

COMPLEMENTS OF CODIMENSION-TWO
SUBMANIFOLDS—
III—COBORDISM THEORY

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Introduction. This paper will study the relationship between the middle-dimensional complementary homology modules of a codimension-two imbedding of compact manifolds and its cobordism class. We will also obtain general results on the cobordism classification of codimension-two imbeddings that extend those of Kervaire ([14]) and Levine ([15]) on knots, Cappell and Shaneson (see [7]) on local knots, parametrized knots and knotted lens spaces as well as results of Ocken (in [19]) and Stoltzfus (in [37]). We also provide an algebraic formulation of general results of Cappell and Shaneson (in [7] and [8]) involving Poincaré imbeddings and the codimension-two splitting problem.

A major tool used in this paper is a type of surgery theory (developed in Chapter I of this paper and called dual surgery theory) that is dual to the homology surgery theory of Cappell and Shaneson in Chapter I of [7] in the following sense: whereas homology surgery theory measures the obstruction of a degree-1 normal map being normally cobordant to a simple homology equivalence, *dual surgery theory starts* with such a map and measures the obstruction to its being homology s -cobordant to a *simple homotopy equivalence*. This paper shows that dual surgery theory provides an algebraic formulation of the problem that solved in Theorem 3.3 of [7] geometrically. This theory is applied to show that, for many classes of codimension-two imbeddings- the set of cobordism classes of imbeddings has a *natural group structure*—in fact it is shown to be canonically isomorphic to a certain subgroup of a dual surgery obstruction group.

The formulation of dual surgery theory in the present paper makes use of Ranicki's algebraic theory of surgery (see [25], [26], and [27]) so that we obtain an "instant dual surgery obstruction" whose computation doesn't require preliminary surgeries below the middle dimension. In many interesting cases the dual surgery obstruction is shown to be expressible as a linking form on a torsion module that directly generalizes the *Blanchfield pairing* (see [4] and [17]) in knot theory. This is applied in Chapter II of this paper to describe the manner in which the cobordism theory and the complementary homology of codimension-two imbeddings interact.

This paper studies condimension-two imbeddings of compact

manifolds from the point of view of *Poincaré imbeddings*, as described in [40], [8] and [34]. Our results apply equally to the smooth, PL or topological categories though in the PL and topological cases we assume that all imbeddings are *locally-flat*.

In Chapter II this theory is applied to study codimension-two imbeddings. It is shown that, for many interesting classes of codimension-two imbeddings, the set of *cobordism classes* has a natural group structure similar to that of knot cobordism classes. The resulting cobordism groups are shown to be isomorphic to certain subgroups of the dual surgery obstruction groups—the map being defined by geometrically associating a dual surgery obstruction to an imbedding. Results are obtained that completely characterize the middle-dimensional complementary homology of certain types of codimension-two imbeddings analogous to *simple knots*.

CHAPTER I—*Dual Surgery Theory*

1. **Introduction.** In this chapter we will develop a form of surgery theory that appears to be particularly suited to the study of codimension-two imbeddings. Its name, dual surgery theory, comes from the fact that it is *dual* to the *homology surgery theory* developed by Cappell and Shaneson in [7] in the following sense: whereas homology surgery theory measures the obstruction to a surgery problem being normally cobordant to a homology equivalence, dual surgery theory *starts* with a surgery problem that is a homology equivalence and measures the obstruction to its being *homology s-cobordant* to a *homotopy equivalence*. Dual surgery theory turns out to be closely related to homology surgery theory—in fact, we show that the dual surgery obstruction groups are *canonically isomorphic* to suitable *relative homology surgery obstruction* groups.

We will make extensive use of Ranicki's algebraic reformulation of surgery theory. Since the definitive work on this subject has not yet been published, we will outline the main ideas and results from [25] that we use.

A version of dual surgery theory (under the name “torsion surgery theory”), in the odd-dimensional case, appeared in my doctoral dissertation [32]. Direct geometric arguments were used there, along the lines of [38].

The following conventions will be in effect throughout this chapter:

1. π denotes a finitely presented group and $w: \pi \rightarrow \mathbb{Z}_2 = \{\pm 1\}$ is a homomorphism. The integral group-ring, $\mathbb{Z}\pi$, will be denoted by Λ and have the conjugation defined by the formula $\bar{g} = w(g)g^{-1}$ for $g \in \pi$;

2. $\mathfrak{F}: A \rightarrow A'$ will denote a local epimorphism of rings in the sense of Cappell and Shaneson in [7], i.e., for every finite set $\lambda_1, \dots, \lambda_k \in A'$, there exists a unit u of A' such that $\lambda_1 u, \dots, \lambda_k u$ is contained in $\mathfrak{F}(A)$;

3. $Wh(\mathfrak{F})$ has exactly the same meaning it had in [7], i.e., it is $K_1(A')/\mathfrak{F}(\pm\pi)$;

4. $(f, b): (M^m, \partial M) \rightarrow (X, Y)$ denotes a dual surgery problem with respect to \mathfrak{F} , i.e., f is a degree-1 normal map being normal map such that $f|_{\partial M}$ is a simple homotopy equivalence and f itself is a simple homology equivalence with respect to local coefficients in A' . Here (X, Y) is an m -dimensional finite simple Poincaré pair in the sense of Chapter 2 of [40] with $\pi_1(X) = \pi$ and with orientation character w .

2. A summary of algebraic surgery. Surgery theory studies the obstruction to a degree-1 normally cobordant to a simple homotopy equivalence by performing surgery on it until it is homotopically connected up to the middle dimension. The algebraic mapping cone of the resulting map has a quadratic structure on it in the middle dimension defined by intersection and self-intersection numbers which consists of either: (a) a quadratic form, in the even-dimensional case or; (b) an automorphism of a quadratic form in the odd-dimensional case—see [13] as a general reference. These quadratic structures are known (see [40], Chapters 5 and 6) to measure the obstruction to performing further surgeries on the normal map to make it a simple homotopy equivalence. One drawback of this theory is that, before the surgery obstruction can be measured one must perform preliminary surgeries to make the map connected up to the middle dimension.

Ranicki has eliminated that difficulty by defining a quadratic structure on the algebraic mapping cone of the *original normal map* that captures the surgery obstruction. In this section we will outline the results of Ranicki's paper, [25], that will be used in the remainder of this paper, and supplement his formulation of *homology surgery theory*¹.

Unless stated otherwise all chain complexes will be assumed to be *finite dimensional, finitely generated, free, and based*. Let ε be an integer equal to $+$ or -1 and *fixed* throughout this chapter.

DEFINITION 2.1. Let $C = (C_*, c_*)$ and $D = (D_*, d_*)$ be n -dimensional chain complexes. Then:

1. $C^* = (C^*, c^*)$ denotes the corresponding dual complex with

¹ Highly detailed expositions of this work have recently appeared—see [26] and [27].

$C^i = \text{Hom}_A(C_i, A)$ (equipped with the dual basis) and dual boundary maps;

2. If $f: C \rightarrow D$ is a chain map the algebraic mapping cone, $C(f)$, is defined as on p. 22 of [40], i.e., $C_n(f) = C_{n-1} \oplus D_n$,

$$c_f = \begin{pmatrix} d & 0 \\ (-1)^{n-1}f & d \end{pmatrix}: C_n(f) \longrightarrow C_{n-1}(f);$$

3. $C^t = (C^t, c^t)$ denotes the chain complex with the same chain modules, bases, and boundary maps as C but with A acting on the right according to the formula $c\lambda = \bar{\lambda}c \in C_i^t, \lambda \in A$;

4. The slant chain map $\backslash: C^t \otimes_A D \rightarrow \text{Hom}_1(C^{n-*}, D); x \otimes y \mapsto (f \mapsto f(x)y)$, where $(C^{n-*})_i = C^{n-i}$, is a simple isomorphism (i.e., it is an isomorphism that preserves bases);

5. Let the generator, T , of Z_2 act on $C^t \otimes_A C$ via the ε -transposition involution: if $x \in C_p^t$ and $y \in C_q$, then $T_i(x \otimes y) = (-1)^{pq}y \otimes \varepsilon x$. The Q -group, $Q_n(C, \varepsilon)$ is defined to be the hyperhomology group (see [10], Chapter XVII) $H_n(Z_2, C^t \otimes_A C)$. An element $\psi \in Q_n(C, \varepsilon)$ will be called a quadratic structure on C , and is represented by a collection of chains $\psi_s \in (C^t \otimes_A C)_{n-s}$ such that:

$$d_{(C^t \otimes_A C)} \psi_s + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1}T_s \psi_{s+1}) = 0$$

and a pair (C, ψ) with $\psi \in Q_n(C, \varepsilon)$ will be called a quadratic complex. Such a quadratic complex will be said to be Poincaré if $(1 + T_i)\psi_0 \in H_n(C^t \otimes C)$ determines a simple homotopy equivalence of chain complexes via the slant product:

$$\backslash: C^r \otimes_Z (C^t \otimes_A C)_n \longrightarrow C_{n-r}; f \otimes (x \otimes y) \longmapsto \overline{f(x)y}. \quad \square$$

REMARKS. 1. In spite of the quadratic nature of their construction, the Q -groups are shown to be homotopy functors of chain complexes in [25]—i.e., a chain homotopy equivalence of chain complexes induces an isomorphism of their Q -groups. If $f: C \rightarrow C'$ is a chain map between complexes we will use the notation $f_\%$ to denote the induced map $Q_n(C, \varepsilon) \rightarrow Q_n(C', \varepsilon)$.

2. Henceforth, unless stated otherwise, chain maps between quadratic complexes will be assumed to preserve the quadratic structure, i.e., if $f: (C, \psi) \rightarrow (C', \psi')$ is a chain map we will assume that $f_\%(\psi) = \psi'$.

3. Ranicki shows (in [25]) that quadratic complexes are direct generalizations of the quadratic forms and formations that appear in [23]. In fact he proves that:

(Proposition 1.5 of [25]): *The homotopy classes of 0-dimensional*

quadratic (Poincaré) complexes over A are in a natural 1-1 correspondence with the isomorphism classes of (nonsingular) quadratic forms over A . Poincaré complexes correspond to nonsingular formations. □

(Proposition 1.8 of [25]): *The homotopy classes of connected 1-dimensional quadratic complexes are in a natural 1-1 correspondence with the stable isomorphism classes of split quadratic formations over A . Poincaré complexes correspond to nonsingular formations.* □

This correspondence maps the quadratic complex (C, ψ) , where C is $0 \rightarrow C_1 \xrightarrow{d} C_0 \rightarrow 0$, to the formation (the notation here follows [25]):

$$(C_1, C^0) = \left(C_1, \left(\begin{pmatrix} \bar{\varepsilon}\psi_0 + \psi_0^* \\ d^* \end{pmatrix}, -(\psi_1 + d\psi_0) \right) C^0 \right)$$

—here a *split* formation is like an element of the *split* unitary group $\widetilde{SU}(A)$ defined in [31]—see § 5 of the present paper for more details.

DEFINITION 2.2. If $C = (C, \psi)$, $C' = (C', \psi')$ are quadratic n -dimensional complexes the direct sum, $C \oplus C'$, is defined to be $(C \oplus C', \psi \oplus \psi')$ where $\psi \oplus \psi' \in \mathcal{Q}_n(C \oplus C', \varepsilon)$. □

If C and C' are Poincaré, $C \oplus C'$ will be Poincaré.

DEFINITION 2.3. 1. Let $f: C \rightarrow D$ be a chain map from an n -dimensional chain complex to an $n + 1$ -dimensional chain complex and define the relative \mathcal{Q} -group, $\mathcal{Q}_{n+1}(f, \varepsilon)$ to be $H_{n+1}(\mathbb{Z}_2, C(f^t \otimes_A f))$, where $C(f^t \otimes_A f)$ is the algebraic mapping cone (see 2.1, part 2) of the $\mathbb{Z}[\mathbb{Z}_2]$ -chain map $f^t \otimes_A f: C^t \otimes_A C \rightarrow D^t \otimes_A D$ (\mathbb{Z}_2 acts on $C(f^t \otimes_A f)$ via T_ε). An element $(\delta\psi, \psi) \in \mathcal{Q}_{n+1}(f, \varepsilon)$ is represented by a collection of chains $(\delta\psi, \psi)_s \in (D^t \otimes_A D)_{n-s+1} \oplus (C^t \otimes_A C)_{n-s}$ such that:

$$\begin{aligned} d_{(D^t \otimes_A D)}(\delta\psi_s) + (-1)^{n-s}(\delta\psi_{s+1} + (-1)^{s+1}T_\varepsilon\delta\psi_{s+1}) + (-1)^n(f^t \otimes_A f)(\psi_s) &= 0 \\ d_{(C^t \otimes_A C)}(\psi_s) + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1}T_\varepsilon\psi_{s+1}) &= 0. \end{aligned}$$

2. A pair $(f, (\delta\psi, \psi))$, where f is a chain map as above and $(\delta\psi, \psi) \in \mathcal{Q}_{n+1}(f, \varepsilon)$ will be called a *quadratic pair* and it will be called *Poincaré* when the relative class $((1 + T_\varepsilon)\delta\psi_0, (1 + T_\varepsilon)\psi_0) \in H_{n+1}(f^t \otimes_A f)$ induces a simple homotopy equivalence via the slant product:

$$\begin{aligned} \backslash: C(f)^r \otimes_{\mathbb{Z}} (C(f^t \otimes_{\mathbb{Z}} f))_{n+1} &\longrightarrow D_{n+1-r}; \\ (g, h) \otimes (u \otimes v, x \otimes y) &\longmapsto \overline{g(u)}v + \overline{h(x)}f(y), \\ (g, h) \in D^t \otimes C^{r-1}, u \otimes v \in (D^t \otimes D)_{n+1}, x \otimes y \in (C^t \otimes C)_n. \end{aligned}$$

If $(f: C \rightarrow D, (\delta\psi, \psi) \in \mathcal{Q}_{n+1}(f, \varepsilon))$ is an $n + 1$ -dimensional quadratic pair, the n -dimensional quadratic complex, (C, ψ) , will be called its *boundary*;

3. If $C=(C, \psi)$ and $C'=(C', \psi')$ are two n -dimensional Poincaré complexes and $(C \oplus C', \psi \oplus -\psi')$ is the boundary of an $n+1$ -dimensional Poincaré pair, C and C' will be said to be *cobordant*. \square

Ranicki has defined several standard constructions, involving quadratic complexes and pairs, that directly correspond to *geometrical* constructions. We will list them, and describe their properties:

DEFINITION 2.4. *The mapping cylinder construction:* Let $f: (C, \psi) \rightarrow (C', \psi')$ be a chain map between n -dimensional quadratic complexes. Then $(C \oplus C', \psi \oplus -\psi')$ is the boundary of the $n + 1$ -dimensional quadratic pair:

$$M(f) = (f \oplus 1: C \oplus C' \longrightarrow C', (0, \psi \oplus -\psi') \in \mathcal{Q}_{n+1}(f \oplus 1, \varepsilon)). \quad \square$$

REMARK. Remember that chain maps are implicitly assumed to preserve quadratic structures. It is not difficult to see that if C and C' are *Poincaré* and f is a *homotopy equivalence*, then $M(f)$ will also be *Poincaré* so that *homotopy-equivalent* quadratic complexes are *cobordant*.

