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SMALL DOWKER SPACES

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We construct a normal, locally compact, first countable, separable, real compact topological space which is not countabley paracompact. This construction is performed under the (relatively consistent) set-theoretic hypotheses: Martin's Axiom plus $\diamondsuit_c(E)$.

Years ago, C. Dowker proved the now well-known theorem that for any Hausdorff topological space X, topological product of X with the closed unit interval is normal iff X is both normal and countably paracompact [3]. Since then researchers have called Hausdorff spaces which are normal but *not* countably paracompact *Dowker spaces.*

Dowker spaces seem to be extremely rare and difficult to construct. In fact, in the literature there is only one which is constructed using just the usual ZFC axioms of set theory [8]. It was discovered by M. E. Rudin who then asked [10] if there were any "small" Dowker spaces, i.e., ones which are, for example, first countable, separable, realcompact, or of small cardinality.

Examples have been constructed, using extra set-theoretic axioms. M. E. Rudin [9] used the existence of a Suslin line to obtain a Dowker space which is hereditarily separable and first countable. In [5] I. Juhász, K. Kunen and M. E. Rudin construct, using CH, a first countable, hereditarily separable, realcompact Dowker space and claim that with the stronger assumption $\diamondsuit$ they could construct one which is locally compact as well.

It was unknown if Martin's axiom plus not CH allowed the construction of small Dowker spaces since Martin's axiom plus not CH often implies topological results contrary to CH or $\diamondsuit$. In this paper we construct a Dowker space using axioms of set theory consistent with Martin's axiom. This Dowker space is locally compact, first countable, separable and realcompact. By the results of [11] we cannot hope to prove that this Dowker space is hereditarily separable.

Recently, M. Bell has constructed a first countable realcompact Dowker space, assuming Martin's axiom. In fact, only a weakened form of Martin's axiom, called MA (σ-centered) is used. We use it repeatedly in our construction.

MA (*σ-centered*). If $\boldsymbol{P}$ is a partial order such that $\boldsymbol{P} = \bigcup_{n\in\omega} \boldsymbol{P}_n$ where each $\boldsymbol{P}_n$ is centered and $\kappa < 2^{\aleph_0}$ and $\{\boldsymbol{D}_\alpha\colon \alpha < \kappa\}$ is a collection of dense subsets of $\boldsymbol{P}$, then $\boldsymbol{P}$ contains a $\{\boldsymbol{D}_\alpha\colon \alpha < \kappa\}$-generic subset.

Recall that a subset S of a partial order $\boldsymbol{P}$ is centered iff each finite subset of S has a lower bound. A subset D of $\boldsymbol{P}$ is called dense iff for each $p \in \boldsymbol{P}$ there is some element of D less than or equal to p. If $\{D_\alpha: \alpha < \kappa\}$ is a collection of dense of subset of $\boldsymbol{P}$, a subset G of $\boldsymbol{P}$ is called $\{D_\alpha: \alpha < \kappa\}$-generic iff $D_\alpha \cap G \neq \emptyset$ for each $\alpha < \kappa$.

Another useful axiom is variation of $\diamondsuit$, called $\diamondsuit_{\omega_2}(E)$. We let $E = \{\alpha \in \omega_2: \mathrm{cf}(\alpha) = \omega\}$.

$\diamondsuit_{\omega_2}(E)$. There is a sequence $\{X_\alpha: \alpha \in E\}$ such that each $X_\alpha \subseteq \alpha$ and for any $X \subseteq \omega_2$ $\{\alpha \in E: X = X \cap \alpha\}$ is stationary in ω_2.

PROPOSITION 1. *If* ZFC *is consistent, so is* ZFC *plus* MA (σ-*centered*) *plus* $2^{\aleph_0} = \aleph_2$ *plus* $\diamondsuit_{\omega_2}(E)$.

Proof. This seems to be folklore. The model needed can be the "usual" model of Martin's axiom plus $2^{\aleph_0} = \aleph_2$ where the iterated forcing is done over a ground model of $V = L$. A modification of the proof of Theorem 7 of Chapter 18 in [2] shows that this model is a model of $\diamondsuit_{\omega_2}(E)$. □

The role of $\aleph_2$ in the collection of axioms in Proposition 1 might be played, in this article, by any regular uncountable cardinal, but for concreteness we choose to assume $2^{\aleph_0} = \aleph_2$. These assumptions are used by Hajnal and Juhász for a topological theorem of a completely different nature in [4].

