

## HOMOGENIZATION OF REGULAR RINGS OF BOUNDED INDEX, II

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**Recently Goodearl and Handelman (2) derived sufficient conditions for a regular ring of bounded index to be a direct product of homogeneous regular rings. In this note we show how their conditions may be modified to yield necessary and sufficient conditions. We also discuss the extension of these results to semisimple  $\pi$ -regular rings of bounded index.**

We shall follow the terminology of [3] and [4]. In particular rings are always associative but need not have a multiplicative identity. We say that a ring  $R$  has *bounded index* if there is a positive integer  $n$  such that  $a^n = 0$  for each nilpotent element  $a$  of  $R$ . The least such  $n$  is called the *index* of  $R$ .

Let  $R$  be a regular ring with index  $n$ . Kaplansky has shown in [4, Theorem 2.3] that if  $P$  is any primitive ideal of  $R$  then  $R/P$  is a simple Artinian ring of length at most  $n$ . (Here the *length* of a simple Artinian ring  $S$  is just the length of a composition series for the module  $S_S$ .) This led Kaplansky to consider how  $R$  may be constructed from certain 'homogeneous' parts. He says that a regular ring is *homogeneous* if all its primitive factor rings are simple Artinian of the same length. A homogeneous ring with an identity element is just a full matrix ring over a strongly regular ring [4, Theorem 4.2]. Furthermore  $R$  can be built up from homogeneous rings by means of a finite number of ring extensions: for each  $k$  let  $I_k$  denote the intersection of all primitive ideals  $P$  of  $R$  for which  $R/P$  has length at most  $k$ ; then  $I_n = 0$  and each  $I_{k-1}/I_k$  is homogeneous (see [4, § 4]). However  $R$  need not be a direct product of these homogeneous rings (see [2, page 64] and [4, page 70] for examples). We shall be considering the complementary picture of  $R$  which may be obtained by using another family of ideals of  $R$ . For each  $k$  let  $J_k$  be the intersection of those primitive ideals  $P$  of  $R$  for which  $R/P$  has length *at least*  $k$ . The semisimplicity of  $R$  now ensures that  $J_1 = 0$  and [4, Theorem 2.3] ensures that  $J_{n+1} = R$ . This time however the rings  $J_{k+1}/J_k$  need not be homogeneous (the examples in [2, page 64] and [4, page 70] show this). Indeed we shall prove (in Theorem 2) that  $R$  is a direct product of homogeneous rings if and only if each  $J_{k+1}/J_k$  is homogeneous. We then use some results of Goodearl and Handelman [2] to relax this condition on  $J_{k+1}/J_k$ . For example if  $R$  is also biregular it turns out that the length of

a primitive factor ring of  $J_{k+1}/J_k$  always divides  $k$  anyway. Thus in this case we just need to assume in Theorem 2 that the length of a primitive factor ring of  $J_{k+1}/J_k$  is never a proper divisor of  $k$  (Theorem 4). A similar, but more complicated, criterion can be used if  $R$  has an identity (Theorem 3). The sufficiency results in [2] are easy consequences of these theorems.

We begin with a result (due to Kaplansky) which relates the primitive factor rings of  $J_{k+1}/J_k$  to those of  $R$ .

**LEMMA 1.** *Suppose  $R$  is a regular ring of bounded index. Let  $I$  be an ideal of  $R$  and  $Q$  an ideal of  $I$ . Then  $Q$  is a primitive ideal of  $I$  if and only if there is a primitive ideal  $P$  of  $R$  such that  $I \not\subseteq P$  and  $Q = I \cap P$ . Furthermore when this is the case the rings  $I/Q$  and  $R/P$  are isomorphic.*

*Proof.* The first statement is proved in [3, page 206] and [4, Theorem 3.1]. For the second statement it is enough to notice that  $I/Q$  and  $R/P$  are simple Artinian rings (by [4, Theorem 2.3]) and that  $I/Q$  is isomorphic to the ideal  $I + P/P$  of  $R/P$ .

This lemma implies that  $J_k$  is the same whether it was defined as an ideal of  $R$  or of  $J_m$  where  $m > k$ . More precisely, if we write  $\text{Spec } S$  for the set of primitive ideals of a ring  $S$ , then for each  $m > k$  we have

$$\begin{aligned} J_k &= \bigcap \{P \in \text{Spec } R: R/P \text{ has length } \geq k\} \\ &= \bigcap \{P \in \text{Spec } R: R/P \text{ has length } \geq m\} \\ &\quad \bigcap \bigcap \{P \in \text{Spec } R: R/P \text{ has length } \geq k\} \\ &= J_m \cap \bigcap \{P \in \text{Spec } R: J_m \not\subseteq P \text{ and } R/P \text{ has length } \geq k\} \\ &= \bigcap \{J_m \cap P: P \in \text{Spec } R, J_m \not\subseteq P \text{ and } R/P \text{ has length } \geq k\} \\ &= \bigcap \{Q \in \text{Spec } J_m: J_m/Q \text{ has length } \geq k\} \end{aligned}$$

as claimed.

