are *homology cobordant* if there is a 4-manifold W , with $\partial W = M_1 \cup M_2$ and the map of $H_*(M_i; \mathbb{Z}) \rightarrow H_*(W; \mathbb{Z})$ an isomorphism.

A 3-manifold M is *irreducible* if every embedded S^2 in M bounds an embedded B^3 . A link L is *irreducible*, or *unsplittable*, if $S^3 - L$

is irreducible.

2. **Preliminary results.** In order to construct homology cobordisms we must use the relationship between knot concordances and homology cobordisms. Proposition 2.1 below describes that relationship. We will then develop the essential tool for showing that a 3-manifold is irreducible, Proposition 2.3.

Given a link L in S^3 of n components, and n diffeomorphisms, h_i , of T^2 to itself, there is a natural way of forming a 3-manifold: remove $N(L)$ from S^3 and replace it according to the h_i 's. In what follows we will be concerned only with *integer surgery*, in which case each h_i maps the longitude to a meridian, and the meridian to a curve which is equivalent in $H_1(\partial N(K); \mathbb{Z})$ to a longitude plus k meridians. If a manifold M^3 is obtained from surgery on a link L_1 , and L_1 is concordant to L_2 , there is a natural way to construct a homology cobordism between M^3 and M^{3*} , where M^{3*} is obtained by surgery on L_2 . That is, remove a tubular neighborhood of the concordance, and replace it according to the maps $h_i \times \text{id}$, mapping $S^1 \times S^1 \times I$ to itself. This construction for knots is described in detail by Gordon in [1]. The following is a generalization to links of the theorem stated there.

PROPOSITION 2.1. *If a manifold W is obtained by surgering a link concordance in $S^3 \times I$, then it is a homology cobordism between its boundary components, M_1 and M_2 .*

Proof. Using Alexander duality it follows that the homology of the complement of the concordance is naturally isomorphic to the homology of the complement of L in S^3 . Then there are natural maps joining the Meyer-Vietoris sequence of M_i to that of W , with almost all maps being isomorphisms. The proof is completed by applying the five lemma to these sequences.

A 3-manifold M is said to have *incompressible* boundary if any embedded S^1 in ∂M which bounds an embedded disk in M also bounds a disk in ∂M . This is equivalent, via Dehn's lemma, to the usual algebraic definition.

LEMMA 2.2. *The complement of an open regular neighborhood of an irreducible link of more than one component in S^3 has an incompressible boundary. If the link has one component which is knotted this result also holds.*

Proof. If for some component K_i of L we have some non-

trivial S^1 in $\partial(N(K_i))$ bounding an embedded disk D in $S^3 - L$, we have immediately that K_i is unknotted. Further, $\partial(N(K_i) \cup N(D)) = S^2$, and K_i is the only component of L in one of the components of S^3 bounded by S^2 . The result readily follows.

PROPOSITION 2.3. *Let $K_i, i = 1, \dots, n$, be a collection of non-trivial knots in S^3 , and L a link of n components in S^3 which is irreducible. Assume that $M = (S^3 - \text{int}(N(L))) \mathbf{U}_{h_1} (S^3 - \text{int}(N(K_1))) \mathbf{U}_{h_2} \dots \mathbf{U}_{h_n} (S^3 - \text{int}(N(K_n)))$ with the h_i diffeomorphisms. Then M is irreducible.*

Proof. Let S be an embedded 2-sphere in M . We first show that S can be isotoped to an embedding such $S \cap \partial(N(L)) = \emptyset$: If $S \cap \partial(N(L))$ is nonempty, take an innermost circle of intersection on S . That circle bounds a disk in either $S^3 - N(L)$ or some $S^3 - N(K_i)$. In either case, by Lemma 2.2 it bounds a disk on the boundary of that space. These two disks form a 2-sphere in S^3 , which bounds an embedded 3-ball. That 3-ball can now be used to isotop S to eliminate at least one circle of intersection. By repeating this process we eventually have $S \cap \partial(N(L)) = \emptyset$. We now have S in one of the original S^3 's. Now, perhaps using that L is irreducible, we have that S bounds a B^3 .

3. Main construction. Our construction proceeds in two steps. We first find a description of a given manifold M as surgery on a link of a given type. A certain concordance is performed which, by Proposition 2.1 gives a homology cobordism. Proposition 2.3 will imply that one end of that cobordism is irreducible.