DEFINITION 2.5. *The union construction:* Let

$$x_1 = (f_C \oplus f_{C'}: C \oplus C' \longrightarrow D, (\delta\psi, \psi \oplus -\psi'))$$

and

$$x_2 = (f'_C \oplus f_{C''}: C' \oplus C'' \longrightarrow D', (\delta\psi', \psi' \oplus -\psi''))$$

be quadratic pairs. Then their *union*, denoted $x_1 \cup x_2$, is defined to be the quadratic pair $(f_C \oplus f_{C''}: C \oplus C'' \rightarrow D'', (\delta\psi'', \psi \oplus \psi''))$ given by:

$$\begin{aligned} d_{D''} &= \begin{pmatrix} d_D & (-1)^{r-1}f_{C'} & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-1)^{r-1}f'_C & d_{D'} \end{pmatrix} \\ : D_r'' = D_r \oplus C'_{r-1} \oplus D'_r &\longrightarrow D''_{r-1} = D_{r-1} \oplus C'_{r-2} \oplus D'_{r-1}; \\ f''_C &= \begin{pmatrix} f_C \\ 0 \\ 0 \end{pmatrix}: C_r \longrightarrow D_r'' = D_r \oplus C'_{r-1} \oplus D'_r; \end{aligned}$$

$$f''_{C'} = \begin{pmatrix} 0 \\ 0 \\ f'_{C''} \end{pmatrix}; C''_r \longrightarrow D''_r = D_r \oplus C'_{r-1} \oplus D'_r;$$

$$\delta\psi''_s = \begin{pmatrix} \delta\psi_s & 0 & 0 \\ (-1)^{n-r}\psi'_s f'^*_{C'} & (-1)^{n-r-s}T_\varepsilon\psi'_{s+1} & 0 \\ 0 & (-1)^s f'_{C'}\psi'_s & \delta\psi'_s \end{pmatrix}$$

$$: D''^{n-r-s+1} = D^{n-r-s+1} \oplus C'^{n-r-s} \oplus D'^{n-r-s+1} \longrightarrow D''_r = D_r \oplus C'_{r-1} \oplus D'_r. \quad \square$$

REMARKS. 1. This is a direct algebraic analogue of the operation of taking the union of cobordisms that have a common boundary—see Proposition 6.7 of [25].

2. At this point it is possible to prove that the relationship of cobordism between Poincaré complexes is an equivalence relation. The mapping cylinder construction implies *reflexivity* and *symmetry* and the union construction implies *transitivity*. The mapping cylinder construction also implies that the set of equivalence classes, equipped with the direct sum operation, forms a *group*. Ranicki proves that this group is isomorphic, in dimension n , to the *Wall group*, $L_n^s(\pi, w)$, where $\varepsilon = (-1)^{\lfloor n/2 \rfloor}$.

Now we will discuss the connection between the algebraic constructions above and the geometric problem of surgery.

DEFINITION 2.6. Let $(f, b): (M^m, \nu_M) \rightarrow (X, \nu_X)$ be a surgery problem, i.e., M is a compact manifold, X is an m -dimensional (geometric) Poincaré complex, f is a degree-1 map and b is a stable isomorphism of the stable normal bundles ν_M and ν_X of M and X , covering f . If C_f is the algebraic mapping cone of f the *quadratic signature*, $q_f = (C_f, \psi_b \in Q_n(C_f, \varepsilon))$, of the map f is defined as follows:

Let $F: \sum^p \tilde{X}_+ \rightarrow \sum^p \tilde{M}_+$ (p a large integer) be the stable map inducing the Umkehr homomorphism on chain complexes defined by Spanier-Whitehead duality applied to $T(b): T(\nu_M) \rightarrow T(\nu_X)$, where \tilde{M}_+ and \tilde{X}_+ are the universal covering spaces of M and X , respectively, with basepoints adjoined. The adjoint, $\bar{F}: X_+ \rightarrow \Omega^\infty \Sigma^\infty M_+$ maps the fundamental class, $[X]$, of X to an element, $F[X]$, which lies in the direct summand, $H_n(S^\infty \times \tilde{M} \times \tilde{M}/Z_2)$ (here Z_2 acts on S^∞ via the antipodal map and on $M \times M$ via transposition). This last group is isomorphic to $Q_n(C(M), \varepsilon)$ ($C(M)$ is the chain complex of M) and ψ_b is defined to be the image of $F[X]$ under the inclusion $Q_n(C(M), \varepsilon) \rightarrow Q_n(C_f, \varepsilon)$. □

REMARKS. 1. The main Theorem (9.1) of Ranicki's paper, [25],

proves that the quadratic signature of a degree-1 normal map coincides with its surgery obstruction under the identification of cobordism classes of Poincaré complexes with elements of $L_n^s(\pi)$. Also see [27] for a more detailed exposition.

The quadratic signature, therefore, constitutes an “instant” surgery obstruction in that no preliminary surgeries have to be performed below the middle dimension in order for it to be defined.

2. Ranicki defines the maps F and \bar{F} on the chain level (Proposition 2.5 in [25]), using an equivariant form of Spanier-Whitehead duality (see § 3 of [25]).

3. Ranicki’s quadratic signature is essentially a generalization of Browder’s definition of the Kervaire invariant in [5].

Two important properties of the quadratic signature of a degree-1 normal map are:

2.7. (PROPOSITION 6.6 of [25])—*Cobordism invariance:* If $(f_i, b_i): M_i^m \rightarrow X$, $i = 1, 2$ are degree-1 normal maps and $(F, B): (W; M_1, M_2) \rightarrow (X \times I; X \times 0, X \times 1)$ is a normal cobordism, then the quadratic signature of (F, B) is a cobordism between the quadratic signatures of the (f_i, b_i) . \square

REMARKS. Strictly speaking, it is necessary to use a relative version of the construction in 2.5 to get the quadratic signature of (F, B) —see § 6 in [25].

2.8. (PROPOSITION 4.4 of [25])—*Compatibility:* Let $(f, b): M^m \rightarrow X$ be a degree-1 normal map that is $[m/2]$ -connected. Then the quadratic signature of (f, b) coincides with the usual surgery obstruction as defined by Wall in Chapters 5 and 6 of [40], under the identification of forms with 0-dimensional and formations with 1-dimensional quadratic complexes described in Remark 3 following 2.1 or Propositions 1.5 and 1.8 of [25]. \square

We will conclude this section by giving an algebraic description of the homology surgery theory of Cappell and Shaneson (see [7]) somewhat more detailed than that in [25].

Recall the geometric problem that homology surgery theory studies. Let

$$(f, b): (M^m, \partial M) \longrightarrow (X, Y)$$

be a degree-1 normal map from a manifold with boundary to a Poincaré pair such that $f|_{\partial M}$ is a simple homology equivalence with local coefficients in A' (with respect to \mathfrak{S}). Then the problem is to characterize the obstruction to f being normally cobordant to

a map that induces a simple A' -homology equivalence.

We will begin by describing the obstruction groups.

DEFINITION 2.9. A quadratic complex whose tensor product with A' is Poincaré will be called an \mathfrak{F} -Poincaré complex. Two n -dimensional \mathfrak{F} -Poincaré complexes, $(C_1, \psi_1), (C_2, \psi_2)$ will be said to be \mathfrak{F} -cobordant if $(C_1 \oplus C_2, \psi_1 \oplus -\psi_2)$ is the boundary of an $n + 1$ -dimensional quadratic pair whose tensor product with A' is Poincaré.

PROPOSITION 2.10. The set of \mathfrak{F} -cobordism classes of n -dimensional \mathfrak{F} -Poincaré complexes, equipped with the direct sum operation, forms an abelian group denoted by $\Gamma_n^r(\mathfrak{F})$. □

At this point we can proceed exactly as in ordinary surgery—i.e., we can define the homology surgery obstruction to be the quadratic signature of the map f . The only problem is that, since $f|\partial M$ is not, in general, a homotopy equivalence, the algebraic mapping cone of f will be a pair of complexes and we will have to use the relative quadratic signature (defined in § 6 of [25]) to get a surgery obstruction that is a Poincaré pair, i.e., $(C(\partial f) \rightarrow C(f), (\partial\psi, \psi))$. In order to handle this additional structural element, we need three more algebraic constructions due to Ranicki (see [25]):

DEFINITION 2.11.—The boundary construction: Let $C=(C, \psi)$ be an n -dimensional quadratic complex. Then the boundary, $\partial C = (\partial C, \partial\psi)$, is defined to be the $n - 1$ -dimensional quadratic complex given by:

$$\begin{aligned}
 d_{\partial C} &= \begin{pmatrix} d_C & (-1)^r(1 + T_\varepsilon)\psi_0 \\ \mathbf{0} & (-1)^r d_C^* \end{pmatrix} \\
 &: \partial C_r = C_{r+1} \oplus C^{n-r} \longrightarrow \partial C_{r-1} = C_r \oplus C^{n-r+1} \\
 \partial\psi_0 &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}: \partial C^{n-r-1} = C^{n-r} \oplus C_{r+1} \longrightarrow \partial C_r = C_{r+1} \oplus C^{n-r} \\
 \partial\psi_s &= \begin{pmatrix} (-1)^{n-r-s} T_\varepsilon \psi_{s-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\
 &: \partial C^{n-r-s-1} = C^{n-r-s} \oplus C_{r+s+1} \longrightarrow \partial C_r = C_{r+1} \oplus C^{n-r}. \quad \square
 \end{aligned}$$

REMARKS. 1. This construction is clearly natural with respect to maps of quadratic complexes. Proposition 5.4, the results of § 7, and Proposition 8.3 from [25] in conjunction with the results of handlebody theory together imply that if (C, ψ) is the quadratic chain complex of $M/\partial M$, where M is a compact manifold with

boundary ∂M , then ∂C is homotopy-equivalent to the Poincaré chain complex of ∂M .

2. In the proof of Proposition 5.4 of [25], Ranicki shows that the quadratic pair $\mathfrak{P}(C) = (0 \oplus 1: \partial C_r = C_{r+1} \oplus C^{n-r} \rightarrow (C^{n-*})_r, (0, \partial\psi_r))$ is *always Poincaré* (regardless of (C, ψ)). This is an algebraic analogue of the well-known geometric fact that any finite CW-complex is homotopy-equivalent to a suitable manifold with boundary (i.e., imbed the CW-complex in a Euclidean space and take a regular neighborhood).

3. In Proposition 5.4 of [25], Ranicki shows that

2.12. *The quadratic complex, C , is Poincaré if and only if ∂C is contractible.*

DEFINITION 2.13.—*The collapsing construction:* Let $\bar{C} = (f: C \rightarrow D, (\delta\psi, \psi))$ be an n -dimensional Poincaré pair. Then $\mathfrak{C}(\bar{C}) = (E, \psi')$ denotes the n -dimensional quadratic complex defined as follows: $E_* = C(f)$, the algebraic mapping cone (see 2.1, part 2);

$$\begin{aligned} \psi'_s &= \begin{pmatrix} \delta\psi_s & 0 \\ (-1)^{n-r-1}\psi_s f^* & (-1)^{n-r-s-1}T_\varepsilon\psi_{s+1} \end{pmatrix} \\ : E^{n-r-s} &= D^{n-r-s} \oplus C^{n-r-s-1} \longrightarrow E_r = D_r \oplus C_{r-1}. \quad \square \end{aligned}$$

REMARKS. 1. The collapsing construction is the algebraic analogue of the geometric operation of collapsing the boundary of a manifold to a point.

2. Ranicki proves (in Proposition 5.4 of [25]) that the operations we have denoted \mathfrak{C} and \mathfrak{P} define *mutually inverse bijections* between the sets of chain homotopy types of *quadratic complexes* and *Poincaré pairs* with quadratic complexes that are *Poincaré mapping* via \mathfrak{P} to *Poincaré pairs* with contractible boundaries; and in fact:

2.13. *If $x = (f: C \rightarrow D, (\delta\psi, \psi))$ is an n -dimensional Poincaré pair, then $\mathfrak{P}(\mathfrak{C}(x))$ is simply homotopy equivalent to x .*

DEFINITION 2.14—*Algebraic surgery:* Let $x = (f: C \rightarrow D, (\delta\psi, \psi))$ be an $n + 1$ -dimensional quadratic pair such that $H_0((1 + T_\varepsilon)\psi_0 f^*) = 0: D^{n-*} \rightarrow C(f)$ (i.e., x is connected). Define the n -dimensional quadratic complex (C', ψ') obtained from (C, ψ) by surgery on x to be given by:

$$\begin{aligned} d_{C'} &= \begin{pmatrix} d_C & 0 & (-1)^{n+1}(1 + T_\varepsilon)\psi_0 f^* \\ (-1)^r f & d_D & (-1)^r(1 + T_\varepsilon)\delta\psi_0 \\ 0 & 0 & (-1)^r d_D^* \end{pmatrix} \\ : C'_r &= C_r \oplus D_{r+1} \oplus D^{n-r+1} \longrightarrow C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n-r+2}; \end{aligned}$$

$$\psi'_0 = \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$: C'^{n-r} = C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \longrightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1}$$

$$\psi'_s = \begin{pmatrix} \psi_s & (-1)^{r+s} T_s \psi_{s-1} f^* & 0 \\ 0 & (-1)^{n-r-s+1} T_s \delta \psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$: C'^{n-r-s} = C^{n-r-s} \oplus D^{n-r-s+1} \oplus D_{r+s+1} \longrightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} .$$

REMARKS. 1. Algebraic surgery, as defined above, corresponds to a *sequence* of geometric surgeries—in fact Ranicki proves (Proposition 8.3 in [25]) that a geometric surgery has the effect, on the quadratic signature of a normal map, of performing an algebraic surgery with the D -complex having only a *single nonvanishing* chain module of rank 1.