**THEOREM 2.** *Let  $R$  be a regular ring of index  $n$  and for each  $k$  write*

$$J_k = \bigcap \{P \in \text{Spec } R: R/P \text{ has length } \geq k\} .$$

*Then  $R$  is a direct product of homogeneous rings if and only if each of the rings  $J_{k+1}/J_k$  is homogeneous.*

*Proof.* The necessity of this condition is straightforward to check. We prove the sufficiency by induction on  $n$ . If  $n = 1$  then  $R$  is already homogeneous [4, page 67] so suppose the result is true

for rings of index at most  $n - 1$  where  $n > 1$ .

By [4, Theorem 2.3] we have  $J_{n+1} = R$  and so our hypotheses imply that  $R/J_n$  is homogeneous. Since  $R/J_n$  has a primitive factor ring of length  $n$  (as may be seen by choosing  $P \in \text{Spec } R$  such that  $a^{n-1} \notin P$ , where  $a \in R$  with  $a^n = 0$  and  $a^{n-1} \neq 0$ ), it follows that every primitive factor ring of  $R/J_n$  has length  $n$ . Let

$$J = \cap \{P \in \text{Spec } R: R/P \text{ has length } < n\}$$

so that  $J \cap J_n = \cap \text{Spec } R = 0$  since  $R$  is semisimple. Suppose that  $J + J_n \neq R$ . Since  $R$  is regular so is  $R/(J + J_n)$  and so there is some primitive ideal  $P$  of  $R$  such that  $J + J_n \subseteq P$ . But then  $R/P$  is a primitive factor ring of both  $R/J_n$  and  $R/J$ . The former implies that  $R/P$  has length  $n$ , but the latter (since  $R/J$  has index at most  $n - 1$ ) implies that  $R/P$  has length at most  $n - 1$  (by [4, Theorem 2.3]). This contradiction shows that  $J + J_n = R$ .

Now  $J \cong R/J_n$  is homogeneous. Furthermore  $J_n \cong R/J$  has index at most  $n - 1$  and the remark after Lemma 1 thus shows that we can apply the induction hypothesis to  $J_n$ . Hence  $R$  is a direct product of homogeneous rings.

REMARKS. (1) Since Kaplansky's original paper actually dealt with semisimple  $\pi$ -regular rings, rather than regular rings, it is probably worthwhile to indicate what scope there is for generalizing Theorem 2 in this direction. In fact Theorem 2 remains true if  $R$  is a semisimple  $\pi$ -regular ring none of whose factor rings is a radical ring (the crucial point in the above proof being the existence of a primitive ideal  $P$  containing  $J + J_n$ ; see [3, Proposition 2, page 208]). Thus, for instance, Theorem 2 is still true if  $R$  is a semisimple  $\pi$ -regular ring having an identity element. The following example, however, shows that Theorem 2 is not true for arbitrary semisimple  $\pi$ -regular rings. Choose a field  $F$  and let  $S = M_2(F)$  and  $T = M_3(F)$ , the  $2 \times 2$  and  $3 \times 3$  matrix rings over  $F$  (respectively). Let  $R$  be the ring of sequences of elements of  $S \times T$  which are eventually constant and of the form

$$\left[ \begin{matrix} 0 & a \\ 0 & 0 \end{matrix} \right], \quad \left[ \begin{matrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right] \text{ where } a \in F.$$

Then  $R$  is a semisimple  $\pi$ -regular ring of index 3 and each  $J_{k+1}/J_k$  is homogeneous (see [4, Remark 2, page 69] for a similar example). However, any ideal of  $R$  containing an element of the above form with  $a \neq 0$  has primitive factor rings of length 2 and of length 3.

Hence  $R$  cannot be a direct product of homogeneous rings.

(2) Goodearl and Handelman prove many of their results for regular rings whose primitive factor rings are all Artinian rather than just those of bounded index. However Theorem 2 does not extend to this generality even if  $R$  has an identity (consider, for instance, the subalgebra of  $\prod_n M_n(F)$  generated by  $\bigoplus_n M_n(F)$  and the identity, where  $F$  is a field). It should be noted though that if a regular ring whose primitive factor rings are all Artinian is a direct product of homogeneous rings it is a finite such direct product and so has finite index anyway (see [1, Theorem 6.2 and the remark following it]).