LEMMA 3.1. *Any 3-manifold can be obtained by integer surgery on an irreducible link.*

Proof. By a result of Lickorish [4] M^3 is obtained by integer surgery on some link. We will now alter that description. (See Rolfsen [5] for a description of these alterations.) Grab a strand from each component of L as indicated in Fig. 1 and add a +1

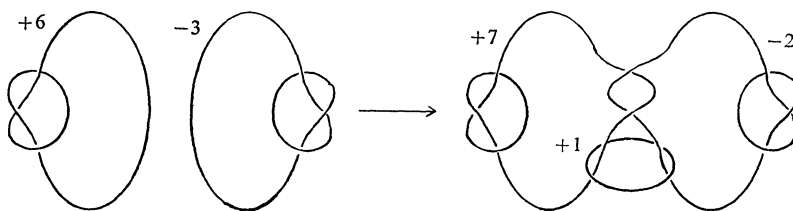


FIGURE 1

linking each one. Each component of L is now geometrically linked to the $+1$. It is now immediate that the new link is irreducible.

Fig. 2 indicates a concordance of 2 knots in $S^1 \times B^2$. This concordance will be essential in this section and in proving that every knot is concordant to a prime knot.

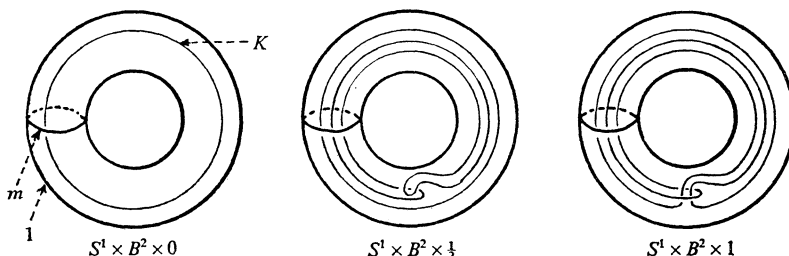


FIGURE 2

The concordance we perform on the link L describing M^3 is just the concordance illustrated in Fig. 2 performed on each component. That is, for each component K_i of L , we map $S^1 \times B^2$ to $N(K_i)$ with $S^1 \times \{0\}$ going to K_i , and similarly map $S^1 \times B^2 \times I$ to $N(K_i \times I) \subset S^3 \times I$. We can now map the concordance over also. For each of these maps there are actually many possible choices, depending on the image of the longitude 1 of $S^1 \times B^2$. In each case pick the map so that the framing on K_i corresponds to -1 surgery on the knot in $S^1 \times B^2$, relative to the longitude indicated in Fig. 2. That is, there is a map as indicated in Fig. 3, preserving meridians.

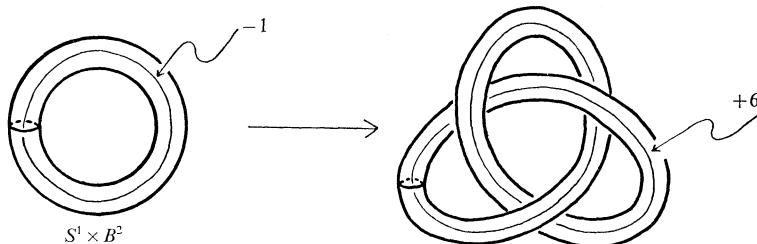


FIGURE 3

THEOREM 3.2. *Every 3-manifold is homology cobordant to an irreducible 3-manifold.*

Proof. Describe M^3 as surgery on an irreducible link. Construct the concordance described above. This gives, using Proposition 2.1, a homology cobordism. The final manifold, M^{3*} is the union of the complement of an irreducible link, and the manifold shown in Fig. 4a. That manifold is the complement of the circle C drawn in Fig. 4b. Redrawing the picture, and then removing the -1 shows that

this manifold is a knot complement. See Fig. 4c and 4d. Hence, applying Proposition 2.3 gives the desired result.