PROPOSITION 2. $\diamondsuit_{\omega_2}(E)$ *implies there exists a sequence of ordered pairs* $\{\langle S_\alpha, T_\alpha\rangle: \alpha \in E\}$ *such that for all* $\alpha \in E$ $S_\alpha \subseteq \alpha$ *and* $T_\alpha \subseteq \alpha$ *and for any two subsets* S *and* T *of* ω_2, $\{\alpha \in E:$ *both* $S_\alpha = S \cap \alpha$ *and* $T_\alpha = T \cap \alpha\}$ *is stationary in* ω_2.

Proof. This too is folklore. Briefly, let $f: \omega_2 \to \omega_2 \times \omega_2$ be any bijection. Let $S_\alpha = \{$first coordinates of elements of $f''X_\alpha\}$ and $T_\alpha = \{$second coordinates of elements of $f''X_\alpha\}$. Then for any two subsets S and T of $\omega_2 \{\alpha \in E: X_\alpha = f^{-1}(S \times T) \cap \alpha\} \cap \{\alpha \in \omega_2: f''\alpha = \alpha \times \alpha\}$ is contained in $\{\alpha \in E: S_\alpha \times T_\alpha = (S \cap \alpha) \times (T \cap \alpha)\}$ which must therefore be stationary. □

We denote the real numbers by $\boldsymbol{R}$ and the rationals by $\boldsymbol{Q}$. In [5], Lusin sets are used to construct a Dowker space. Here, we use:

PROPOSITION 3. MA (σ-*centered*) *plus* $2^{\aleph_0} = \aleph_2$ *implies there is an* $\boldsymbol{L} \subseteq \boldsymbol{R}$ *such that*

(i) $\boldsymbol{L} \cap \boldsymbol{Q} = \emptyset$,

(ii) *for any open* $U \subseteq \boldsymbol{R}$, $|\boldsymbol{L} \cap U| = \aleph_2$,
(iii) *for any nowhere dense* $N \subseteq \boldsymbol{R} |\boldsymbol{L} \cap N| < \aleph_2$.

Proof. Folklore again. Note that the hypotheses imply that $\boldsymbol{R}$ is not the union of $\leqq \aleph_1$ nowhere dense sets and then make a simple modification to the usal inductive construction of a Lusin set. □

We now do the promised construction. We will eventually define a topology τ on a subset X of $\boldsymbol{L} \cup \boldsymbol{Q}$, where $\boldsymbol{L}$ is as given in Proposition 3. Fix $\leqq$ as a well ordering of $\boldsymbol{L}$. Let $\{A_\alpha: \alpha \in \omega_2 \backslash E\}$ enumerate $\{A \subseteq \boldsymbol{L}: |A| < \aleph_2\}$ with each A being repeated $\aleph_2$ times. In addition, let $\{B_\alpha: \alpha \in \omega_2 \backslash E\}$ enumerate all countable sequences of subsets of $\boldsymbol{Q}$. Finally, let $\{\langle S_\alpha, T_\alpha \rangle: \alpha \in E\}$ be as in Proposition 2.

From now on we shall assume ZFC plus MA (σ-centered) plus $2^{\aleph_0} = \aleph_2$ plus $\diamondsuit_{\omega_2}(E)$.

We shall define the topology τ on $X \subseteq \boldsymbol{L} \cup \boldsymbol{Q}$ by recursively defining a sequence of points $\{x_\alpha: \alpha \in \omega_2\} \subseteq \boldsymbol{L}$ and topologies τ_β on $X_\beta = \{x_\alpha: \alpha \in \beta\} \cup \boldsymbol{Q}$, for all $\beta \in \omega_2$. Simultaneously we define a collection $\{C_\alpha: \alpha \in \omega_2 \backslash E\}$ of subsets of $\boldsymbol{L} \cup \boldsymbol{Q}$ and a function $\rho: X \to \omega$ recursively.