2. As Ranicki points out, the boundary construction is the result of performing suitable algebraic surgeries on the *empty complex*.

The following result is an adaptation of Proposition 7.1 in [25]:

PROPOSITION 2.15. *Algebraic surgery preserves the simple homotopy type of the boundary and \mathfrak{F} -Poincaré complexes $C = (C, \psi)$, $C' = (C', \psi')$ are \mathfrak{F} -cobordant if and only if C' can be obtained from C by surgery and a simple \mathfrak{F} -homology equivalence.*

Proof. 1. The statement about the boundaries of quadratic complexes was proved by Ranicki in [25]—the remaining statement is thus a consequence of 2.11.

2. In the proof of 7.1 in [25], Ranicki showed that complexes resulting from surgery performed on other complexes are cobordant to them (if the complexes are Poincaré)—the *same argument* implies that if surgery is performed on an \mathfrak{F} -Poincaré complex the result will be \mathfrak{F} -cobordant to it. If there exists a chain map between \mathfrak{F} -Poincaré complexes inducing a simple A' -homology equivalence, the *mapping cylinder* (see 2.3) will be an \mathfrak{F} -cobordism between the complexes.

All that remains to be proved is that, if C and C' are \mathfrak{F} -cobordant, then it is possible to [perform surgery on C in such a way that the result is simply A' -homology equivalent to C' . Our argument will be similar to that used in the proof of 7.1 in [25]. Let the \mathfrak{F} -cobordism between C and C' be $(f \oplus f': C \oplus C' \rightarrow D, (\delta\psi, \psi \oplus -\psi'))$ and let (C'', ψ'') be the Poincaré complex obtained from (C, ψ) by surgery on the connected $(n + 1)$ -dimensional quadratic

pair $(g: C \rightarrow D', (\partial\psi', \psi'))$ defined by:

$$\begin{aligned}
 d_{D'} &= \begin{pmatrix} d_D & (-1)^{r-1} f' \\ 0 & d_{C'} \end{pmatrix}: D'_r = D_r \oplus C'_{r-1} \longrightarrow D'_{r-1} = D_{r-1} \oplus C'_{r-2} \\
 g &= \begin{pmatrix} f \\ 0 \end{pmatrix}: C_r \longrightarrow D'_r = D_r \oplus C'_{r-1} \\
 \partial\psi'_s &= \begin{pmatrix} \partial\psi_s & (-1)^s f \psi'_s \\ 0 & (-1)^{n-r-s+1} T_\varepsilon \psi'_{s+1} \end{pmatrix} \\
 &: D'^{n-r-s+1} = D^{n-r-s+1} \oplus C'^{n-r-s} \longrightarrow D'_r = D_r \oplus C'_{r-1}.
 \end{aligned}$$

Then the chain map

$$h = 0 \oplus 0 \oplus 1 \oplus 0 \oplus -T_\varepsilon \psi'_0: C''_r = C_r \oplus D_{r+1} \oplus C'_r \oplus D^{n-r+1} \oplus C'^{n-r} \longrightarrow C'_r$$

defines a simple A' -homology equivalence of n -dimensional \mathfrak{F} -Poincaré complexes $h: (C'', \psi'') \rightarrow (C', \psi')$. This completes the proof of the proposition. □

As one might expect:

COROLLARY 2.16. *An n -dimensional \mathfrak{F} -Poincaré complex represents the zero element of $\Gamma_n^s(\mathfrak{F})$ if and only if it is possible to perform surgery on it in such a way that the result becomes simply acyclic when the tensor product with A' is taken.* □

Now we will return to the geometric situation connected with homology surgery theory. It is not difficult to see that, if we take the *relative* quadratic signature of a homology surgery problem and perform the *collapsing construction* on its boundary, we will get an \mathfrak{F} -Poincaré complex which we can define to be the homology surgery obstruction of the normal map. As Ranicki points out in § 17 of [25], the same methods that are used to show that ordinary algebraic surgery theory corresponds to the geometric theory *also* apply to this formulation of *homology surgery theory*. See Appendix A for an analogous description of *relative* homology surgery obstructions.

Now recall the natural map $j: L_n^s(\pi) \rightarrow \Gamma_n^s(\mathfrak{F})$ defined in Chapter I of [7], which results from regarding an ordinary surgery problem as a homology surgery problem. In the context of algebraic surgery this map can be defined by taking a Poincaré complex representing an element of the Wall group and considering the corresponding element of the Γ -group it defines. We will conclude this section by proving a technical result that characterizes elements of $\Gamma_n^s(\mathfrak{F})$ that are in the image of the map j :

LEMMA 2.17. *Let $x = (C, \psi)$ represent an element of $\Gamma_n^s(\mathfrak{F})$. Then x represents an element in the image of the map j defined above if and only if $\partial x = (\partial C, \partial \psi)$ is simply homotopy equivalent to the boundary of a quadratic complex, $y = (D, \psi')$ such that $D \otimes_A A'$ is simply acyclic.*

REMARK. For an analogous result, corresponding to the usual interpretation of Γ -groups given in §1 of [25], see 5.6 in the present paper.

Necessity. Suppose that x does represent an element in the image of j . It follows that there exists a Poincaré complex (not just an \mathfrak{F} -Poincaré complex), $p = (P, \psi'')$, such that x is \mathfrak{F} -cobordant to p or $x \oplus -p = (C \oplus P, \psi \oplus -\psi'')$ is \mathfrak{F} -cobordant to 0. Corollary 2.16 then implies that it is possible to perform surgery (see 2.14) on $x \oplus -p$ to obtain a complex (D, ψ') such that $D \otimes_A A'$ is simply acyclic. The conclusion follows from the facts that the boundary of p is *simply acyclic* (since p is Poincaré—see 2.11) and surgery preserves boundaries, up to simple homotopy equivalence.

Sufficiency. Suppose that ∂x is simply homotopy equivalent to ∂y , where y is acyclic with coefficients in A' .

We can replace x and y by Poincaré pairs $\bar{x} = (f_x: C \rightarrow C^*, (0, \psi))$ and $\bar{y} = (f_y: D \rightarrow D^*, (0, \psi'))$ such that the result of collapsing C and D gives us x and y , respectively, up to *simple homotopy equivalence* (by Remark 2 following 2.12). Since ∂x is simply-homotopy equivalent to ∂y (by hypothesis) we can form the *union* of \bar{x} and \bar{y} *along* ∂x (see 2.4)—we can replace f_y by its composite with a simple homotopy equivalence between ∂x and ∂y and regard ∂x as the boundary of \bar{y} . The result, $z = \bar{x} \cup \bar{y} = (E, \psi'')$ will be given by:

$$d_E = \begin{pmatrix} d_x^* & (-1)^{r-1} f_x & 0 \\ 0 & d_{\partial x} & 0 \\ 0 & (-1)^{r-1} f_y & d_y^* \end{pmatrix}$$

$$: E_r = C^{n-r} \oplus \partial C_{r-1} \oplus D^{n-r} \longrightarrow E_{r-1} = C^{n+1-r} \oplus \partial C_{r-2} \oplus D^{n+1-r}$$

$$\psi_s'' = \begin{pmatrix} 0 & 0 & 0 \\ (-1)^{n-r} \partial \psi_s f_x^* & (-1)^{n-r-s} T_s \partial \psi_{s+1} & 0 \\ 0 & (-1)^s f_y^* \partial \psi_s & 0 \end{pmatrix}.$$

Note that z *doesn't* have a boundary—i.e., it represents an element of $\Gamma_n^s(\mathfrak{F})$ in the *image* of the map j . The result of collapsing ∂x in \bar{x} —which, as remarked before, is simply homotopy equivalent to

x —is given by:

$$\begin{aligned} \mathfrak{C}(x) &= (F, \bar{\psi}) \\ d_F &= \begin{pmatrix} d_x^* & (-1)^{r-1} f_x^* \\ 0 & d_{\partial x} \end{pmatrix}: F_r = C^{n-r} \oplus \partial C_{r-1} \longrightarrow F_{r-1} = C^{n+1-r} \oplus \partial C_{r-2} \\ \bar{\psi}_s &= \begin{pmatrix} 0 & 0 \\ (-1)^{n-r-1} \partial \psi_s f_x^* & (-1)^{n-r-s-1} T_s \partial \psi_{s+1} \end{pmatrix}. \end{aligned}$$

The map

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}: E_r = C^{n-r} \oplus \partial C_r \oplus D^{n-r} \longrightarrow F_r = C^{n-r} \oplus \partial C_{r-1}$$

is clearly a chain map preserving quadratic structures. Since its kernel is precisely D^* , which is acyclic with coefficients in A' by hypothesis, it follows that the map defined above is a simple A' -homology equivalence and z represents the *same* element of $\Gamma_n^s(\mathfrak{F})$ as $\mathfrak{C}(\bar{x})$, by 2.15. This completes the proof of the lemma. \square

3. An algebraic formulation of dual surgery. Throughout this section n will be a positive integer and ε will equal $(-1)^{\lfloor n/2 \rfloor}$.

DEFINITION 3.1. Poincaré complexes and pairs will be said to be *relatively acyclic* if their tensor products with A' (with A acting on A' via multiplication by the image under \mathfrak{F}) are simply acyclic with respect to a preferred base.

Two relatively acyclic n -dimensional Poincaré complexes will be said to be *homology s -cobordant* if they are cobordant via cobordism that is relatively acyclic.

We are now in a position to define the dual surgery obstruction groups:

PROPOSITION 3.2. *Homology s -cobordism is an equivalence relation on the set of relatively acyclic n -dimensional Poincaré complexes such that simply homotopy equivalent complexes are equivalent.*

The set of equivalence classes forms a group with respect to the direct sum operation which we will denote by $D_n^s(\mathfrak{F}, w)$ or just $D_n^s(\mathfrak{F})$.

REMARK. We will use the notation $C_1, \psi_1 \smile_s (C_2, \psi_2)$ to indicate that the Poincaré complexes (C_1, ψ_1) and (C_2, ψ_2) are homology

s-cobordant.

Proof. All of the statements are immediate consequences of the fact that the union of two homology s-cobordisms and the mapping cylinder of a simple homotopy equivalence between relatively acyclic complexes are homology s-cobordism (see 2.3 and 2.4 and the remarks following them). \square

We will need the following lemma in the proof of the main geometric result:

LEMMA 3.3. *Let (C_1, ψ_1) be an n -dimensional relatively acyclic Poincaré complex that represents the zero element of $D_n^s(\mathfrak{S})$. Then $(C_1, \psi_1)_s \simeq 0$.*

Proof. The statement that (C_1, ψ_1) represents the zero element of $D_n^s(\mathfrak{S})$ implies that there exists a relatively acyclic Poincaré complex, (C_2, ψ_2) such that $(C_1 \oplus C_2, \psi_1 \oplus \psi_2)_s \simeq (C_2, \psi_2)$. By the definition of cobordism (2.2, part 3) it follows that $(C_1 \oplus C_2 \oplus C_2, \psi_1 \oplus \psi_2 \oplus -\psi_2)_s \simeq 0$. The mapping cylinder of the identity map (see 2.3) of $(C_1 \oplus C_2, \psi_1 \oplus \psi_2)$ defines a homology s-cobordism between (C_1, ψ_1) and $(C_1 \oplus C_2 \oplus C_2, \psi_1 \oplus \psi_2 \oplus -\psi_2)$ and the lemma follows from the union construction (2.4). \square

The following corollary will be important in the algebraic analysis of the groups $D_n^s(\mathfrak{S})$:

COROLLARY 3.4. *Let $x_i = (C_i, \psi_i)$, $i = 1, 2$, be n -dimensional relatively acyclic Poincaré complexes. Then the x_i represent the same element of $D_n^s(\mathfrak{S})$ if and only if there exist $(C'_i, \psi'_i)_s \simeq 0$, $i = 1, 2$, such that $x_1 \oplus (C'_1, \psi'_1)$ is simply isomorphic to $x_2 \oplus (C'_2, \psi'_2)$.*

Proof. The statement is clearly sufficient. To prove it is necessary, suppose x_1 and x_2 represent the same element of $D_n^s(\mathfrak{S})$. Then $x_1 \oplus -x_2$ represents 0 so that $(C_1 \oplus C_2, \psi_1 \oplus -\psi_2) = (C'_2, \psi'_2)_s \simeq 0$, and $(C_1 \oplus C_2 \oplus C_2, \psi_1 \oplus -\psi_2 \oplus \psi_2) = x_1 \oplus (C_2 \oplus C_2, -\psi_2 \oplus \psi_2) = x_2 \oplus (C'_2, \psi'_2)$. \square

THEOREM 3.5. *Let $(f, b): (M^n, \partial M) \rightarrow (X, Y)$ be a dual surgery problem and let (f, b) be the quadratic signature of the normal map, f (see 2.5). Then $\sigma(f, b)$ is a relatively acyclic n -dimensional Poincaré complex and:*

1. *The element of $D_n^s(\mathfrak{S})$ represented by $\sigma(f, b)$ depends only upon the homology s-cobordism class of f ;*
2. *$\sigma(f, b)$ represents the zero element of $D_n^s(\mathfrak{S})$ if and only if*

the map f is (geometrically) homology s -cobordant to a simple homotopy equivalence.

Proof. The fact that $\sigma(f, b)$ is relatively acyclic follows from the fact that f induces a simple homology equivalence with respect to local coefficients in A' . The homology s -cobordism invariance of the element of $D_n^s(\mathfrak{F})$ defined by $\sigma(f, b)$ is an immediate consequence of the cobordism invariance of the quadratic signature—see 2.6. This also implies that $\sigma(f, b)$ will be equivalent to zero in $D_n^s(\mathfrak{F})$ if f is homology s -cobordant to a simple homotopy equivalence.