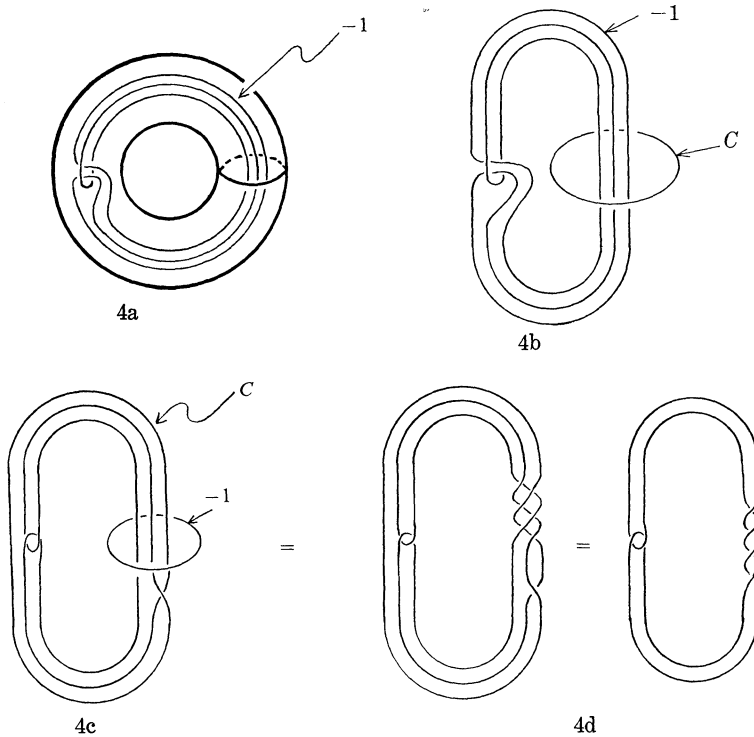


FIGURE 4

REMARK. Akbulut has informed me of a proof that integral surgery on any composite knot gives an irreducible 3-manifold. A similar argument to his, along with Proposition 2.3, gives the result that integral surgery on an irreducible link with all components knots gives an irreducible manifold. Hence another proof of Theorem 3.2 can be given, by changing the concordance used above to one taking each component to a composite knot. It is interesting to note that in the proof given earlier, all components finish as prime knots, as will be proven later.

Akbulut's construction is as follows. Let K_1 and K_2 be knots in S^3 . Form a 4-manifold W by attaching one copy of B^4 to another, identifying $N(K_1)$ in the first copy to $N(K_2)$ in the second. (The choice of attaching maps will determine the framing of the final knot.) The boundary of this manifold is clearly the union of 2 knot complements and is hence irreducible. However, this 4-manifold is the same one as is obtained by surgery on $K_1 \# K_2$. This can be seen by "sliding" the cone point of K_2 in the second B^4 into the first. See Fig. 5.

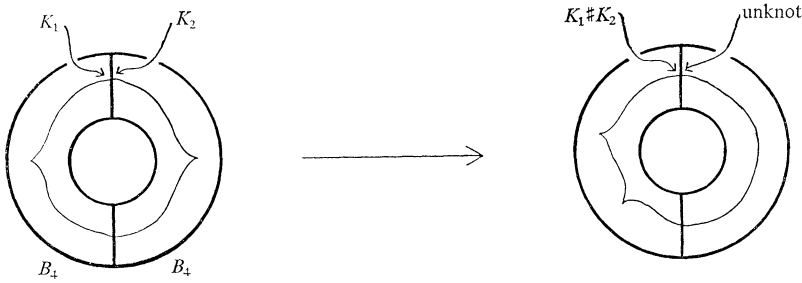


FIGURE 5

Another way of seeing this is to construct W by first identifying the 2 copies of B^4 along 2 copies of B^3 in $N(K_1)$ and $N(K_2)$. This new manifold is just B^4 . The identification is completed by adding a 2-handle to B^4 along $K_1 \# K_2$. See Fig. 6.