The inductive hypothesis is that for all $\alpha < \beta$

(i) $\tau_\alpha = \tau_\beta \cap \mathscr{P}(X_\alpha)$,
(ii) τ_β refines the Euclidean topology on X_β,
(iii) $\boldsymbol{Q}$ is dense in $\langle X_\beta, \tau_\beta \rangle$,
(iv) τ_β is a locally compact, locally countable topology,
(v) each C_α is a clopen subset of $\langle X_\beta, \tau_\beta \rangle$,
(vi) $\rho(x_\alpha) = n$ implies there exists $U \in \tau_{\alpha+1}$ such that for all $y \in U$ $\rho(y) \leqq n$.
(vii) for all finite $F \subseteq \beta$, $\boldsymbol{Q} \backslash \bigcup_{\alpha \in F} C_\alpha$ is dense in the Euclidean topology on $\boldsymbol{R}$.

We shall make the construction so that each subset of $\boldsymbol{L}$ of cardinality $< \aleph_2$ is contained in some C_α.

We begin the construction by letting $X_0 = \boldsymbol{Q}$ and τ_0 the discrete topology on $\boldsymbol{Q}$.

At limit ordinals λ, including $\lambda = \omega_2$, we let τ_λ be the topology on $X_\lambda = \bigcup \{X_\beta: \beta \in \lambda\}$ generated by $\bigcup \{\tau_\beta: \beta \in \lambda\}$. The inductive hypothesis is easily seen to be satisfied. All the action takes place at successor stages and there are two cases to consider: first, when $\beta = \alpha + 1$ for some $\alpha \in E$ and second, when $\beta = \alpha + 1$ for some $\alpha \notin E$.

At stage $\alpha + 1$ when $\alpha \in E$ we consider $\langle S_\alpha, T_\alpha \rangle$ and an increasing sequence $\{\alpha_n: n \in \omega\}$ converging to α.

Case (a). There is $x \in \boldsymbol{L} \backslash X_\alpha$, there is a sequence $\{s_n: n \in \omega\} \subseteq S_\alpha$

and a sequence $\{t_n: n \in \omega\} \subseteq T_\alpha$ such that:

(i) each $s_n \in X_\alpha \backslash X_{\alpha_n}$ and each $t_n \in X_\alpha \backslash X_{\alpha_n}$,

(ii) there exists $l \in \omega$ such that $\{\rho(s_n): n \in \omega\} \subseteq l$ and $\{\rho(t_n): n \in \omega\} \subseteq l$,

(iii) in the Euclidean topology on $\boldsymbol{R}$ $\{S_n\} \to x$ and $\{t_n\} \to x$.

In this case, let $x_\alpha = x$. We need a lemma.

LEMMA 1. *There exist open $U_n \in \tau_\alpha$ containing s_n such that for all $\xi < \alpha\{n \in \omega: C_\xi \cap U_n \neq \varnothing\}$ is finite.*

Proof. Since $\langle X_\alpha, \tau_\alpha \rangle$ is first countable, let $V(n, m)$ be the mth basic open neighborhood of s_n.

Let $\boldsymbol{P}$ be the set of all pairs $\langle f, g \rangle$ such that

(i) f is a finite partial function from α into ω,

(ii) g is a finite partial function from ω into ω, and

(iii) for all $\xi \in \operatorname{dom} f$, for all $n \in \operatorname{dom} g$, $n > f(\xi)$ implies

$$s_n \in C_\xi \quad \text{or} \quad V(n, g(n)) \cap C_\xi = \varnothing ,$$

with the partial ordering of function extension in both coordinates.

Let $\boldsymbol{D}_n^1 = \{\langle f, g \rangle: n \in \operatorname{dom} g\}$. Since each C_ξ is closed in $\langle X_\alpha, \tau_\alpha \rangle$, each $\boldsymbol{D}_n^1$ is dense in $\boldsymbol{P}$.

Let $\boldsymbol{D}_\xi^2 = \{\langle f, g \rangle: \xi \in \operatorname{dom} f\}$. Clearly each $\boldsymbol{D}_\xi^2$ is dense in $\boldsymbol{P}$.