Now suppose $\sigma(f, b)$ represents the zero element of $D_n^s(\mathfrak{F})$. It follows, from 3.3 that $\sigma(f, b)$ is the boundary of a relatively acyclic $(n + 1)$ -dimensional Poincaré pair. In particular, the ordinary surgery obstruction of f vanishes, by 2.7, so that there exists a normal cobordism $F: (W; M, M') \rightarrow (X, Y)$ such that $F|M = f$ and $F|M'$ is a simple homotopy equivalence. We can regard F as an $(n + 1)$ -dimensional surgery problem (using a Morse function on W , for instance):

$$F: (W; M, M') \longrightarrow (X \times I, X \times 0, X \times 1) .$$

Consider the homology surgery obstruction, $x \in \Gamma_{n+1}^s(\mathfrak{F})$, of F (as defined in the preceding section). This will be represented by an $(n + 1)$ -dimensional \mathfrak{F} -Poincaré complex whose boundary is $\sigma(f, b)$. Since $\sigma(f, b)$ is also the boundary of a relatively acyclic complex it follows, from 2.17, that x is in the image of the natural map $j: L_{n+1}^s(\pi) \rightarrow \Gamma_{n+1}^s(\mathfrak{F})$; suppose $y \in L_{n+1}^s(\pi)$ maps to x . The realization theorems of [40] (5.8 in the even dimensional and 6.5 in the odd-dimensional case) imply that there exists a degree 1 normal map $G: (W'; M', M'') \rightarrow (M' \times I; M' \times 0, M' \times 1)$ with $G|M'$ the identity map, $G|M''$ a simple homotopy equivalence, and with surgery obstruction $-y \in L_{n+1}^s(\pi)$. Form the union of W' with W along the common boundary M' and map the result to $X \times [0, 2]$. The resulting cobordism

$$F \cup G: (W \cup W'; M, M'') \longrightarrow (X \times [0, 2]; X \times 0, X \times 2)$$

has a *homology surgery obstruction* (rel $M \cup M''$) of 0 so that we can perform surgery to get

$$F': (W''; M, M'') \longrightarrow (X \times [0, 2]; X \times 0, X \times 2) .$$

The result, W'' , is the required (geometric) homology s -cobordism from f to a simple homotopy equivalence. This completes the proof of the theorem. We will conclude this section by stating a realization theorem similar to 5.8 and 6.5 in [40] and corresponding results

in [7]. The proof will be given in the next section.

THEOREM 3.6. *Let $(M^{n-1}, \partial M)$ be a compact manifold with $\pi_1(M) = \pi$ and orientation character $w: \pi \rightarrow \mathbb{Z}_2$, and $n \geq 6$ and let $\sigma \in D_n^s(\mathfrak{F})$ be an arbitrary element. Then there exists a degree-1 normal map $F: (W; M, M') \rightarrow (M \times I, M \times 0, M \times 1)$ such that $\partial W = \partial M \times I \cup M \cup M'$ and:*

1. $F|(\partial M \times I \cup M)$ is the identity map;
2. F is a simple A' -homology equivalence and $F|M': M' \rightarrow M \times 1$ is a simple homotopy equivalence;
3. the dual surgery obstruction of F (rel ∂W) is equal to σ . \square

4. Relations with other surgery theories. The main result of this section is to prove the exactness of the following sequence of abelian groups:

$$4.1. \quad \dots \longrightarrow L_{n+1}^s(\pi) \xrightarrow{i} \Gamma_{n+1}^s(\mathfrak{F}) \xrightarrow{\partial} D_n^s(\mathfrak{F}) \xrightarrow{i} L_n^s(\pi) \xrightarrow{j} \Gamma_n^s(\mathfrak{F}) \longrightarrow \dots .$$

Here the L -, Γ - and D -groups are as defined in §§2 and 3 of this paper—i.e., they are regarded as algebraic cobordism groups of suitable quadratic complexes—and the maps i , j , and ∂ are defined as follows:

1. i and j are induced by the inclusion of quadratic complexes representing elements of *one* surgery group in the *other* group;
2. ∂ is induced by the *boundary construction* (see 2.10).

For the time being we will simply *assume* the maps in 4.1 are *well-defined* and that the sequence is *exact* and draw conclusions from it.

First of all, it is not hard to see that the maps i and j in 4.1 *coincide* with the *geometrically* defined maps that result from giving the surgery groups a bordism-theoretic description along the lines of §9 of Wall, [40]. The *same* is true of the map ∂ , however, since:

LEMMA 4.2. *Let $f: (M^{n+1}, \partial M) \rightarrow (X, Y)$ be a homology surgery problem, i.e., f is a degree-1 normal map with $f|\partial M$ a simple A' -homology equivalence and $\pi_1(X) = \pi$. If $\sigma \in \Gamma_{n+1}^s(\mathfrak{F})$ is the homology surgery obstruction of f , then $\partial(\sigma) \in D_n^s(\mathfrak{F})$ is the dual surgery obstruction of $f|\partial M$.*

Proof. This is an immediate consequence of the way we have defined the algebraic homology surgery obstruction at the end of §2 and of 2.13 (we are implicitly assuming that the map ∂ in 4.1 is well-defined). \square

Our first application of this result is in a proof of Theorem 3.6. Recall that $(M^{n-1}, \partial M)$ is a compact manifold of dimension at least 5 with fundamental group π and we want to construct a cobordism $F: (W^n; M, M') \rightarrow (M \times I; M \times 0, M \times 1)$ whose dual surgery obstruction is an arbitrarily prescribed element, σ , of $D_n^s(\mathfrak{F})$. The realization theorems of [40] (5.8 and 6.5) imply that we can construct a cobordism

$$F': (W'^n; M, M'') \longrightarrow (M \times I; M \times 0, M \times 1)$$

whose ordinary surgery obstruction is $i(\sigma) \in L_n^s(\pi)$ and we can assume, without loss of generality, that F' is a simple A' -homology equivalence (since its homology surgery obstruction vanishes). It follows that $i(\sigma - \sigma(F')) = 0$, where $\sigma(F')$ is the dual surgery obstruction of F' . This implies that there exists an element $h \in \Gamma_{n+1}^s(\mathfrak{F})$ such that $\partial(h) = \sigma - \sigma(F')$. Now we use the realization theorems (1.8 and 2.2) of [7] to construct a cobordism $G: (Z; W', W) \rightarrow (W' \times I; W' \times 0, W' \times 1)$ such that the *homology* surgery obstruction of G is equal to h . Lemma 4.2 implies that the dual surgery obstruction of $G|W: W \rightarrow W' \times 1 \rightarrow M \times I$ is precisely σ . \square

Throughout the remainder of this section we will assume that $A' = \mathbf{Z}\pi'$ where π' is a group and $\mathfrak{F}: A \rightarrow A'$ is induced by a homomorphism of groups $f: \pi \rightarrow \pi'$:

Let Φ denote the following diagram of rings:

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow 1 & & \downarrow \mathfrak{F} \\ A & \xrightarrow{\mathfrak{F}} & A' \end{array}$$

Recall, from Chapter I of [7], that we can define a *relative* Γ -group, $\Gamma_n^s(\Phi)$ which solves a *relative* homology surgery problem described in Theorem 3.3. of the paper [7] of Cappell and Shaneson.

If $(f, b): (M^n; \partial_-M, \partial_+M) \rightarrow (X; Y_-, Y_+)$ is a normal map into the simple Poincaré triad $(X; Y_-, Y_+)$, with $\pi_1(Y_+) = \pi$, $\pi_1(X) = \pi'$, and with $f|_{\partial_-M}$ a simple A' -homology equivalence, then f is normally cobordant to $g: (N; \partial_-N, \partial_+N) \rightarrow (X; Y_-, Y_+)$ with g a simple A' -homology equivalence and $g|_{\partial_+N}$ a simple homotopy equivalence if and only if the homology surgery obstruction of f in $\Gamma_n^s(\Phi)$ vanishes.

It is not hard to see that if $f: M^n \rightarrow X$ is a *dual* surgery problem and we form the product with a unit interval, $f \times 1: M^n \times I \rightarrow X \times 1$, we get a relative homology surgery problem and that this gives to a well-defined homomorphism

$$h: D_n^s(\mathfrak{F}) \longrightarrow \Gamma_{n+1}^s(\Phi).$$

THEOREM 4.3. *The map defined above, $h: D_n^s(\mathfrak{F}) \rightarrow \Gamma_{n+1}^s(\Phi)$, is an isomorphism.*

REMARKS. 1. In spite of the existence of the isomorphism above, we will continue to use the notation $D_n^s(\mathfrak{F})$ for dual surgery obstruction groups, since the geometric significance of dual surgery is different from that of relative homology surgery.

2. It is possible to prove this theorem algebraically and thus eliminate the requirement that A' be a group ring.

3. The identification of $D_n^s(\mathfrak{F})$ with $\Gamma_{n+1}^s(\Phi)$ implies the existence of a homomorphism $p: D_n^s(\mathfrak{F}) \rightarrow L_{n+1}^s(\mathfrak{F})$ (see [7], p. 300) which, when combined with the homomorphism $p: \Gamma_n^s(\mathfrak{F}) \rightarrow L_n^s(\pi')$, gives rise to a map of long exact sequences.

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & L_{n+1}^s(\pi) & \longrightarrow & \Gamma_{n+1}^s(\mathfrak{F}) & \longrightarrow & D_n^s(\mathfrak{F}) & \longrightarrow & L_n^s(\pi) & \longrightarrow & \Gamma_n^s(\mathfrak{F}) & \longrightarrow & \dots \\
 & & \downarrow 1 & & \downarrow p & & \downarrow p & & \downarrow 1 & & \downarrow p & & \\
 \dots & \longrightarrow & L_{n+1}^s(\pi) & \longrightarrow & L_{n+1}^s(\pi') & \longrightarrow & L_{n+1}^s(\mathfrak{F}) & \longrightarrow & L_n^s(\pi) & \longrightarrow & L_n^s(\pi') & \longrightarrow & \dots
 \end{array}$$

The map from $D_n^s(\mathfrak{F})$ to $L_{n+1}^s(\mathfrak{F})$ will turn out to have important geometric applications. For a detailed algebraic description of the homomorphism $p: D_n^s(\mathfrak{F}) \rightarrow L_{n+1}^s(\mathfrak{F})$ see Appendix A.

Proof. This follows from the 5-Lemma upon comparing 4.1 with the long exact sequence in homology surgery theory:

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & L_{n+1}^s(\pi) & \longrightarrow & \Gamma_{n+1}^s(\mathfrak{F}) & \xrightarrow{\partial} & D_n^s(\mathfrak{F}) & \longrightarrow & L_n^s(\pi) & \longrightarrow & \dots \\
 & & \downarrow u & & \downarrow 1 & & \downarrow h & & \downarrow u & & \\
 \dots & \longrightarrow & \Gamma_{n+1}^s(A \xrightarrow{1} A) & \longrightarrow & \Gamma_{n+1}^s(\mathfrak{F}) & \longrightarrow & \Gamma_{n+1}^s(\Phi) & \longrightarrow & \Gamma_n^s(A \xrightarrow{1} A) & \longrightarrow & \dots
 \end{array}$$

and noting that the maps, u , are all isomorphisms (see [7], Chapter I), and that the map h commutes with all of the other maps. \square

COROLLARY 4.4. *Taking the product with CP^2 induces isomorphisms $D_n^s(\mathfrak{F}) \rightarrow D_{n+4}^s(\mathfrak{F})$.* \square

COROLLARY 4.5. (*The $\pi-\pi$ theorem*): *Let $f: (M^n, \partial M) \rightarrow (X, Y)$, $n \geq 6$ be a degree-1 normal map from the compact manifold $(M, \partial M)$ to the Poincaré pair (X, Y) and suppose:*

1. f and $f|_{\partial M}$ are simple A' -homology equivalences;
2. $\pi_1(X) = \pi_1(Y) = \pi$.

Then there exists a normal cobordism $F: (W^{n+1}, U) \rightarrow (X, Y)$ such that

1. $\partial U = \partial M \cup \partial M'$;
2. $\partial W = U \cup M \cup M'$;

- 3. F is a simple A' -homology equivalence; □
- 4. $F|M'$ and $F|\partial M$ are simple homotopy equivalences.

REMARK. This result implies that we can define *relative* dual surgery obstruction groups in the usual way. They will be *canonically* isomorphic to suitable relative homology surgery obstruction groups.

Throughout the remainder of this section we will assume that the group π' is a finite extension of a polycyclic group and that the kernel of $f: \pi \rightarrow \pi'$ is a finitely generated nilpotent group. We will relate the dual surgery theory developed in this paper with the local surgery theory of Pardon (in [21] and [22] and Ranicki (in [25]). First we must recall the results of [35] regarding “acyclic localizations”. Let I be the kernel ideal of the homomorphism $\mathfrak{F}: A \rightarrow A'$, and define S to be the multiplicatively closed set of elements of A of the form $1 + i$, $i \in I$.

LEMMA 4.5. *The localization $\bar{A} = A[S^{-1}]$ is well-defined and if C_* is a finitely generated finite dimensional right projective chain complex $C_* \otimes_A A'$ is acyclic if and only if $C_* \otimes_A \bar{A}$ is acyclic, i.e., if and only if $H_i(C_*) \otimes_A \bar{A} = 0$. □*

REMARK. This is Theorem 1 of [35].

Recall that the ring $\bar{A} = A[S^{-1}]$ has the following *universal property*: If $g: A \rightarrow \theta$ is a homomorphism that maps the elements of S to *invertible* elements of θ , then g has a *unique* extension to $\bar{g}: \bar{A} \rightarrow \theta$ —see p. 50 of [36]. In what follows, let $\tilde{\mathfrak{F}}: \bar{A} \rightarrow A'$ be the extension of \mathfrak{F} (since all of the elements of S map to $1 \in A'$), and let $\tilde{\mathfrak{F}}_*: K_1(\bar{A}) \rightarrow K_1(A')$ be the induced map in algebraic K -theory.

LEMMA 4.6. *Let $A = \tilde{\mathfrak{F}}_*^{-1}(\pm\pi') \in K_1(A')$. Then the canonical homomorphisms*

- (a) $j_i: \Gamma_i^A(A \rightarrow \bar{A}) \rightarrow L_i^A(\bar{A})$ (defined in [7]) and
- (b) $p_i: \Gamma_i^A(A \rightarrow \bar{A}) \rightarrow \Gamma_i^s(\mathfrak{F})$ (induced by the identity map of A and $\tilde{\mathfrak{F}})$ are isomorphisms for all i . Consequently, $\Gamma_i^s(\mathfrak{F})$ is isomorphic to $\Gamma_i^A(\bar{A})$. □

REMARK. This is Theorem 6 of [35].