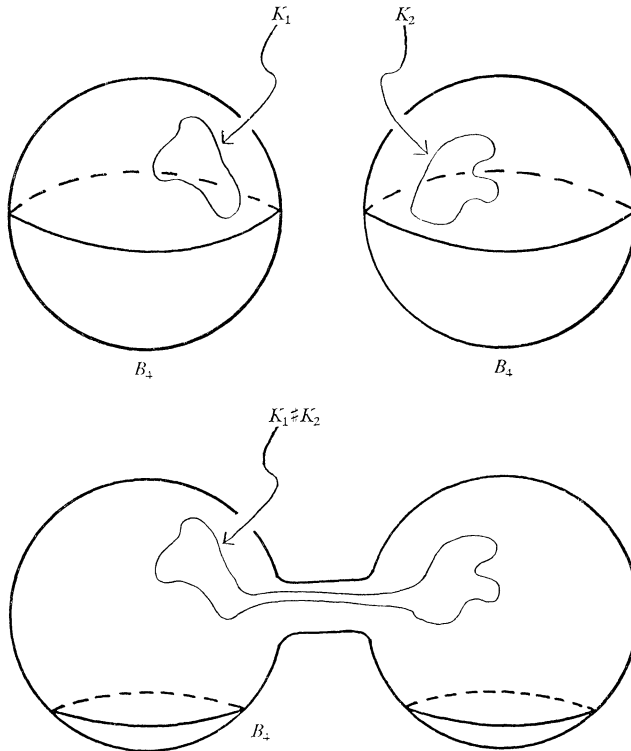


FIGURE 6

The same technique illustrated in Fig. 4 can be used to show that -1 surgery on the knot in $S^1 \times B^3$ illustrated in Fig. 7a gives the complement of the knot C in Fig. 7b. Hence all the 3-manifolds obtained by surgery on the knot K_1 in Fig. 7c and -1 surgery on any knot J_n in Fig. 7d is the union of the complement of the knot C with a second copy of itself.

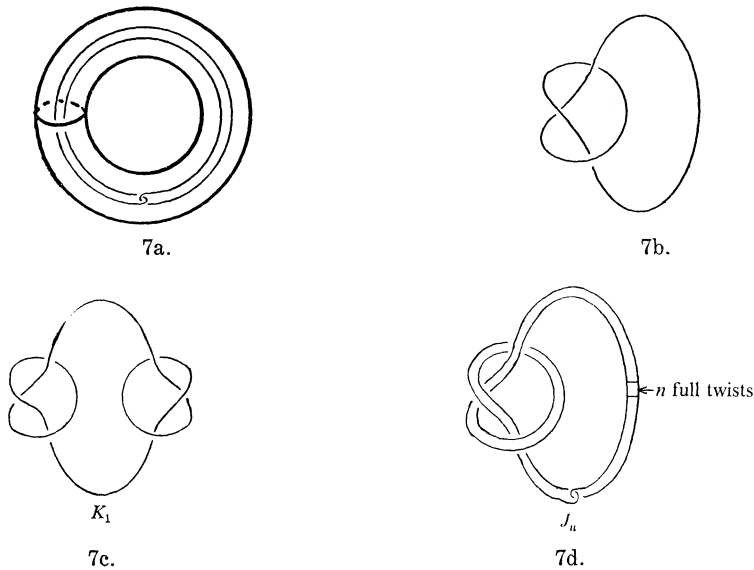
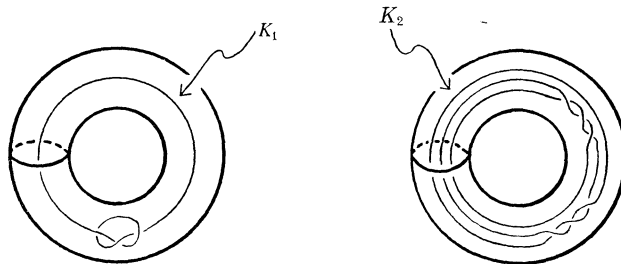


FIGURE 7

4. Generalized cable knots. Before proving our main theorem concerning prime knots, we will set up some notation. By a *splitting- S^2* , S , for a knot K (in S^3 or $S^1 \times B^2$) we indicate an embedded 2-sphere, S , intersecting K in exactly 2 points. A knot in either S^3 or $S^1 \times B^2$ is *prime* if for every splitting- S^2 , S , S bounds some 3-ball, B , with $(B, B \cap K)$ a trivial pair. See Fig. 8. The *winding number* of a knot K in $S^1 \times B^2$ is that element z of $H_1(S^1 \times B^2; \mathbb{Z})$ with $z \geq 0$ and K representing z . The *wrapping number* of a K is the minimum number of intersections of K with a disk D in $S^1 \times B^2$ with $\partial D = \text{meridian}$. See Fig. 9. If K_1 is a knot in $S^1 \times B^2$ and K_2 is a knot in S^3 , the K_1 *cable* of K_2 is the knot in S^3 formed by mapping $S^1 \times B^2$ into $N(K_2)$, and considering the image of K_1 under this map. The only restriction on the map of $S^1 \times B^2$ into $N(K_2)$ is that it maps a meridian to a meridian. See Fig. 10.



K_1 is prime in S^3 , but not on $S^1 \times B^2$.
 K_2 is prime in $S^1 \times B^2$, but not in S^3 . (See §5).