Now, if $\langle f_2, g \rangle$ and $\langle f_2, g \rangle$ are elements of $\boldsymbol{P}$ and f_1 and f_2 are compatible as functions, then $\langle f_1 \cup f_2, g \rangle$ is in $\boldsymbol{P}$. Hence for each $\tilde{g}$, $\{\langle f, g \rangle: g = \tilde{g}\}$ is isomorphic to the partial order of finite partial functions from α into ω ordered by set containment, and so $\boldsymbol{P}$ is σ-centered.

By MA (σ-centered), there is a $\boldsymbol{G} \subseteq \boldsymbol{P}$ generic for all $\boldsymbol{D}_n^1$ and $\boldsymbol{D}_\xi^2$. Let $F = \bigcup\{f: \text{for some } g, \langle f, g \rangle \in \boldsymbol{G}\}$ and $G = \bigcup\{g: \text{for some } f, \langle f, g \rangle \in \boldsymbol{G}\}$.

Let $U_n = V(n, G(n))$. For each $\xi \in \alpha$ there is some $n \in \omega$ such that $C_\xi \subseteq X_{\alpha_{n_\xi}}$. If $n > \max\{n_\xi, F(\xi)\}$, then $U_n \cap C_\xi = \varnothing$. □

Returning to the construction, let $\{O_s(n): n \in \omega\}$ refine $\{U_n: n \in \omega\}$ of Lemma 1 such that

(i) $s_n \in O_s(n) \subseteq U_n$, $\rho'' O_s(n) \subseteq l$,

(ii) $O_s(n)$ is countable, compact and open in $\langle X_\alpha, \tau_\alpha \rangle$,

(iii) the Euclidean diameter $\operatorname{diam} O_s(n) < 1/n$ for each n.

Similarly obtain $\{O_t(n): n \in \omega\}$ with each $t_n \in O_t(n)$. Let $V_m = \{x_\alpha\} \cup \bigcup\{O_s(n): n \geqq m\} \cup \bigcup\{O_t(n): n \geqq m\}$. Let $\tau_\alpha \cup \{V_m: m \in \omega\}$ generate $\tau_{\alpha+1}$. Let $\rho(x_\alpha) = l$. A routine check shows that the inductive hypothesis is still satisfied.

Case (b). If the conditions for Case (a) are not satisfied, we waste time with the help of this lemma.

LEMMA 2. *There is a* $Q^* \subseteqq Q$ *which is dense in the Euclidean topology and for all* $\xi < \alpha$, $(Q^* \cap C_\xi)$ *is finite.*

Proof. Enumerate Q as $\{q_n: n \in \omega\}$. Let P be the set of all ordered pairs $\langle M, f\rangle$ such that

(i) M is a finite subset of Q,

(ii) f is a finite partial function from α into ω,

(iii) if $\xi \in \operatorname{dom} f$ then $C_\xi \cap M \subseteqq \{q_n: n < f(\xi)\}$ with the partial order given by extension of each coordinate.

Let $D_\xi = \{\langle M, f\rangle: \xi \in \operatorname{dom} f\}$. Clearly D_ξ is dense in P for each $\xi < \alpha$.

Let, for each pair of rationals p, r with $p < r$, $D_{p,r} = \{\langle M, f\rangle: M \cap (p, r) \neq \emptyset\}$, where (p, r) is the open interval. Condition (vii) of the inductive hypothesis ensures that each $D_{p,r}$ is dense.

For each $\tilde{M}$, $\{\langle M, f\rangle: M = \tilde{M}\}$ is isomorphic to the partial order of finite partial functions from α into ω ordered by set containment, and hence P is σ-centered.

Hence by MA (σ-centered) there is a $G \subseteqq P$ which is generic with respect to the above-mentioned dense sets. Let $Q^* = \bigcup\{M$: there is some f such that $\langle M, f\rangle \in G\}$, and let $F = \bigcup\{f$: there is some M such that $\langle M, f\rangle \in G\}$. Then Q^* is dense and for each $\xi < \alpha$, $(Q^* \cap C_\xi) \subseteqq \{q_n: n < F(\xi)\}$. □

We can now pick any $x \in L\backslash X_\alpha$ and find $\{p_n: n \in \omega\} \subseteqq Q^*$ from Lemma 2 such that $\{p_n\} \to x$ in the Euclidean topology. Let $x_\alpha = x$ and let $V_m = \{x_\alpha\} \cup \{p_n: n \geqq m\}$. Let $\tau_{\alpha+1}$ be generated by $\tau_\alpha \cup \{V_m: m \in \omega\}$ and let $\rho(x_\alpha) = 0$ to satisfy the inductive hypothesis in this case.