Let $D_i^A(A \rightarrow \bar{A})$, A a subgroup of $K_1(\bar{A})$, be dual surgery obstruction groups in which we have dropped the requirement that complexes representing elements be *simply-acyclic* over \bar{A} and that equivalent complexes be homology *s-cobordant*—instead we only require the complexes to be acyclic over \bar{A} with Whitehead torsion

in A and equivalent complexes may be only homology h -cobordant with Whitehead torsion in A . The following is an immediate consequence of 4.6 and the 5-Lemma:

THEOREM 4.7. *The homomorphisms $D_i^A(A \rightarrow \bar{A}) \rightarrow L_{i+1}^A(A \rightarrow \bar{A})$ induced by the maps*

$$D_i^A(A \longrightarrow \bar{A}) = \Gamma_{i+1}^A \left[\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow 1 & & \downarrow \\ A & \longrightarrow & \bar{A} \end{array} \right] \longrightarrow L_{i+1}^A(A \longrightarrow \bar{A})$$

are isomorphisms for all i . □

REMARK. The map from the relative Γ -group to the relative Wall group is defined on p. 300 of [7].

COROLLARY 4.8. *The homomorphisms $D_i^A(A \rightarrow \bar{A}) \rightarrow D_i^s(\mathfrak{S})$, induced by the identity map of A and $\mathfrak{S}: \bar{A} \rightarrow A'$, are isomorphisms for all i , where $A = \mathfrak{S}_*^{-1}(\pm\pi')$. If the set S consists of non zero-divisors, it follows that $D_i^s(\mathfrak{S}) = L_i^t(\bar{A}/A)$ (in the notation of Pardon—[21] and [22]) for all i . □*

REMARKS. 1. The local surgery groups, $L_i^t(\bar{A}/A)$, that appear here are not quite the same as those defined by Pardon and Ranicki—our local surgery groups take Whitehead torsion into account with respect to the subgroup $A = \mathfrak{S}_*^{-1}(\pm\pi') \in K_1(\bar{A})$ like the groups $D_i^A(A \rightarrow \bar{A})$.

2. These results imply the dual surgery groups generalize local surgery groups—at least if one follows the geometric definition given in Pardon’s thesis [20], or Ranicki’s paper [25].

We will conclude this section with a proof of its main result—namely that the sequence 4.1 is exact.

1. *The maps i , j , and ∂ are well-defined:* The fact that i and j are well-defined is clear: in both cases the equivalence relation in the target group is weaker than that in the domain. It only remains to prove that the map ∂ is well-defined. Since it clearly preserves direct sums, it suffices to show that boundaries of complexes representing the zero element of $\Gamma_{n+1}^s(\mathfrak{S})$ are homology s -cobordant to 0. But, if $x = (C, \psi)$ represents 0 in $\Gamma_{n+1}^s(\mathfrak{S})$ it is possible to perform algebraic surgery on it (see 2.14), without altering the boundary (up to simple homotopy equivalence), to get a relatively acyclic complex, by 2.15, and this proves the claim.

2. *Exactness at $D_n^s(\mathfrak{F})$* : First of all, it is clear that the composite of ∂ and i is the zero map. If $x \in D_n^s(\mathfrak{F})$ maps to zero in $L_n^s(\pi)$ a relatively acyclic complex, (C, ψ) , representing x is the boundary of an $n + 1$ -dimensional Poincaré pair $(C \rightarrow D, (\partial\psi, \psi))$. Since (C, ψ) is relatively acyclic, we can collapse it (regarding it as the boundary of $(C \rightarrow D, (\partial\psi, \psi))$ —see 2.12) to get an \mathfrak{F} -Poincaré complex whose boundary is simply-homotopy equivalent to (C, ψ) , by 2.13. This proves the result.

3. *Exactness at $L_n^s(\pi)$* : This is clear: a complex representing an element of $L_n^s(\pi)$ maps to 0 in $L_n^s(\mathfrak{F})$ if and only if it has a *relatively acyclic* representative.

4. *Exactness at $L_{n+1}^s(\mathfrak{F})$* : This is a direct consequence of 2.17 and the fact that the map, ∂ , is well-defined.

5. *The highly connected case.* In this section we will give a somewhat simplified algebraic description of the dual surgery obstructions and the maps that appear in the exact sequence 4.1.

The following result is a consequence of an algebraic argument analogous to the proof of Lemma 4.3 in [7].

LEMMA 5.1. *Let $x \in D_n^s(\mathfrak{F})$. Then x has a representative that has:*

- (a) *only two nonvanishing chain modules if n is odd;*
- (b) *only three nonvanishing chain modules if n is even.*

Furthermore, if $f: M^n \rightarrow X$ is a dual surgery problem, $n \geq 5$, then f is homology s -cobordant to a map $f': M' \rightarrow X$ that is $[(n-1)/2]$ -connected. □

Recall the result of Ranicki in [25] (quoted in the third remark following 2.1 in the present paper) relating 1-dimensional quadratic complexes with stable isomorphism classes of split quadratic formations (see [23] for definitions related to formations).

Before we can proceed we must recall the definition and some of the properties of *split formations*, defined by Ranicki in [25]:

DEFINITION 5.2. A *split formation*, $x = \left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta \right) G \right)$, consists of:

- 1. a kernel $H_e(F)$ (see [40], p. 47), where F is a based free A -module;
- 2. a pair of self-annihilating submodules (in the bilinear and quadratic forms on $H_e(F)$), F , and $\text{im} \begin{pmatrix} \gamma \\ \mu \end{pmatrix}: G \rightarrow F \oplus F^*$ —if the

second submodule is a subkernel, the split formation will be said to be nonsingular;

3. a $(-\varepsilon)$ -quadratic form $\theta \in Q_{-\varepsilon}(G)$ such that $\gamma^*\mu = \theta - \varepsilon\theta^* \in \text{Hom}(G, G^*)$. □

REMARKS. The third statement is the *only* difference between a *split* formation and the formations defined by Ranicki in [23]. Essentially, split formations are to ordinary formations as elements of the split special unitary group defined by Sharpe (see [31], § 3 or [30], § 3) are to elements of the special unitary group of Wall (see [40], p. 57). In fact a nonsingular split formation $(F, \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}, \theta\right)F)$ defines an element of $\widetilde{SU}(A)$, (modulo *right* multiplication by arbitrary elements of $\widetilde{EU}(A)$), namely, $\left(\left(\begin{smallmatrix} \gamma & r \\ \mu & s \end{smallmatrix}, \begin{smallmatrix} a \\ \theta \end{smallmatrix}\right)\right)$ (in the notation of [31], § 3), where r and s are chosen so that $\text{im}(r, s): G \rightarrow F \oplus F^*$ is self-annihilating, $\text{im}(\gamma, \mu) + \text{im}(r, s) = F \oplus F^*$ (see Remark 2 following 9.5 in the present paper) and $[a]$ is any quadratic form such that $a - \varepsilon a^* = r^*s$.

As shown by Sharpe (in [31]) and Ranicki (in [25]), the additional structure contained in a split formation is not needed to describe the Wall surgery obstruction (though it *is* needed to capture the *relative* surgery obstruction in the bounded even-dimensional case and we will need it for the *dual* surgery obstruction).

DEFINITION 5.3. A *simple isomorphism* of split formations

$$(\alpha, \beta, \psi): \left(F, \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}, \theta\right)G\right) \longrightarrow \left(F', \left(\begin{smallmatrix} \gamma' \\ \mu' \end{smallmatrix}, \theta'\right)G'\right)$$

is a triple consisting of simple A -module isomorphisms (preserving preferred bases) $\alpha: F \rightarrow F'$, $\beta: G \rightarrow G'$, and a $(-\varepsilon)$ -quadratic form $(F^*, \psi \in Q_{-\varepsilon}(F^*))$ such that:

1. $\alpha\gamma + \alpha(\psi - \varepsilon\psi^*)^*\mu = \gamma'\beta: G \rightarrow F'$;
2. $\alpha^{*-1}\mu = \mu'\beta: G \rightarrow F'^*$;
3. $\theta + \mu^*\psi\mu = \beta^*\theta'\beta \in Q_{-\varepsilon}(G)$. □

REMARK. In 1.8 of [25], Ranicki shows that this is an isomorphism of the underlying formations and, conversely, that any isomorphism of formations is covered by an isomorphism of suitable *split* formations.

DEFINITION 5.4. The split formation, $T = \left(P, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}, 0\right)P^*\right)$, where P is a based free module, will be called a *trivial split formation*. A *stable simple isomorphism* of split formations is a simple isomorphism

of the type $x \oplus T \rightarrow x' \oplus T'$, for some trivial formations T and T' .

The following lemma is Proposition 1.8 of [25]—it will play an important part in the remainder of this chapter:

LEMMA 5.5. *The simple homotopy classes of connected (i.e. $H_0((1 + T_\varepsilon)\psi_0: C^{1-*} \rightarrow C_*) = 0$) 1-dimensional quadratic complexes are in a natural 1-1 correspondence with the stable simple isomorphism classes of split formations over Λ . Poincaré complexes correspond to nonsingular split formations. \square*

REMARKS. 1. The correspondence maps the quadratic complex:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^0 & \xrightarrow{d^*} & C^1 & \longrightarrow & 0 \\ & & \psi_0 \downarrow & \searrow \psi_1 & \downarrow \tilde{\psi}_0 & & \\ 0 & \longrightarrow & C_1 & \xrightarrow{d} & C_0 & \longrightarrow & 0 \end{array}$$

to the split formation $(C_1, \left(\left(\begin{smallmatrix} \bar{\varepsilon}\psi_0 & \\ & d^* \end{smallmatrix} \right) + \tilde{\psi}_0^* \right), -(\psi_1 + d\psi_0)C^0)$.

2. This result provides the link between the algebraic theory of odd-dimensional surgery of Ranicki and the standard theory due to Wall since each split formation defines (in a natural way) an element of the special unitary group $SU(\Lambda)$ and, hence, an element of the Wall group $L_n^s(\pi)$ —in [25], Ranicki shows that this procedure respects surgery obstructions—i.e., the surgery obstruction defined by Ranicki (see 2.5 in the present paper) maps to the Wall surgery obstruction.

3. Let $x = (F, \theta \in Q_{-\varepsilon}(F))$ be a free based module equipped with a quadratic form θ , (this is a matrix that gives rise, via symmetrization and restriction to the diagonal, to bilinear and quadratic forms λ and μ , respectively, on F see—[39], Theorem 1), and regard x as defining an $n + 1$ -dimensional quadratic complex with a single nonvanishing chain module, F , in the middle dimension. Now perform the boundary construction (2.10) on this complex and map the result to the set of split formations by the correspondence defined above. The result will be the split formation

$$\left(F, \left(\left(\begin{smallmatrix} 1 \\ \lambda \end{smallmatrix} \right), \mu \right) F \right)$$

—see the discussion preceding 5.5 in [25]. Such split formations will be called *split graph formations* in analogy to a similar construction in [23].

4. Note that the split formation $(F, \left(\left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right), \theta \right) G)$ is *trivial* if

and only if the map $\mu: G \rightarrow F^*$ is a *simple isomorphism*—this follows from Lemma 2.1 in [23] or directly from Lemma 5.5 above.

Recall that, in the description of even-dimensional homology surgery obstructions in Chapter I of [7], a special $(-\varepsilon)$ -form (F, λ, μ) (see p. 286 of [7]) representing an element of $\Gamma_{n+1}^s(\mathfrak{S})$ was said to be *strongly equivalent to zero* if it contains a submodule, H , such that:

1. $\lambda, \mu|_H$ are identically zero;
2. the image of H in $F \otimes_A A'$ is a subkernel.

In Lemma 1.3 (in [7]), Cappell and Shaneson prove that a special form represents the zero element of $\Gamma_{n+1}^s(\mathfrak{S})$ if and only if its direct sum with some kernel is *strongly equivalent to zero*.

The following lemma is a version of 2.17 corresponding to the description of elements of $\Gamma_{n+1}^s(\mathfrak{S})$ by quadratic forms (where n is odd):

LEMMA 5.6. *Let (F, λ, μ) be a special hermitian form representing an element, x , of $\Gamma_{n+1}^s(\mathfrak{S})$. Then x is in the image of the natural map $j: L_{n+1}^s(\pi) \rightarrow \Gamma_{n+1}^s(\mathfrak{S})$ if and only if the split graph formation $(F, \left(\begin{smallmatrix} 1 \\ \lambda \end{smallmatrix}\right), \mu)F$, is stably simply isomorphic to a split graph formation of the form $(F', \left(\begin{smallmatrix} 1 \\ \lambda' \end{smallmatrix}\right), \mu')F'$, where (F', λ', μ') is a special $(-\varepsilon)$ -form that is strongly equivalent to zero.*

Proof. This follows from Lemma 1.3 of [7] (quoted above), 2.17, 5.5 and the third remark following that lemma. □

This implies that:

COROLLARY 5.7. *Let n be odd and let D denote the Grothendieck group of stable isomorphism classes of nonsingular split quadratic formations over A (with $\varepsilon = (-1)^{(n-1)/2}$) whose tensor product with A' is trivial and let K be the subgroup generated by graph formations of the form $x = (F, \left(\begin{smallmatrix} 1 \\ \lambda \end{smallmatrix}\right), \mu)F$, where (F, λ, μ) is a quadratic form representing an element of $\Gamma_{n+1}^s(\mathfrak{S})$ that is strongly equivalent to zero in the sense of p. 286 of [7].*

Then the map $D \rightarrow D_n^s(\mathfrak{S})$ defined by mapping formations to the highly-connected complexes that correspond to them by 5.5 is surjective and its kernel is K .

Proof. First of all, it is clear that the elements of D correspond to the relatively acyclic Poincaré complexes under the correspondence described in 5.5. This implies that there exists a surjective homomorphism $D \rightarrow D_n^s(\mathfrak{S})$. That the kernel is precisely

K is an immediate consequence of Remark 3 following 5.5, and 5.1, and 2.17, and 5.6. \square

At this point we are in a position to describe the maps that appear in the exact sequence 4.1.