FIGURE 8

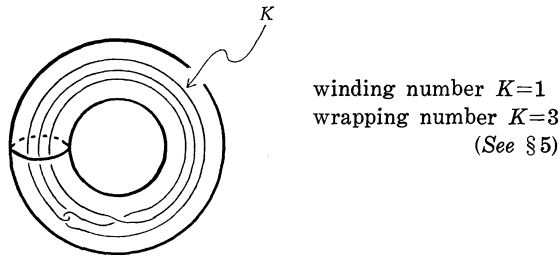


FIGURE 9

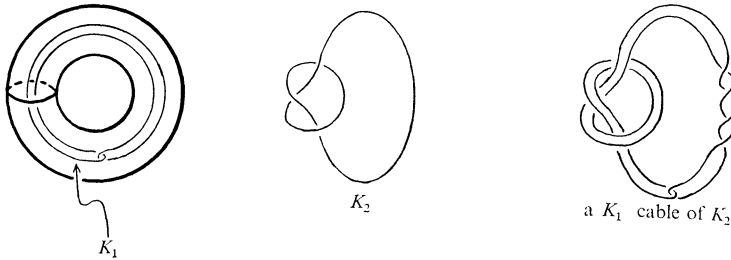


FIGURE 10

LEMMA 4.1. *If K_1 is a knot in $S^1 \times B^2$ with wrapping number $K_1 \geq 1$, and K_2 is any nontrivial knot in S^3 , the K_1 cable of K_2 is nontrivial in S^3 .*

Proof. If the K_1 cable of K_2 , K , is trivial, then the longitude of K bounds an embedded disk, D , in $S^3 - K$. We will show first that $D \cap \partial N(K_2)$ can be made empty. If $D \cap \partial N(K_2)$ is nontrivial, let S^1 be an innermost circle, on D , of intersection. S^1 bounds an embedded disk, either in $S^3 - N(K_2)$ or $S^1 \times B^2 - K_1$. Using either the fact that K_2 is nontrivial or that wrapping number $K_1 \geq 1$, we have that S^1 is trivial on $\partial N(K_2)$. Hence the disk it bounds on $\partial N(K_2)$ along with the one it bounds on D form an S^2 . That S^2 bounds a B^3 which can be used to isotop D to eliminate a circle of intersection. Hence, proceeding in this manner we eventually have $D \cap \partial N(K_2) = \emptyset$, so D is contained in $S^1 \times B^2$. But this would imply that wrapping number K_1 is 0, contradicting the hypothesis.

THEOREM 4.2. *Let K_1 be knot in $S^1 \times B^2$ with wrapping number > 1 , and let K_2 be any nontrivial knot in S^3 . The K_1 cable of K_2 is prime if and only if K_1 is prime in $S^1 \times B^2$.*

Proof. If K_1 is not prime in $S^1 \times B^2$, then K_1 is obtained from a knot K'_1 in $S^1 \times B^2$ by connect summing. K'_1 has wrapping number > 1 , as any isotopy of K'_1 in $S^1 \times B^2$ could be extended to a neighborhood of K'_1 , and then to K'_1 connect sum any knot. We then have

that the K_1 cable of K_2 is the connect sum of some nontrivial knot and the K'_1 cable of K_2 , which, by Lemma 4.1 is nontrivial.

To prove the converse, assume that S is a splitting 2-sphere for the K_1 cable of K_2 . We will show that there is an isotopy such that $S \cap \partial N(K_2) = \emptyset$. Then we will have $S \subset S^1 \times B^2$. The primeness of K_1 will then imply that the B^3 bounded by S in $S^1 \times B^2$ satisfies $(B^3, B^3 \cap K) = \text{trivial pair}$.

Denote by \underline{A} S with small open neighborhoods of $S \cap K$ removed. $A \cap \partial N(K_2)$ consists of a collection of circles. If any such circle is null homologous on A , let S^1 denote an innermost one. S^1 bounds an embedded disk in either $S^3 - N(K_2)$ or $S^1 \times B^2 - N(K_2)$. In either case we get S^1 is null homologous on $\partial(N(K_2))$. (K_2 is knotted and wrapping number $K_1 \geq 1$.) The disk bounded by S on $\partial(N(K_2))$ along with the one it bounds on A form an embedded S^2 in S^3 , which bounds a B^3 in S^3 . This B^3 can be used to isotop S to eliminate (at least) one circle of intersection. See Fig. 11.