When the ring in Theorem 2 has an identity we can relax the conditions on  $J_{k+1}/J_k$  somewhat.

**THEOREM 3.** *Let  $R$  be a regular ring with identity and suppose that  $R$  has bounded index. The following conditions are equivalent.*

- (a)  $R$  is a direct product of homogeneous rings.
- (b) For each  $k$ , if  $J_{k+1}/J_k$  has primitive factor rings of lengths  $k_1, k_2, \dots, k_t$  such that  $m_1 k_1 + \dots + m_t k_t = k$  for suitable positive integers  $m_1, m_2, \dots, m_t$  then  $t = 1$  and  $k_1 = k$ .

*Proof.* (a)  $\Rightarrow$  (b) is clear. Since a direct application of Theorem 2 does not seem to be possible here, we once more use induction on  $n$ , the index of  $R$ , to show that (b)  $\Rightarrow$  (a). We have  $J_{n+1} = R$  and

$$\bigcap \{P \in \text{Spec}(R/J_n) : (R/J_n)/P \text{ has length } n\} = 0.$$

Hence [2, Lemma 4] and our assumption (b) imply that  $R/J_n$  is homogeneous. The argument in Theorem 2 now shows that  $R = J \oplus J_n$  where  $J$  is homogeneous and  $J_n$  has index at most  $n - 1$ . Since  $J_n$  now has an identity we can use induction to finish the proof as in Theorem 2.

**COROLLARY** [2, Corollary 7]. *Let  $R$  be a regular ring with identity and suppose that  $R$  has bounded index. Let  $\mathcal{L}$  be the set of lengths of primitive factor rings of  $R$ . Assume that if  $m_1 k_1 + \dots + m_t k_t = k$  for some  $k, k_1, \dots, k_t \in \mathcal{L}$  and some positive integers  $m_1, \dots, m_t$  then  $t = 1$  and  $k_1 = k$ . Then  $R$  is a finite direct product of full matrix rings over strongly regular rings.*

*Proof.* This now follows immediately from Theorem 3, Lemma 1 and [4, Theorem 4.2].

**REMARKS.** (1) Theorem 3 is false if  $R$  does not have an iden-

tity. To see this choose a field  $F$  and let  $R$  be the ring of all sequences in  $M_3(F)$  which are eventually constant and of the form

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(2) Theorem 3 is, however, still true if we assume that  $R$ , instead of being regular, is merely semisimple and  $\pi$ -regular. This may be seen by considering “pseudo-rank functions” on a  $\pi$ -regular ring (these functions should be defined as for regular rings [1, Chapter 16] except that the additional property “ $N(x + y) \leq N(x) + N(y)$  for all  $x, y$ ” should be included). Suitable versions of Lemmas 2, 3 and 4 of [2] then follow after only minor modifications of their proofs. The details may safely be left to the reader.

Finally Theorem 2 can be improved even further if we already know that  $R$  is biregular.

**THEOREM 4.** *Let  $R$  be a regular biregular ring of bounded index. Then  $R$  is a direct product of homogeneous rings if and only if, for each  $k$ , no primitive factor ring of  $J_{k+1}/J_k$  has length  $m$  where  $m|k$  and  $m \neq k$ .*

*Proof.* Suppose no primitive factor ring of  $J_{k+1}/J_k$  has length  $m$  where  $m|k$  and  $m \neq k$ . As  $R$  is regular and biregular so is  $J_{k+1}/J_k$ . Using Lemma 1 and the remark following it, it is not hard to see that

$$\cap \{P \in \text{Spec}(J_{k+1}/J_k) : (J_{k+1}/J_k)/P \text{ has length } k\} = 0.$$

By [2, Lemma 9] (if suitable care is taken, the proof in [2] is still valid even if  $R$  does not have an identity), it follows that every primitive factor ring of  $J_{k+1}/J_k$  has length dividing  $k$ . By our hypothesis  $J_{k+1}/J_k$  must be homogeneous and so the result follows from Theorem 2.

**COROLLARY [2, Corollary 12].** *Let  $R$  be a regular biregular ring of bounded index and let  $\mathcal{L}$  be the set of lengths of primitive factor rings of  $R$ . If  $s \nmid t$  for all distinct  $s, t \in \mathcal{L}$  then  $R$  is a direct product of homogeneous rings.*

*Proof.* This follows immediately from Theorem 4 and Lemma 1.

**REMARK.** In this instance there is nothing to be gained by replacing “regular” by “ $\pi$ -regular”. Indeed by [4, Theorem 4.4] any

$\pi$ -regular biregular ring of bounded index is regular anyway.

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