THEOREM 5.8. *The maps, i and ∂ , in 4.1 can be described as follows:*

1. i carries a split formation representing an element of $D_n^s(\mathfrak{F})$ to the class of the automorphism of the standard kernel defined by it in $SU_c(A)/RU_c(A) = L_n^s(\pi)$;
2. ∂ carries a quadratic form (F, λ, μ) representing an element of $\Gamma_{n+1}^s(\mathfrak{F})$ to the class of the split graph formation $(F, \left(\begin{smallmatrix} 1 \\ \lambda \end{smallmatrix}\right), \mu)F$ in $D_n^s(\mathfrak{F})$. \square

We will conclude this section by giving a similar description in the even-dimensional case. Consider the map of long exact sequences of surgery obstruction groups in Remark 3 following 4.3:

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & L_{2k+1}^s(\pi) & \longrightarrow & \Gamma_{2k+1}^s(\mathfrak{F}) & \longrightarrow & D_{2k}^s(\mathfrak{F}) & \longrightarrow & L_{2k}^s(\pi) & \longrightarrow & \Gamma_{2k}^s(\mathfrak{F}) & \longrightarrow & \cdots \\
 & & \downarrow 1 & & \downarrow \textcircled{1} & & \downarrow p & & \downarrow 1 & & \downarrow \textcircled{2} & & \\
 \cdots & \longrightarrow & L_{2k+1}^s(\pi) & \longrightarrow & L_{2k+1}^s(\pi') & \longrightarrow & L_{2k+1}^s(\mathfrak{F}) & \longrightarrow & L_{2k}^s(\pi) & \longrightarrow & L_{2k}^s(\pi') & \longrightarrow & \cdots
 \end{array}$$

PROPOSITION 5.9. *The homomorphism $p: D_{2k}^s(\mathfrak{F}) \rightarrow L_{2k+1}^s(\mathfrak{F})$ is injective.*

Proof. This is an immediate consequence of the map of exact sequences above, the 5-Lemma, and the facts that the homomorphisms labeled $\textcircled{1}$ and $\textcircled{2}$ are *injective* and *surjective*, respectively (see Cappell and Shaneson [7], Chapter 1). We will, consequently, recall Wall’s definition of relative odd-dimensional surgery obstruction groups, in [40], Chapter 7:

5.10. *The elements of $L_{2k+1}^s(\mathfrak{F})$ are equivalence classes of quadruples (F, λ, μ, K) where:*

- (a) (F, λ, μ) is a special ε -Hermitian form over A in the sense of Chapter 5 of [40] such that $(F, \lambda, \mu) \otimes_A A'$ is a kernel;
- (b) $K \subset (F, \lambda, \mu) \otimes_A A'$ is a subkernel.

Two such quadruples (F_i, λ_i, μ_i) $i = 1, 2$, are equivalent if and only if there exists a kernel H , over A , with subkernel S such that:

- (a) $(F_1, \lambda_1, \mu_1) \oplus H \oplus (F_2, -\lambda_2, -\mu_2) = H_1$ is a kernel, with subkernel S_1 ;
- (b) an automorphism of $H_1 \otimes_A A'$ taking $S_1 \otimes_A A'$ to $K_1 \oplus (S \otimes_A A') \oplus K_2$ is stably in $RU(A')$.

The following theorem is an immediate consequence of the definition of odd-dimensional homology surgery groups due to Cappell and Shaneson in §2 of [7], and Wall's description of the maps $L_{2k+1}^s(\mathfrak{F}) \rightarrow L_{2k}^s(\pi)$ and $L_{2k+1}^s(\pi') \rightarrow L_{2k+1}^s(\mathfrak{F})$:

THEOREM 5.11. 1. *The group $D_{2k}^s(\mathfrak{F})$ is the subgroup of $L_{2k+1}^s(\mathfrak{F})$ generated by quadruples (F, λ, μ, K) such that the ε -Hermitian form (F, λ, μ) regarded as representing an element of $\Gamma_{2k}^s(\mathfrak{F})$, is strongly equivalent to zero with a pre-subkernel, P , such that the image of P in $F \otimes_A A'$ is K —see [7], p. 286;*

2. *the map $D_{2k}^s(\mathfrak{F}) \rightarrow L_{2k}^s(\pi)$ in 4.1 carries an element represented by a quadruple (F, λ, μ, K) to the element of the L -group represented by the form (F, λ, μ) ;*

3. *the map $\partial: \Gamma_{2k+1}^s(\mathfrak{F}) \rightarrow D_{2k}^s(\mathfrak{F})$ carries the element represented by an automorphism $\alpha \in SU(A)$ of the standard kernel S , to the element of the D -group represented by the quadruple $(F \oplus F^*, \lambda, \pi, K)$, where $(F \oplus F^*, \lambda, \mu)$ is a standard kernel with subkernel F such that $(F \oplus F^*, \lambda, \mu) \otimes_A A' = S$ and $K = \alpha(F \otimes A')$. \square*

It is not difficult to see how the dual surgery obstruction is computed in this case:

Given a highly-connected dual surgery problem (see 5.1), $f: M^{2k} \rightarrow X$, $k \geq 3$, perform surgery upon M to obtain $f': M' \rightarrow X$, where f' is k -connected. Calculate the ordinary surgery obstruction of f' —this will be an ε -Hermitian form $x = (F, \lambda, \mu)$, where F is the k -dimensional homology kernel and λ and μ are determined by intersection and self-intersection numbers—see [40], Chapter 5. This form, x , will be *strongly equivalent to zero* (after stabilization, if necessary) since the original map was a simple A' -homology equivalence—see Proposition 1.6 of [7]. If P is the pre-subkernel constructed in the *proof* of that proposition and its image in $x \otimes_A A'$ is K the quadruple (F, λ, μ, K) represents the dual surgery obstruction of f .

6. **A geometric description of odd-dimensional dual surgery obstructions.** In this section we will give an interpretation of the odd-dimensional dual surgery obstruction, in the highly connected case, as linking and self-linking forms on the middle dimensional homology. These linking forms are defined geometrically and they are direct generalizations of the Seifert form in knot theory. This treatment has the advantage of associating dual surgery obstructions to specific homotopy-theoretic invariants that occur in a dual surgery problem.

We will essentially use 4.7 coupled with the algebraic descriptions and geometric interpretations of local surgery groups due to

Pardon (see [21]) and Ranicki (see § 13 of [25]). Though a similar formulation is possible in the even-dimensional case it does not appear to offer any advantages in our geometric applications over that at the end of the preceding section—the interested reader is referred to [21] and § 13 of [25] for more details.

The following conventions will be in effect throughout this section:

1. The map $\mathfrak{F}: A \rightarrow A'$, where as before $A = \mathbf{Z}G$ and $A' = \mathbf{Z}H$, is induced by $\mathfrak{F}_0: G \rightarrow H$, H is a finite extension of a polycyclic group and $\ker \mathfrak{F}_0$ is a finitely generated torsion-free nilpotent group;

2. S is the multiplicatively closed set in A of elements of the form $1 + i$, $i \in I$, where I is the kernel of \mathfrak{F} ;

3. $\bar{A} = A[S^{-1}]$ —the existence of this localization is guaranteed by Theorem 1 of [35];

4. $n = 2k + 1$ and $\varepsilon = (-1)^k$;

5. $B = \mathfrak{F}_*(\pm\pi')$ where $\mathfrak{F}_*: K_1(\bar{A}) \rightarrow K_1(A')$ is induced by $\mathfrak{F}: \bar{A} \rightarrow A'$ extending \mathfrak{F} —see the remark following 4.5.

We will need the following technical result:

PROPOSITION 6.1. *The elements of the set S , defined above, are non zero-divisors.*

Proof. Let I' be the augmentation ideal of $\mathbf{Z}K$, where K is the kernel of \mathfrak{F}_0 , and let I be the kernel of \mathfrak{F} , as above. Proposition 1.2 in [35] implies that the ideals I and I' both satisfy the *symmetric Artin-Rees* condition in their respective rings—i.e., given a finitely generated $\mathbf{Z}G$ -module M and submodule L , $M \cdot I^n \cap L \subset L \cdot I^n$ for all n , if M and L are *right* modules and $I^n \cdot M \cap L \subset I^n \cdot L$ if M and L are *left* modules, similarly for I' . This implies, by the Krull intersection theorem on p. 171 of [36], that the (two sided) ideal of (right and left) annihilators of elements of S is precisely $X_1 = \bigcap_{j=1}^{\infty} I^j$ and that the ideal of annihilators of elements of $\mathbf{Z}K$ of the form $1 + i'$, $i' \in I'$, is $X_2 = \bigcap_{j=1}^{\infty} (I')^j$. Since Corollary 3 in [6] implies that the ring $\mathbf{Z}K$ doesn't have any zero-divisors it follows that $X_2 = 0$. Since K is normal in G it follows that $I^n = \mathbf{Z}G \cdot (I')^n$ so that $X_1 = \bigcap_{j=1}^{\infty} (\mathbf{Z}G \cdot (I')^j) = \bigoplus_{i=1}^{G, K1} (X_2)_i = 0$ (regarding $\mathbf{Z}G$ as a free $\mathbf{Z}K$ -module) and this proves the proposition. \square

The following corollary is the reason for the assumption that K be torsion-free:

COROLLARY 6.2. *If C_* is a finitely generated $2k + 1$ -dimensional $k - 1$ -connected relatively acyclic projective A -complex, then*

the only nonvanishing homology module is $H_k(C_*)$ —and this is \bar{A} -torsion.

Proof. The statement that the homology of C_* is \bar{A} -torsion follows from 4.5. The only possible homology module other than $H_k(C_*)$ is $H_{k+1}(C_*)$ and this will be a submodule of C_{k+1} . Proposition 6.1 implies that no nonzero submodule of a free module can be annihilated by elements of S so $H_{k+1}(C_*)$ must be zero.

DEFINITION 6.3. A t -module consists of a \bar{A} -torsion module, K , with a short free resolution: $0 \rightarrow F_1 \xrightarrow{j} F_2 \rightarrow K \rightarrow 0$ such that:

1. bases for F_1 and F_2 are specified;
2. $j \otimes 1: F_1 \otimes_A \bar{A} \rightarrow F_2 \otimes_A \bar{A}$, regarded as an element of $K_1(\bar{A})$, is contained in B . □

DEFINITION 6.4. A simple isomorphism of t -modules K_1 and K_2 is an isomorphism $f: K_1 \rightarrow K_2$ such that f lifts to a simple chain equivalence of the presentations of K_1 and K_2 (regarded as chain-complexes, and with respect to the preferred bases):

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
 0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & K_2 & \longrightarrow & 0 .
 \end{array}
 \quad \square$$

REMARKS. 1. Note that t -modules contain considerably more structural information than the underlying module. For example, the set of simple isomorphism classes of t -module over ZG whose underlying module is zero is in a 1-1 correspondence with the elements of $Wh(G)$.

2. Simplicity of isomorphisms of t -module is a well-defined concept since an isomorphism between K_1 and K_2 lifts to a unique chain homotopy class of chain equivalence between presentations—[10] p. 77.

3. t -modules play the same part in local surgery theory as based free modules in ordinary surgery theory.

We will always assume that when t -modules are discussed, that the base ring is a group-ring.

The following lemma is similar to Proposition 1.4 in [21]:

LEMMA 6.5. Let K be a t -module. Then:

1. $\text{Hom}_A(K, \bar{A}/A)$ has the natural structure of a t -module;
2. A simple isomorphism of t -modules $f: K_1 \rightarrow K_2$ induces a natural simple isomorphism $f^d: \text{Hom}_A(K_2, \bar{A}/A) \rightarrow \text{Hom}_A(K_1, \bar{A}/A)$.

Proof. Let K be presented by $0 \rightarrow F_1 \rightarrow F_2 \rightarrow K \rightarrow 0$.

Claim. $\text{Hom}_A(K, \bar{A}/A)$ is presented by:

$$0 \longrightarrow \text{Hom}_A(F_2, A) \xrightarrow{j^*} \text{Hom}_A(F_1, A) \longrightarrow \text{Hom}_A(K, \bar{A}/A) \longrightarrow 0$$

where we use the dual bases for the $\text{Hom}_A(F_i, A)$. Assuming this is true, the statement about maps of t -modules follows from the fact that a simple chain equivalence of chain complexes induces a simple equivalence of the dual complexes (the condition on B implies that the dual $\text{Hom}_A(K, \bar{A}/A)$ is a t -module).

We will now prove the claim:

A. If K is \bar{A} -torsion, $\text{Hom}_A(K, \bar{A}/A) = \text{Ext}^1(K, A)$. This follows from the long exact sequence induced by

$$0 \longrightarrow A \longrightarrow \bar{A} \longrightarrow \bar{A}/A \longrightarrow 0$$

which is $\text{Hom}_A(K, \bar{A}) \rightarrow \text{Hom}_A(K, \bar{A}/A) \rightarrow \text{Ext}_A^1(K, A) \rightarrow \text{Ext}_A^1(K, \bar{A})$. But since \bar{A} is flat $\text{Ext}_A^i(M, N) = \text{Ext}^i(M \otimes_A \bar{A}, N)$ (see [10], Chapter 6, § 4), thus

$$\text{Ext}_A^i(K, \bar{A}) = \text{Ext}_A^i(K \otimes_A \bar{A}, A)$$

which vanishes if K is \bar{A} -torsion and statement A follows.

Now, applying this to the sequence $0 \rightarrow F_1 \rightarrow F_2 \rightarrow K \rightarrow 0$ we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(F_2, A) & \longrightarrow & \text{Hom}_A(F_1, A) & \longrightarrow & \text{Ext}_A^1(K, A) \longrightarrow 0 \\ & & & & & & \parallel \\ & & & & & & \text{Hom}_A(K, \bar{A}/A) \end{array}$$

which implies the lemma. □

DEFINITION 6.6. The following conventions will be in effect throughout this section. If F is a free module $F^* = \text{Hom}_A(F, A)$, with the dual basis;

2. If K is a t -module $K^d = \text{Hom}_A(K, \bar{A}/A)$ with the dual equivalence class of bases defined in Lemma 6.5. □

DEFINITION 6.7. Define:

- (a) $Q^\varepsilon(A, S) = \{b \in \bar{A} \mid b - \varepsilon b = a - \varepsilon \bar{a}, a \in A\}/A$;
- (b) $Q_\varepsilon(A, S) = \{b \in \bar{A} \mid b = \varepsilon \bar{b}\}/\{a + \varepsilon \bar{a} \mid a \in A\}$;
- (c) $Q_\varepsilon(\bar{A}/A) = \bar{A}/\{a + b - \varepsilon \bar{b} \mid a \in A, b \in \bar{A}\}$;
- (d) $\mathcal{L}: Q_\varepsilon(\bar{A}/A) \rightarrow Q_\varepsilon(A, S)$ sending x to $x + \varepsilon \bar{x}$;
- (e) $\mathcal{P}: Q_\varepsilon(A, S) \rightarrow Q^\varepsilon(A, S)$ sending x to its class in the target.