We can now proceed to stage $\alpha + 1$ where $\alpha \notin E$. We must do two things: define x_α and $\tau_{\alpha+1}$ and define C_α.

Embarking on the first task, let's consider $B_\alpha = \langle B_\alpha^0, B_\alpha^1, B_\alpha^2, \cdots\rangle$. Using Lemma 2 obtain $Q^* \subseteqq Q$, dense, such that $C_\xi \cap Q^*$ is finite for each $\xi < \alpha$. There are now two cases to consider.

Case (a). Each $x \in L\backslash X_\alpha$ is in the Euclidean closure of only finitely many of the sets $B_\alpha^j \cap Q^*$, $j \in \omega$. In this case let x_α be any element of $L\backslash X_\alpha$. Pick $\{p_n: n \in \omega\} \subseteqq Q^*$ such that $\{p_n\} \to x_\alpha$ in the Euclidean topology. Let $\tau_\alpha \cup \{\{x_\alpha\} \cup \{p_n: n \supseteqq m\}: m \in \omega\}$ generate $\tau_{\alpha+1}$. Let $\rho(x_\alpha) = 1 + \max\{j: x_\alpha$ is in the Euclidean closure of $B_\alpha^j \cap Q^*\}$. Note that the inductive hypothesis is satisfied and that for all $j \supseteqq \rho(x_\alpha)$, $x_\alpha \notin \tau_{\alpha+1}$-closure of B_α^j.

Case (b). There is some $x \in L\backslash X_\beta$ such that x is in the Euclidean closure of infinitely many of the sets $B^j_\alpha \cap \boldsymbol{Q}^*$. In this case pick x_α to be some such x. Pick an infinite $J \subseteq \omega$ such that for each $j \in J$, x_α is in the Euclidean closure of $B^j_\alpha \cap \boldsymbol{Q}^*$. For each $j \in J$ pick $\{p^j_n : n \in \omega\} \subseteq B^j_\alpha \cap \boldsymbol{Q}^*$ such that $\{p^j_n\} \to x_\alpha$ in the Euclidean topology. Let $\{r_k : k \in \omega\}$ be a subsequence of $\{p^j_n : n \in \omega, j \in J\}$ such that for each $j \in J$ each tail of $\{r_k\}$ contains an infinite subset of $\{p^j_n : n \in \omega\}$ and such that $\{r_k\} \to x_\alpha$ in the Euclidean topology. Let $\tau_\alpha \cup \{\{x_\alpha\} \cup \{r_k : k \geqq m\} : m \in \omega\}$ generate $\tau_{\alpha+1}$. Let $\rho(x_\alpha) = 0$. Note that the inductive hypothesis is satisfied and that x_α is in the $\tau_{\alpha+1}$-closure of infinitely many E^j_α.

We now proceed to define C_α. Consider A_α. If $A_\alpha \not\subseteq X_{\alpha+1}$, let $C_\alpha = \varnothing$. If $A_\alpha \subseteq X_{\alpha+1}$, then C_α is given by the following lemma.

LEMMA 3. *Suppose $A_\alpha \subseteq (X_{\alpha+1} \cap \boldsymbol{L})$ and $\{C_\xi : \xi \in \alpha \backslash E\}$ is a collection of clopen subsets of $X_{\alpha+1}$ such that for each finite $F \subseteq (\alpha \backslash E)$, $(Q \backslash \bigcup \{C_\xi : \xi \in F\})$ is dense in the Euclidean topology. Then there exists a clopen $C_\alpha \subseteq X_{\alpha+1}$ such that $A_\alpha \subseteq C_\alpha$ and for each finite $F \subseteq ((\alpha + 1) \backslash E)$, $(Q \backslash \bigcup \{C_\xi : \xi \in F\})$ is dense in the Euclidean topology.*

Proof. From Lemma 1 obtain $\boldsymbol{Q}^*$ such that for each $\xi \in (\alpha \backslash E)$, $(\boldsymbol{Q}^* \cap C_\xi)$ finite, and $\boldsymbol{Q}^*$ is dense in the Euclidean topology. It now suffices to construct a clopen C_α such that $A_\alpha \subseteq C_\alpha \subseteq X_{\alpha+1}$ and $(\boldsymbol{Q}^* \backslash C_\alpha)$ is dense.