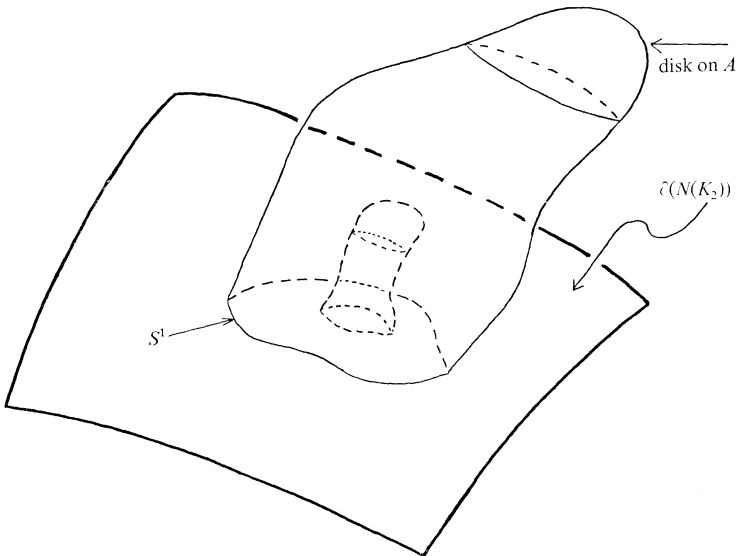


FIGURE 11

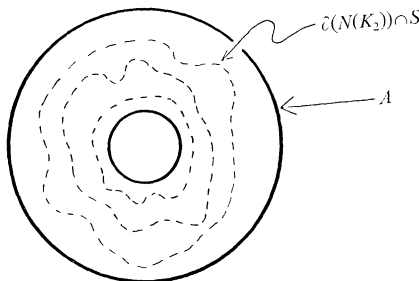


FIGURE 12

After repeating this procedure we have that all circles of intersection on A are parallel to the boundary of A . See Fig. 12. If any such circle existed, however, by considering an innermost one we would get wrapping number $K_1 = 1$, contradicting the hypothesis. By the initial remark, we have completed the proof.

5. Prime knots in $S^1 \times B^2$. In order to apply Theorem 4.2 we need examples of prime knots in $S^1 \times B^2$. In this section we will provide a large number of such examples. Many of these examples were known to Schubert [6]. In what follows we will consider $S^1 \times B^2$ embedded in S^3 in a standard way. Hence any knot K in $S^1 \times B^2$ gives rise to a knot K^* in S^3 . Basic examples derive from the following result.

PROPOSITION 5.1. *If K is a knot in $S^1 \times B^2$ and K^* is trivial in S^3 , then K is prime in $S^1 \times B^2$.*

Proof. Let S be a splitting- S^2 for K in $S^1 \times B^2$. It is also a splitting- S^2 for K^* . K^* is trivial implies that for any ball bounded by S, B^3 , $(B^3, B^3 \cap K^*)$ is trivial, so $(B^3, B^3 \cap K)$ is also trivial.

COROLLARY 5.2. *Doubled knots are prime.*

Proof. Doubled knots are obtained by taking the K_1 cable about some knot, where K_1 is pictured in Fig. 13. Note that K_1^* is trivial. (This result also follows from a genus argument.)

COROLLARY 5.3. *Every knot is concordant to a prime knot.*

Proof. Apply the concordance used in proving Theorem 3.2. The new knot is the K_2 cable about some knot, with K_2 as in Fig. 13. Note that K_2^* is trivial.

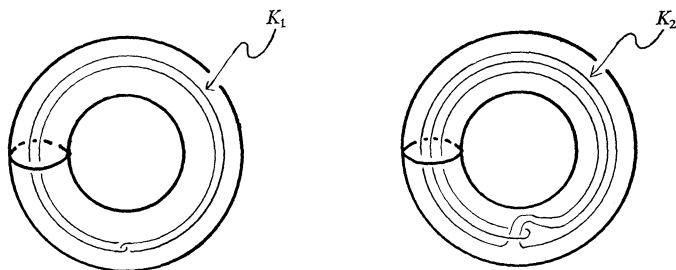


FIGURE 13

REMARK. The only prime knot in $S^1 \times B^2$ with wrapping number

0 is trivial, the only prime knot in $S^1 \times B^2$ with wrapping number 1 is the core. Both of these results are straightforward. Hence to see that K_1 above does not have wrapping number 0 and that K_2 does not have wrapping number 1, note that -1 surgery on each gives a nontrivial knot complement. See Figs. 4 and 7. This is not the case for either the core of $S^1 \times B^2$ or a trivial circle.

LEMMA 5.4. *Let K be a knot in $S^1 \times B^2$ with wrapping number = n . Let D be a disk in $S^1 \times B^2$ with $\partial D = \text{meridian of } S^1 \times B^2$ and $D \cap K = n \text{ points}$. If K is not prime, there is a splitting $-S^2, S$, for K , with $S = \partial B^3, (B^3, B^3 \cap K)$ nontrivial, and $S \cap D = \emptyset$.*

Proof. See Fig. 14. Denote by \underline{A} $\{S\text{-neighborhoods of } K \cap S\}$. We will use an innermost circle argument to eliminate circles of intersection.