If K is a t -module and $\varepsilon = (-1)^k$ then a $2k + 1$ -dimensional t -form over K , (K, b, q) consists of:

- (a) a \mathbf{Z} -bilinear pairing $b: K \times K \rightarrow \bar{A}/A$ such that $b(xa, y) = \bar{a} \cdot b(x, y)$ and $b(y, x) = \overline{\varepsilon b(x, y)}$ for $x, y \in K$;
- (b) a function $q: K \rightarrow Q_\varepsilon(\bar{A}/A)$ such that $q(xa) = \bar{a} \cdot q(x) \cdot a \in Q_\varepsilon(\bar{A}/A)$,
 $q(x + y) - q(x) - q(y) = b(x, y)$, and $b(x, x) = \mathcal{P}(\mathcal{P}(q(x)))$. □

REMARKS. 1. A t -form in the present paper is equivalent to a split ε -quadratic linking form in the terminology of [28]. It is proved in [28] that a t -form as defined here is equivalent to an ε -Hermitian form over a torsion module if $2 \in S$.

2. I am indebted to Professor Ranicki for pointing out the need for the definition of a t -form given above (which differs from the definition in an earlier version of this paper).

3. A t -form (K, b, q) will be called *nonsingular* if $\text{ad } b: K \rightarrow K^d$ is a simple isomorphism. Henceforth, all t -forms will be assumed to be nonsingular, unless stated otherwise.

DEFINITION 6.8. Let $x_i = (K_i, b_i, q_i)$, $i = 1, 2$ be t -forms. Then a *simple isometry* of x_1 with x_2 is a simple isomorphism of t -modules $f: K_1 \rightarrow K_2$ such that

$$\begin{array}{ccc} K_1 & \xrightarrow{\text{ad } b_1} & K_1^d \\ f \downarrow & & \downarrow f^d \\ K_2 & \xrightarrow{\text{ad } b_2} & K_2^d \end{array}$$

commutes and $q_2(f(x)) = q_1(x)$ for all $x \in K_1$. □

The connection between $D_{2k+1}^s(\mathfrak{F})$ and $L_{2k+1}^t(\bar{A}/A)$ is described by the following result due to Ranicki (§ 3.4 of [28]):

PROPOSITION 6.9. *The following equivalence classes are in one-one correspondence with each other:*

- 1. *the simple isometry classes of (nonsingular) $2k + 1$ -dimensional t -forms;*
- 2. *the stable isomorphism classes of nonsingular split formations whose tensor products with A' (via \mathfrak{F}) are trivial;*
- 3. *the simple homotopy classes of $k - 1$ -connected $2k + 1$ -dimensional relatively acyclic Poincaré complexes.* □

REMARKS. 1. Our statement differs slightly from that of Proposition 13.2 in [25]. This difference is justified by 4.5 and other results in [35].

2. The relation between relatively acyclic Poincaré complexes and split formations has already been given in 5.5. We will describe the remaining relations:

A. A $k - 1$ -connected $2k + 1$ -dimensional Poincaré complex:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C^k & \xrightarrow{d^*} & C^{k+1} & \longrightarrow & M \longrightarrow 0 & \varphi_0 = \psi_0 - \varepsilon\tilde{\psi}_0^* \\
 & & \varphi_0 \downarrow & \searrow \psi_1 & \downarrow \tilde{\varphi}_0 & & \downarrow b & \\
 0 & \longrightarrow & C_{k+1} & \xrightarrow{d} & C_k & \longrightarrow & M^d \longrightarrow 0 & \tilde{\varphi}_0 = \tilde{\psi}_0 - \varepsilon\psi_0^*
 \end{array}$$

maps to the t -form (M, b, q) , where $M = H^{k+1}(C_*)$, $M^d = H_k(C_*)$
 $b: M \rightarrow M^d$ maps $[x]$ to $([y] \mapsto x((\psi_0 - \varepsilon\tilde{\psi}_0^*)(z))/s) \in M^d$; $q: M \rightarrow \bar{A}_k$
maps $[y]$ to $(1/\bar{s})(z(\psi_1 + d\psi_0)(z))(1/s)$; where $x, y \in C^{k+1}$, and $s \in S$,
 $z \in C^k$ are such that $ys = d^*z$.

B. A nonsingular split formation $(F, \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}\right), \theta)G$ maps to the
 t -form (M, b, q) where: $M = \text{coker } (\mu: G \rightarrow F^*)$, $b: M \rightarrow M^d$ maps $[x]$
to $([y] \mapsto x(\gamma(g))/s)$, $q: M \rightarrow \bar{A}_k$ maps $[y]$ to $(1/\bar{s})\theta(g)(g)(1/s)$, where
 $x, y \in F^*$, and $s \in S, g \in G$ are such that $ys = \mu(g)$.

This is proved in § 3.4 of [28] and in 13.2 of [25].

We will now define an important class of t -forms. The following definition is due to Pardon, Ranicki, and Karoubi:

DEFINITION 6.10. A t -form (M, b, q) is said to be a t -kernel if there exists a t -submodule $L \subset M$ such that:

- (i) $b, q|L = 0$;
- (ii) the map $b: M/L \rightarrow L^d; [x] \mapsto ([y] \mapsto b(x, y))$

is a simple isomorphism of t -modules. □

REMARKS. 1. This definition is analogous to Wall's definition of a *kernel*—see 5.3 in [40].

2. Ranicki's Proposition 3.4.5 in [28] proves (among other things) that a t -form is a t -kernel if and only if the $k - 1$ -connected $2k + 1$ -dimensional Poincaré complex corresponding to it under the bijection described in 6.9 is the boundary of a $2k + 2$ -dimensional Poincaré pair that becomes acyclic when the tensor product is taken with \bar{A} . This, coupled with 4.5, implies the following *equivalent* definitions of a t -kernel:

DEFINITION 6.10a. A t -form is a t -kernel if and only if it corresponds under the bijection described in 6.9 to a $k - 1$ -connected $2k + 1$ -dimensional relatively acyclic Poincaré complex that is homology s -cobordant to 0. □

DEFINITION 6.10b. A t -form, (M, b, q) is a t -kernel if and only if there exists a special $(k + 1)$ -Hermitian form, (F, λ, μ) that is strongly equivalent to zero (when regarded as defining an element

of $\Gamma_{2k+2}^s(\mathfrak{S})$ —see p. 286 of [7]) and such that:

1. $M = \text{coker} (\text{ad } \lambda: F \rightarrow F^*)$;
2. $b: M \rightarrow M^d$ maps $[y]$ to $([x] \mapsto x(z)/s)$;
3. $q: M \rightarrow \bar{A}_k$ maps $[y]$ to $(1/\bar{s})\mu(z)(1/s)$

where $x, y \in F^*$, and $s \in S, z \in F$ are such that $ys = (\text{ad } \lambda)(z)$. □

Now we are in a position to describe the group $D_{2k+1}^s(\mathfrak{S})$ in terms of t -forms:

THEOREM 6.11. *Let T be the semigroup of simple isometry classes of t -forms with the operation: $(M_1, b_1, q_1) + (M_2, b_2, q_2) = (M_1 \oplus M_2, b_1 \oplus b_2, q_1 \oplus q_2)$. If \tilde{T} is the associated Grothendieck group of stable simple isometry classes and \tilde{K} is the subgroup generated by t -kernels, then $T/K \cong D_{2k+1}^s(\mathfrak{S})$, the isomorphism being induced by the bijection described in 6.9.*

REMARKS. 1. Pardon ([21]), Ranicki ([25]) and Karoubi ([13]) show that for any t -form (M, b, q) , $(M, b, q) + (M, -b, -q)$ is simply isometric to a t -kernel so that every element of \tilde{T}/\tilde{K} has a representative that is an *actual t -form* rather than just a *formal difference* of t -forms.

2. Suppose $f: M^{2k+1} \rightarrow X, k \geq 2$, is a highly connected dual surgery problem with $\pi_1(X) = \pi$ and satisfying the conditions in effect in this section and let (C, ψ) be the dual surgery obstruction. Then, by the theorem above, we can also regard the dual surgery as a t -form, (T, b, q) where, by statement B following 6.9, $M = H^{k+1}(C^*) = H_k(C^*) = \ker f_*: H_k(M) \rightarrow H_k(X)$ (since f is a *degree-1 map*—see [40], Chapter 2). Furthermore b is induced by the map of chain complexes corresponding to the symmetrization of ψ which, since (C, ψ) is a *quadratic signature* of a *normal map*, coincides with *Poincaré duality* and therefore b is identical to the geometric linking form defined by Wall in his original formulation of odd-dimensional surgery theory.

We can also give a geometric description of $\not\sim(q(x))$ similar to Wall's definition of the self-linking form on p. 251 of [38]. Let $x \in H_k(C_*) = T$ be represented by an immersed sphere $g: S^k \rightarrow M$ and suppose $s \in S$ is a right annihilator of x . Then we can form the immersion $g \cdot s$ (i.e., regard g as an element of $\pi_k(M)$ —this is a module over $Z\pi_1(M)$) and this will be a boundary ∂p . Now define $\hat{q}(x)$ to be the element $(p \cdot g')/s$ of \bar{A}/A , where g' is the immersion, g , displaced a short distance along the first vector of the framing and $p \cdot g'$ is the intersection number. Our main result is:

THEOREM 6.12. *Let $f: M^{2k+1} \rightarrow X$ be a dual surgery problem,*

with $k \geq 2$, and suppose that all of the assumptions at the beginning of this section are in effect, and let (T, b, q) be the dual surgery obstruction of f , where $T = K_k(M)$. Then $\not\prec(q(x)) = \hat{q}(x)$ for all $x \in T$ (see 6.7 for a definition of $\not\prec: Q_s(\bar{A}/A) \rightarrow Q_s(A, S)$).

REMARK. This result can clearly be *relativized*.

Proof. We begin by forming the product $f \times 1: M \times I \rightarrow X \times I$. Perform surgery on $f \times 1$ rel $M \times \partial I$ so that it becomes $k + 1$ -connected and let the result be $F: (W; M \times 0, M \times I) \rightarrow X \times I$. Let $(K_{k+1}(W), \lambda', \mu')$ (here we are using the common surgery-theoretic notational device of denoting homology kernels by K) be the quadratic signature of F —see the remarks following 2.1. Remark 3 following 5.5 and 6.9 imply that the self-linking form on the dual surgery obstruction, $(T, b, q) \oplus (T, -b, -q)$, of $F|_{M \times \partial I}$ is induced by the exact sequence

$$0 \longrightarrow K_{k+1}(W) \xrightarrow{\text{ad } \lambda'} K_{k+1}(W, M \times \partial I) = K_{k+1}(W)^* \longrightarrow K_k(M \times \partial I) \longrightarrow 0$$

in the following sense: if $x \in K_k(M \times \partial I) = T \oplus T$ is annihilated by $s \in S$ and lifts to x' in $K_{k+1}(W, M \times \partial I)$ and if $z \in K_{k+1}(W)$ is such that $\text{ad } \lambda'(z) = x' \cdot s$, then $\not\prec(q(x)) = \lambda'(x', z)/s$ —this results from the fact that $(T, b, q) \oplus (T, -b, -q)$ corresponds (via 6.9) to the *boundary* of $(K_{k+1}(W), \lambda', \mu')$. Now note that the proof of Lemma 5.4 in [38] of Wall implies an analogous statement to the above except that \hat{q} and the *geometric* intersection form, λ , must be substituted for $\not\prec(q(x))$ and λ' , respectively.

The conclusion now follows from the proof of the compatibility theorem for quadratic signatures in §4 of [25], which implies that $\lambda'(x, z) = \lambda(x, z)$ for all $x, z \in K_{k+1}(W)$. □

DEFINITION 6.13. Let $f: (M^{2k+1}, \partial M) \rightarrow (X, Y)$ be a dual surgery problem, where $f|_{\partial M}$ is a simple homotopy equivalence. Then the invariant $(K_k(M), b, q)$, where $K_k(M) = \ker f_*$, and b and q are as described in the second remark following 6.11 and in 6.12 will be called the *t-signature* of the map f . □

REMARK. The *t-signature* of a dual surgery problem is nothing but the *particular representative* of the dual surgery obstruction (where $D_{2k+1}^s(\mathfrak{F})$ is regarded as a group of stable isometry classes of *t-forms* as in 6.11) that arises from the *geometric context* of the problem.

In the geometric applications to follow we will need the following realization theorem for *t-signatures*:

THEOREM 6.14. *Let $f: (M^{2k+1}, \partial M) \rightarrow (X, Y)$, $k \geq 2$, $\pi_1(X) = \pi$, and with $f|_{\partial M}$ a simple homotopy equivalence, be a dual surgery problem with f k -connected and suppose $x \in D_{2k+1}^s(\mathfrak{F})$ is its dual surgery obstruction rel ∂M . If (T, b, q) is any t -form that represents x , then there exists a homology s -cobordism of f , rel ∂M , to a dual surgery problem $f': (M', \partial M) \rightarrow (X, Y)$ with f' also k -connected and with t -signature isometric to (T, b, q) .*

Proof. Suppose the t -signature of f is (T', b', q') . Since this represents the same class in $D_{2k+1}^s(\mathfrak{F})$ as (T, b, q) , it follows that there exists t -kernels (K_i, b_i, q_i) , $i = 1, 2$, such that $(T, b, q) \oplus (T_2, b_2, q_2) \cong (T', b', q') \oplus (T_1, b_1, q_1)$. Since (T_1, b_1, q_1) is a t -kernel it follows that there exists a special $(-1)^{k+1}$ -Hermitian form (F, λ, μ) that is strongly equivalent to zero in the sense of Cappell and Shaneson (see [7], p. 286) such that $(T_1, b_1, q_1) = \partial(F, \lambda, \mu)$, i.e., it satisfies the conditions of 6.10b. The realization theorem of Cappell and Shaneson (Theorem 1.8 of [7]) implies that there exists a homology surgery problem

$$F'_1: (W'_1; M, M_1) \longrightarrow M \times I$$

with homology kernel in dimension $k + 1$ equal to F' and with intersection and self-intersection forms λ and μ , respectively. Definition 2.10 and Lemma 4.2 imply that $F'_1|_{M_1}$ is k -connected and its t -signature is (T_1, b_1, q_1) . The fact that the homology surgery obstruction of F'_1 vanishes implies that we may replace F'_1 with $F''_1: (W_1; M, M_1) \rightarrow M \times I$, where W_1 is a homology- s -cobordism rel ∂M , and its composite with f gives

$$F_1: (W_1; M, M_1) \longrightarrow (X, Y)$$

where the t -signature of $F_1|_{M_1}$ is $(T', b', q') \oplus (T_1, b_1, q_1)$, and W_1 is still a homology s -cobordism. Let (T_1, b_1, q_1) be induced by the $-\varepsilon$ -Hermitian form (F, λ, μ) in the sense of 6.10b and let $\{a_i\} \in T_1$ be a generating set that is in the image of the canonical basis for F^* (see 6.10b). Now perform surgery on imbeddings of spheres whose homology classes are the $\{a_i\}$ (this is always possible because we are working below the middle dimension). Let the trace of the surgery be

$$F_2: (W; M_1, M') \longrightarrow (X, Y)$$

Then 7.3 in [27] and 2.14 in the present paper imply that the t -signature of $F_2|_{M'}$ is precisely (T', b', q') —i.e., we have *killed* (T_1, b_1, q_1) . An argument like that used in 3.5 implies that the homology surgery obstruction of F_2 rel $M_1 \cup M'$ is in the image of

the map from a Wall group so that (as in 3.5) it can be killed. This implies that M_1 and M' are homology s-cobordant and this completes the proof. \square

CHAPTER II. *Cobordism Theory*

In this chapter we will apply our results on dual surgery theory to the cobordism theory of codimension-two imbeddings and its relation to the middle-dimensional complementary homology. We begin by describing a technical device used to bring imbedding problems into the framework of surgery theory.