To this end let $\boldsymbol{P}$ be the set of all triples $\langle a, b, c \rangle$ such that:

(i) a and b are clopen, compact subsets of $X_{\alpha+1}$,

(ii) c is a finite subset of $\boldsymbol{Q}^*$,

(iii) $(a \cup A) \cap b = \varnothing$,

(iv) $a \cap c = \varnothing$

with the partial order of set containment in each coordinate.

$\{\langle a, b, c \rangle : x \in a \cup b\}$ is dense for each $x \in X_{\alpha+1}$. For each interval (p, q) with rational endpoints, $\langle a, b, c \rangle : (p, q) \cap c \neq \varnothing\}$ is dense.

For each finite union I of interval with rational endpoints and each finite $\tilde{c} \subseteq \boldsymbol{Q}^*$, $\{\langle a, b, c \rangle : a \subseteq I, (b \cap I) = \varnothing$ and $c = \tilde{c}\}$ is centered; hence $\boldsymbol{P}$ is σ-centered.

Let $\boldsymbol{G} \subseteq \boldsymbol{P}$ be generic with respect to the above dense sets. Let $C_\alpha = \bigcup \{a :$ for some b and c, $\langle a, b, c \rangle \in \boldsymbol{G}\}$. It is straightforward to show that C_α is clopen, $A_\alpha \subseteq C_\alpha$ and $(\boldsymbol{Q}^* \backslash C_\alpha)$ is dense. □

This completes the construction. We let $X = X_{\omega_2}$ and $\tau = \tau_{\omega_2}$. Clearly τ is a locally compact, T_2, first countable, separable and submetrizable topology on X. Since $\langle X, \tau \rangle$ is first countable and submetrizable, it is real-compact [5] also. It remains to show that $\langle X, \tau \rangle$ is a Dowker space.

We show that $\langle X, \tau \rangle$ is not countably compact by showing that the open cover $\{O_j : j \in \omega\}$ has no precise locally finite refinement, where $O_j = \{x \in X: \rho(x) \leqq j\}$. Due to (vi) of the inductive hypothesis $\{O_j: j \in \omega\}$ is an open cover. In order to achieve a contradiction assume $\{W_j: j \in \omega\}$ is a locally finite precise refinement. For some $\alpha \in (\omega_2 \backslash E)$, $\langle (W_0 \cap \boldsymbol{Q}), (W_1 \cap \boldsymbol{Q}), (W_2 \cap \boldsymbol{Q}), \cdots \rangle = B_\alpha$ and we look at what happened at stage $\alpha + 1$. Under Case (b) x_α witnesses that $\{W_j: j \in \omega\}$ is not locally finite. Under Case (a) x_α witnesses that $\{W_j: j \in \omega\}$ is not a cover, since if $j < \rho(x_\alpha)$ then $x_\alpha \notin O_j$ and if $j \geqq \rho(x_\alpha)$ then $x_\alpha \notin \overline{W_j \cap \boldsymbol{Q}} \supseteqq W_j$.

In order to show that $\langle X, \tau \rangle$ is normal, we need a short lemma.