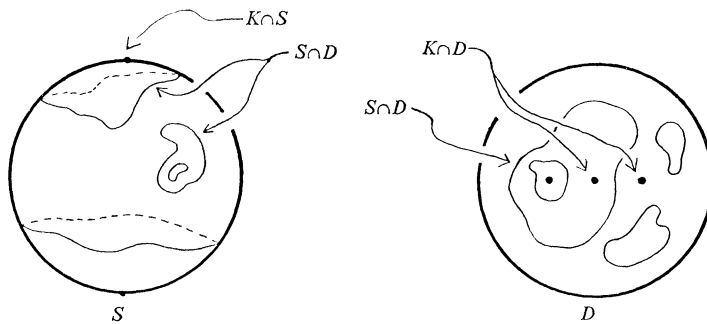


FIGURE 14

If any circle of intersection is null homologous on A , take an innermost such circle, S^1 . If S^1 is null homologous on $D - K$, we can eliminate that circle of intersection using an isotopy. See Fig. 15. If S^1 is not null homologous on $D - K$, by replacing the disk it bounds on D with the one it bounds on S , we would get a

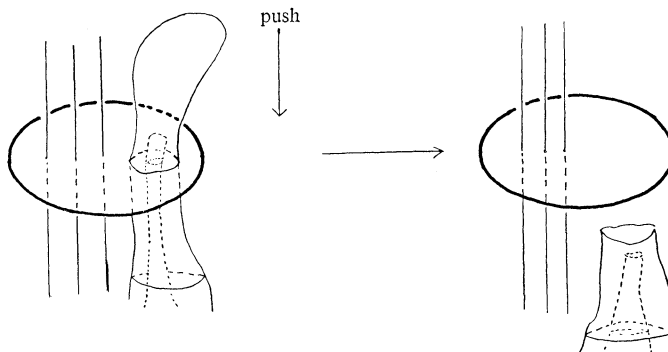


FIGURE 15

meridional disk meeting K in fewer than n points, contradicting the fact that wrapping number $K = n$. Hence, after an isotopy, all circles of intersection can be assumed to be parallel to ∂A .

Now consider an innermost circle of intersection on A , that is one closest to the boundary, S^1 . S^1 links $D \cap K$ algebraically once on D . But it also must link it geometrically once. If it linked more than once we could remove the disk on K bounded by S^1 with the one it bounds on S . This would again contradict the wrapping number of K .

The disk bounded by S^1 on D along with the one bounded by S^1 on S form a splitting $-S^2, S'$, for K , which after a small isotopy misses D . It bounds a 3-ball, B^3 , missing D . If $(B^3, B^3 \cap K)$ is nontrivial we are done. If it is trivial we can use B to isotop S (fixing K) to eliminate at least one circle of intersection. After repeated application of this procedure we get the desired result. Fig. 16 illustrates this procedure.

REMARK. This procedure can be repeated to eliminate intersections with any disjoint, finite collection of such disks.

