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CARTAN SUBALGEBRAS OF A LIE ALGEBRA AND ITS IDEALS II

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Cartan subalgebras H of a Lie algebra L and Cartan subalgebras $\hat{H}$ of its p-closure $\bar{L}$ are related. This is used to prove that $I_0(\mathrm{ad}\, H \cup I)$ is a Cartan subalgebra of I if $p=0$ or $(\mathrm{ad}_I I)^p \subset \mathrm{ad}_I I$, by reduction to the known case $(\mathrm{ad}\, L)^p \subset \mathrm{ad}\, L$.

In Winter [3], the following theorem is proved about a Lie algebra L with Cartan subalgebra H over a field of characteristic $p \geqq 0$.

THEOREM 1. *Let I be an ideal of L. Then $I_0(\mathrm{ad}(H \cap I))$ is a Cartan subalgebra of I if either $p = 0$ or $(\mathrm{ad}\, L)^p \subset \mathrm{ad}\, L$ and $(\mathrm{ad}_I I)^p \subset \mathrm{ad}_I I$.*

The purpose of this note is to relate Cartan subalgebras of L and those of the p-closure $\bar{L}$ of L, in Theorem 2 below, and use this to show in Theorem 3 that the hypothesis $(\mathrm{ad}\, L)^p \subset \mathrm{ad}\, L$ in Theorem 1 can be dropped. This result is used in Winter [5].

We refer the reader to Jacobson [1] for preliminaries on Lie p-algebras (restricted Lie algebras) for $p > 0$.

THEOREM 2. *Let L be a subalgebra of a Lie p-algebra M and let H be a Cartan subalgebra of L. Let $\bar{L}$ be the p-closure $\bar{L} = \sum_{p=0}^{\infty} L^{p^e}$ of L in M where L^{p^e} is the span of $\{x^{p^e} \mid x \in L\}$. Then*

(1) *every ideal I of L is an ideal of $\bar{L}$ and $[\bar{L}, \bar{I}] \subset I$;*

(2) *$\bar{L} = \hat{H} + L$ for any Cartan subalgebra $\hat{H}$ of $\bar{L}$;*

(3) *for any Cartan subalgebra H of L, $\bar{L}$ has a Cartan subalgebra $\hat{H}$ such that $[\hat{H}, H] \subset H$ and $\hat{H} \cap L \subset H$.*

Proof. Statements (1) and (2) are proved in Winter [2], §7.1. For (3), note that $T = \bar{H}^{p^\infty} = \bigcap_{e=0}^{\infty} \bar{H}^{p^e}$ is a torus and $L_0(\mathrm{ad}\, T) = H$, as proved in Winter [4]. Letting $\hat{T}$ be a maximal torus of $\bar{L}$ containing T, and letting $\hat{H} = \bar{L}_0(\mathrm{ad}\, \hat{T})$, $\hat{H}$ is a Cartan subalgebra of $\bar{L}$, by Winter [4]. Since $T \subset \hat{T}$, we have $\hat{H} = \bar{L}_0(\mathrm{ad}\, \hat{T}) \subset \bar{L}_0(\mathrm{ad}\, T)$. Thus, $\hat{H}$ normalizes $\bar{L}_0(\mathrm{ad}\, T) \cap L = H$ in the sense that $[\hat{H}, H] \subset H$; and $\hat{H} \cap L \subset \bar{L}_0(\mathrm{ad}\, T) \cap L = H$. □

THEOREM 3. *Let I be an ideal of L and suppose that either $p=0$ or $(\mathrm{ad}_I I)^p \subset \mathrm{ad}_I I$. Then $H_I = I_0(\mathrm{ad}(H \cap I))$ is a Cartan subalgebra of I.*

Proof. If, furthermore, $(\operatorname{ad} L)^p \subset \operatorname{ad} L$, this is Theorem 1. In order to bypass this additional assumption, let M be a Lie p-algebra containing L as subalgebra. Then, by Theorem 2, there is a Cartan subalgebra $\hat{H}$ of $\bar{L}$ such that $[\hat{H}, H] \subset H$ and $\hat{H} \cap L \subset H$, and $\bar{L} = \hat{H} + L$. By the Theorem 1, $\hat{H}_I = I_0(\operatorname{ad}(\hat{H} \cap I))$ is a Cartan subalgebra of I. But $\hat{H}_I = I_0(\operatorname{ad}(\hat{H} \cap L \cap I)) \supset I_0(\operatorname{ad}(H \cap I)) = H_I$ since $\hat{H} \cap L \subset H$. Thus, $H_I \subset \hat{H}_I$ and H_I is nilpotent. Since $H_I = I_0(\operatorname{ad}(H \cap I))$, H_I is also selfnormalizing in I and is therefore a Cartan subalgebra of I; e.g., $x \in I$ and $[x, H_I] \subset H_I$ implies $x \in I_0(\operatorname{ad}(H \cap I)) = H_I$. □

Note that H_I in the above proof is also maximal nilpotent in I, so that $H_I = \hat{H}_I = I_0(\operatorname{ad}(\hat{H} \cap I))$.

We can now consolidate and supplement some of our conclusions as follows.

THEOREM 4. *Let L be a subalgebra of a Lie p-algebra M, let H be a Cartan subalgebra of L and choose (using Theorem 2) a Cartan subalgebra $\hat{H}$ of $\bar{L}$ such that $[\hat{H}, H] \subset H$. Then*

(1) *$\bar{L} = \hat{H} + L$ and $\hat{H} \cap L \subset H$;*

(2) *if I is an ideal of L and $p = 0$ or $(\operatorname{ad}_I I)^p \subset \operatorname{ad}_I I$, then $I_0(\operatorname{ad} H \cap I)$, $I_0(\operatorname{ad} \hat{H} \cap I)$ are equal and are Cartan subalgebras of I;*

(3) *if $p = 0$ or $(\operatorname{ad}_L L)^p \subset \operatorname{ad}_L L$, then $H = L_0(\operatorname{ad} \hat{H} \cap L)$.*

Proof. For (1), note that $\bar{L} = \hat{H} + L$ by Theorem 2 and $\hat{H} \subset \bar{L}_0(\operatorname{ad} H)$ since $[\hat{H}, H] \subset H$, so that $\hat{H} \cap L \subset \bar{L}_0(\operatorname{ad} H) \cap L = L_0(\operatorname{ad} H) = H$. And (2) follows from the observation following Theorem 3. Finally, (3) follows from (2), taking $L = I$. □

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