LEMMA 4. *If H and K are closed disjoint subsets of X, one of which has cardinality $< \aleph_2$, then there are disjoint open sets separating H and K.*

Proof. Assume $|H| < \aleph_2$ so that there is some clopen $C \subseteqq X$ such that $|C| < \aleph_2$ and $H \subseteqq C$. Let $K^* = K \cap C$. By MA (σ-centered), since X is locally compact and submetrizable there exist disjoint open U and V such that $H \subseteqq U$ and $K^* \subseteqq V$ [1, 6]. Then $H \subseteqq U \cap C$ and $K \subseteqq U \cup (X \backslash C)$ separating H and K. □

Now let H and K be arbitrary closed disjoint subsets of X. Without loss of generality we assume $H \cup K \subseteqq \boldsymbol{L}$ and we find disjoint open sets separating H and K. Let $H_i = \{x \in H: \rho(x) \leqq i\}$ and $K_i = \{x \in K: \rho(x) \leqq i\}$. Define U_i^H to the compliment in $\boldsymbol{R}$ of the Euclidean closure of $\bigcup \{V: V$ is Euclidean open and $|V \cap H_i| < \aleph_2\}$. Note that for each nonempty Euclidean open $V \subseteqq U_i^H |V \cap H_i| = \aleph_2$. Similarly define U_i^K for each $i \in \omega$.

LEMMA 5. *With U_i^H and U_i^K defined as above for each $i \in \omega$ we have $\bigcup_{i \in \omega} U_i^K \cap \bigcup_{i \in \omega} U_i^H = \emptyset$.*

Proof. Suppose $U_i^H \cap U_j^K \neq \emptyset$. Then there is some interval $I \subseteqq U_i^H \cap U_j^K$. Let $\hat{H} = H_i \cap I$ and $\hat{K} = K_j \cap I$. Let $Y_H = \{\alpha \in \omega_2$: for all $\beta < \alpha \hat{H} \cap [\beta, \alpha)$ is dense in $I\}$ and $Y_K = \{\alpha \in \omega_2$: for all $\beta < \alpha \hat{K} \subset [\beta, \alpha)$ is dense in $I\}$. Y_H and Y_K are both cub subsets of ω_2. So there exists, by $\diamondsuit_{\omega_2}(E)$, some $\alpha \in E \cap Y_H \cap Y_K$ such that $\hat{H} \cap \alpha = S_\alpha$ and $\hat{K} \cap \alpha = T_\alpha$.

Let's see what happened at stage $\alpha + 1$. We had $\{\alpha_n: n \in \omega\}$ increasing up to α. Since $\alpha \in Y_H \cap Y_K$ we can find $x \in \boldsymbol{L} \backslash X_\alpha$ and sequences $\{s_n: n \in \omega\}$ and $\{t_n: n \in \omega\}$ such that the conditions for Case (a) hold. Hence $x_\alpha \in \bar{S}_\alpha \cap \bar{T}_\alpha \subseteqq \bar{H} \cap \bar{K}$ which is a contradiction. □

Now, since $\boldsymbol{L}$ has the properties from Proposition 3, both $H \backslash \bigcup_{i \in \omega} U_i^H$ and $K \backslash \bigcup_{i \in \omega} U_k^K$ have cardinality less than $\aleph_2$. Hence by Lemma 4 there are open V_H and V_K such that:

$$(H\backslash \bigcup_{i\in\omega} U_i^H) \subseteq V_H \subseteq \bar{V}_H \subseteq (X\backslash K)$$

and

$$(K\backslash \bigcup_{i\in\omega} U_i^K) \subseteq V_K \subseteq \bar{V}_K \subseteq (X\backslash H) .$$

Hence

$$H \subseteq (V_H\backslash \bar{V}_K) \cup \bigcup_{i\in\omega} U_i^H$$

and

$$K \subseteq (V_K\backslash \bar{V}_H) \cup \bigcup_{i\in\omega} U_i^K$$

which separates H and K. The proof is complete.

We could do the above constructions and proofs with $\aleph_2$ replaced everywhere by $\aleph_1$ thus reducing the set theoretic assumptions to ZFC plus $\diamondsuit$. A Dowker space with properties similar to X was claimed in [5] to follow from $\diamondsuit$. The additional property of hereditary separability mentioned in [5] can be obtained by weaving in the idea of constructing hereditarily separable spaces from $\diamondsuit$ in [7] to the above construction.

The space $\langle X, \tau\rangle$ above is normal and each bounded or closed unbounded (referring to the indices) subspace is also normal. However, there is no reason to believe that $\langle X, \tau\rangle$ is hereditarily normal.

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