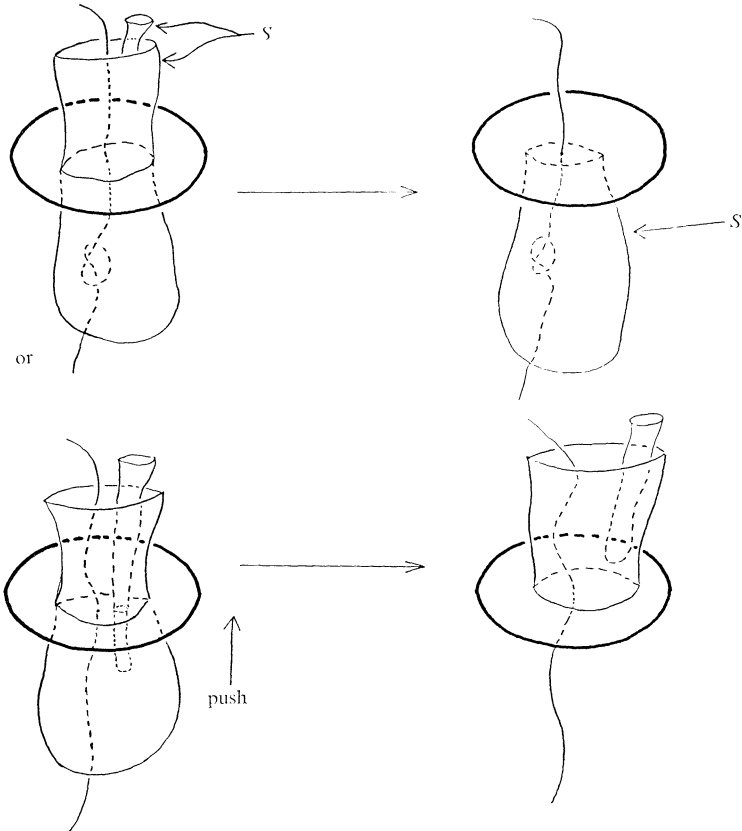


FIGURE 16

Let K denote a knot in $S^1 \times B^2$. Cutting $S^1 \times B^2$ along a meridional disk gives a B^3 with a collection of embedded arcs. Any such collection of embeddings will be called *prime* if for any splitting- S^2 , for an arc, S bounds a 3-ball, B^3 , with $(B^3, B^3 \cap \text{arc})$ trivial. See Fig. 17.

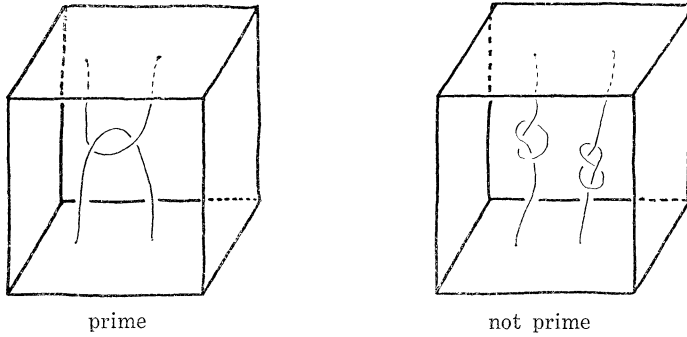


FIGURE 17

PROPOSITION 5.5. *Let K be a knot in $S^1 \times B^2$ and $\{D_i\}$ be a collection of disjoint disks with $\partial D_i = \text{meridian}$, and each D_i intersecting K in a minimum number of points. If each B^3 obtained by cutting $S^1 \times B^2$ along the D_i 's contains a prime collection of arcs then K is prime, and conversely.*

Proof. If K is not prime, by Lemma 5.4 and the remark following it, there is a splitting- S^2 , S , for K , with $S \cap D = \emptyset$ for all i , and with $S = \partial B^3$, where $(B^3, B^3 \cap K)$ is nontrivial. Cutting $S^1 \times B^2$ along the D_i 's does not effect S , B^3 , or $B^3 \cap K$, so one of the collection of arcs is not prime. The converse is immediate.

Schubert's results [6] that if cutting along a single disk yields a B^3 with each arc trivial, then the K cable about any knot is prime follows immediately. For example:

COROLLARY 5.6. *Any braid in $S^1 \times B^2$ is prime.*

COROLLARY 5.7. *Any Mazur knot in $S^1 \times B^2$ is prime. See Fig. 18.*

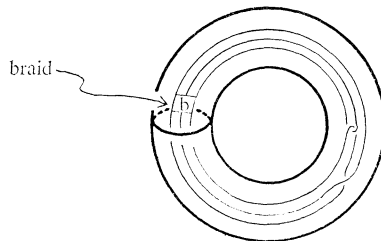


FIGURE 18

As a final example, the knot in Fig. 19 is prime. If it were not prime we would have the collection of arcs in Fig. 20a not prime, in which case those in Fig. 20b would not be prime. But they are prime as the knot in Fig. 20c is prime, which follows from Proposition 5.1.

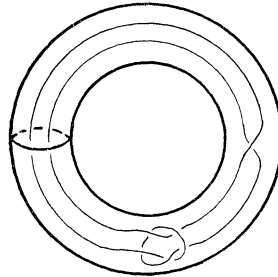
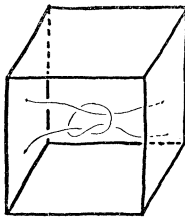
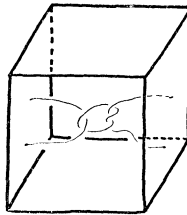


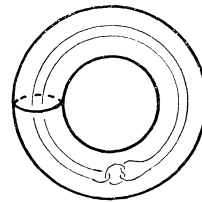
FIGURE 19



20a.



20b.



20c.

FIGURE 20

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