7. Definitions. We will define a homotopy-theoretic analogue to an imbedding of compact manifolds, known as a *Poincaré imbedding*. This paper will study actual imbeddings of manifolds that are modeled upon a given Poincaré imbedding—these will be called *realizations* of the Poincaré imbedding.

For a more detailed treatment (and proofs of the results stated here) see [7], [8], [12], and [34].

DEFINITION 7.1. Let M^m and V^{m+2} be compact manifolds. Then a *Poincaré imbedding* $\theta = (E, \xi, h)$ of M in V consists of

1. a 2-plane bundle ξ over M with associated unit circle and unit disk bundles $S(\xi)$, $T(\xi)$ respectively;
2. a finite *CW*-pair $(E, S(\xi))$ and a simple homotopy equivalence $h: V \rightarrow E \mathbf{U}_{S(\xi)} T(\xi)$ with the homology class

$$\text{im}(h([V])) \in H_{m+2}(E \cup T(\xi), E)$$

going by excision to a generator of the top-dimensional homology of $(T(\xi), S(\xi))$; in the nonorientable case we use homology with twisted integer coefficients. \square

REMARKS. 1. If the map h is a homotopy equivalence with Whitehead torsion an element, g , of $Wh(\pi_1(V))$ we will call θ a *g-Poincaré imbedding*.

2. If M and V have boundaries we will assume that E is a quadrad and $h: (V, \partial V) \rightarrow (E \mathbf{U}_{S(\xi)} T(\xi), F \mathbf{U}_{S(\xi)|_{\partial M}} T(\xi)|_{\partial M})$ is a simple homotopy equivalence of pairs.

3. The definition above is due to Cappell and Shaneson (see [8], § 5) and is a specialization of the usual definition found in [40].

4. Condition 2 above and Proposition 2.7 in [40] imply that $(E, S(\xi))$ is a *Poincaré pair* with local coefficients in $\mathbf{Z}\pi_1(V)$. The Poincaré imbedding θ will be called *regular* if $(E, S(\xi))$ satisfies Poincaré duality with local coefficients in $\mathbf{Z}\pi_1(E)$ —through the

remainder of this paper all Poincaré imbeddings will be assumed to be regular.

5. The composite $h^{-1}z: M \rightarrow V$, where z is the inclusion of M in $T(\xi)$ as zero-section, will be called the *underlying map* of θ ; if this map preserves orientation characters θ will be said to be *orientable*.

6. Clearly any *actual* locally-flat imbedding f of M in V induces a Poincaré imbedding $\theta_f = (E, \xi, h) \rightarrow T(\xi)$ is a tubular neighborhood of $f(M)$ and E is its complement.

7. Throughout the remainder of this paper we will make the assumption that $\ker \pi_1(E) \rightarrow \pi_1(E) \bigcup_{S(\xi)} T(\xi)$ is a *cyclic group*, which will be denoted C_θ . See [9] or [34] for a proof that this is no significant loss of generality in making this assumption. We will also use the notation G_θ for $\pi_1(E)$ —this is clearly a $\pi_1(V)$ -extension of C_θ . The class of this extension as well as C_θ are *completely determined* in Proposition 1 of [33], in the case where the underlying map of θ induces an *isomorphism* of fundamental groups and a *surjection* to second homotopy groups.

DEFINITION 7.2. Let M^m and V^{m+2} be compact manifolds, let $\theta = (E, \xi, h)$ be a g -Poincaré imbedding of M into V , and let $f: M \rightarrow V$ be an actual locally-flat imbedding with normal bundle η and associated unit disk and unit circle bundles $T(\eta)$ and $S(\eta)$, respectively. Then f will be called a *realization* of θ if there exists a map $c: E' \rightarrow E$, where $E' = \overline{V - T(\xi)}$ (identifying $T(\eta)$ with a tubular neighborhood of $f(M)$ in V) such that:

- (a) $c|S(\eta): S(\eta) \rightarrow S(\xi)$ is a bundle isomorphism;
- (b) $c \cup 1: E' \bigcup_{S(\eta)} T(\eta) = V \rightarrow E \bigcup_{E(\xi)} T(\xi)$ is a homotopy equivalence, with Whitehead torsion g , that is homotopic to h . □

REMARKS. 1. The map, c , appearing in the definition above, will be called the *complementary map* of the realization.

2. Excision and the additivity of Whitehead torsion over finite unions imply that the complementary map of any realization of a g -Poincaré imbedding will be a *homology equivalence* with respect to local coefficients in $\mathbb{Z}\pi_1(V)$ with Whitehead torsion equal to g (using the same local coefficients).

3. The remark above implies that the complementary map of any realization of a g -Poincaré imbedding is always an *integral* homology equivalence so that it induces an isomorphism in real K -theory. It follows that there exists a *unique* (up to isotopy) map of stable normal bundles covering such a complementary map. This implies that, if we regard a complementary map as defining a surgery (or dual surgery) problem there will be a *unique framing*.

With this in mind, we will omit all mention of the framing in the future.

Note that, in this definition, V' can be any manifold homotopy equivalent to V . Since we will often want to insure that V' is *homeomorphic* to V we make the following definition:

DEFINITION 7.3. Let θ and f be as in 7.1—then f will be called a *normal realization* if $(V', c \cup 1)$, where c is the complementary map of f , and (V, h) are s -cobordant. \square

One important property of *regular* Poincaré imbeddings is:

PROPOSITION 7.4. Let c be the complementary map of a realization $f: M^m \rightarrow V'^{m+2}$ of a regular Poincaré imbedding $\theta = (E, \xi, h)$ of M into V^{m+2} . Then c induces split surjections in homology and, in particular, if E' is the complement of $f(M)$ in V' , $H_i(E'; \mathbb{Z}\pi_1(E)) = H_i(E; \mathbb{Z}\pi_1(E)) \oplus K_i$ for all i , where K_i are the homology modules of the mapping cone of f . \square

Our main results in [34] and in the present paper actually characterize the *kernel modules* $\{K_i\}$, of the complementary map of realizations of a Poincaré imbedding and the Poincaré imbeddings are required to be *regular*.

Here are some examples of Poincaré imbeddings and their realizations:

EXAMPLE 7.5 (Classical Knots). Let $\theta_i = (S^1 \times D^{m+1}, \xi, h)$ be the Poincaré imbedding defined by the standard inclusion of spheres $i: S^m \rightarrow S^{m+2}$. It is well-known that all imbeddings of S^m in S^{m+2} are normal realizations of θ_i .

EXAMPLE 7.6 (Local Knots). Let $T(\xi)$ be the total space of the unit disk bundle associated to a 2-plane bundle ξ over a manifold M^m , and let $z: M \rightarrow T(\xi)$ be the inclusion as zero-section. Then Cappell and Shaneson show, in [7] that all locally-flat imbeddings of M in $T(\xi)$ homotopic to z are normal realizations of the Poincaré imbedding $\theta_z = (S(\xi) \times I, \xi, h)$ defined by z , where $S(\xi)$ is the unit circle bundle associated to ξ .

EXAMPLE 7.7 (Parametrized Knots). Let $f: S^n \times M^m \rightarrow S^{n+2} \times M^m$ denote the imbedding $i \times 1$, where i is the standard inclusion of S^n in S^{n+2} . Imbeddings homotopic to f were first studied by Cappell and Shaneson in [7] in the case where M is simply-connected and closed. The general case was studied by Ocken in his thesis

[19] under the additional assumptions that the imbedding is homotopic to $i \times 1$ relative to $S^n \times \partial M$. They showed that all imbeddings of this type are normal realizations of the Poincaré imbedding $\theta_f = (D^{n+1} \times M \times S^1, \xi, h)$, where ξ is a trivial bundle.

Before we can state an example for knotted lens spaces we must discuss some of the algebraic invariants of homotopy lens spaces. Let n be an odd integer and let R_n be the ring of algebraic integers in a cyclotomic field generated by a primitive n th root of unity, τ , (which will be fixed for the remainder of this discussion). If L^{2k-1} is a homotopy lens space of index n , $\Delta(L)$ will denote its Reidemeister torsion (see [40] for a definition) and $d(L) \in \mathbf{Z}_n$ will denote its image in I_n^k/I_n^{k+1} , where I_n is the principal ideal of R_n generated by $\tau - 1$; see [40], p. 205 for a proof that $I_n^k/I_n^{k+1} = \mathbf{Z}_n$. Theorem 14E.3 on p. 207 of [40] proves that $d(L)$ determines the homotopy type of L in a given dimension and $\Delta(L)$ determines its simple homotopy type. The exact sequence on p. 32 of [18] shows that $Wh(\mathbf{Z}_n)$ is isomorphic to the quotient of the subgroup of the group of units of R_n mapping to 1 under $f: R_n \rightarrow R_n/I_n = \mathbf{Z}_n$ by the subgroup of n th roots of unity, i.e., the Reidemeister torsion of a complex that is acyclic over $\mathbf{Z}[\mathbf{Z}_n]$ will be a unit of R_n . We will usually regard elements of $Wh(\mathbf{Z}_n)$ as multiples of units of R_n by arbitrary n th roots of unity.

Our main result is:

EXAMPLE 7.8 (Knotted Lens Spaces). Let L_1^{2k-1} and L_2^{2l+1} be homotopy lens spaces of index n , i.e., quotients of spheres by free \mathbf{Z}_n -actions, and suppose there exists a locally-flat imbedding of L_1 in L_2 . Then all locally-flat imbeddings of L_1 in L_2 are normal realizations of the g -Poincaré imbedding $\theta_l = (S^1 \times D^{2k}, \xi, h)$, where $g = \Delta(L_1)(\tau^d - 1)\Delta(L_2)^{-1}$, $d \cdot d(L_1) \equiv d(L_2) \pmod{n}$, and ξ is the 2-disk bundle over L_1 with Euler class e with $ed \equiv 1 \pmod{n}$.

REMARK. The discussion on p. 205 of [40] implies that the d -invariant of a homotopy lens space is always a unit of \mathbf{Z}_n so that e and d are well-defined.

See § 9 of [7] and I.9 of [34] for proofs.

We will conclude this section by defining some important geometric concepts used in classifying codimension-two imbeddings:

DEFINITION 7.9. Two imbeddings of compact manifolds, $f_i: M^m \rightarrow$

V^{m+2} , $i = 1, 2$, are said to be:

A. *conjugate* if there exist homeomorphisms $h_1: M \rightarrow M$ and $h_2: V \rightarrow V$ both pseudo-isotopic to the identity, such that $h_2 \circ f_2 = f_1 \circ h_1$;

B. *concordant* if there exists an imbedding $F: M \times I \rightarrow V \times I$ with $F(M \times \partial I) \subset V \times \partial I$, and such that $F|_{M \times 0} = f_1$ and $F|_{M \times 1} = f_2$;

C. *cobordant* if they are conjugate to concordant imbeddings. \square

This is essentially the usual geometric definition of cobordism of imbeddings (through in the case of parametrized knots it is somewhat stronger than the definition given by Cappell and Shaneson in [7] or by Ocken in [19]). In the framework of Poincaré imbeddings used in the present paper, however, we will need a stronger definition:

DEFINITION 7.10. Two realizations f_i , $i = 0, 1$, of θ will be said to be:

1. θ -*concordant* if there exists a realization, F , of the Poincaré imbedding $\theta \times I = (E \times I, \xi \times I, h \times I)$ of $M \times I$ into $V \times I$ such that $F|_{M \times \{i\}}$ and the restriction of the complementary map of F to $M \times \{i\}$ agrees with the complementary map of f_i , $i = 0, 1$;

2. θ -*cobordant* if they are conjugate to θ -concordant realizations of θ . \square

The set of θ -cobordism classes of normal realizations of θ will be denoted by $C(\theta)$.

REMARKS. The papers cited in this section prove that, for the Examples 7.5–7.8 of Poincaré imbeddings, θ -cobordism is *equivalent* to ordinary cobordism (defined in 7.9). In addition, obstruction theory and the definition of a *normal* realization of a Poincaré imbedding imply that:

PROPOSITION 7.11. *Suppose a meridian class in $\pi_1(E)$ acts trivially on $\{\pi_i(E)\}$ for $2 \leq i \leq m + 3$, so that the action factors through $\pi_1(V)$, where $\theta = (E, \xi, h)$ is a Poincaré imbedding of M^m into V^{m+2} , M and V being compact manifolds. If the map $H^i(E, S(\xi); \pi_i(E)) \rightarrow H^i(V; \pi_i(E))$, $2 \leq i \leq m + 2$, induced by collapsing $T(\xi)$ and excision, is injective, then θ -cobordism is equivalent to cobordism. \square*

8. Dual surgery theory and codimension-two imbeddings. In this section we will use dual surgery theory to study the complementary homology of *simple realizations* of a Poincaré imbedding.

These realizations are similar to *simple knots*—their complementary maps are highly-connected. We will use ideas first applied by Cappell and Shaneson to the codimension-two splitting problem and to the study of invariant spheres under finite cyclic group actions in [7].

Throughout this section and the next the following conventions will be in effect:

8.1. 1. As in the preceding section, $\theta = (E, \xi, h)$ will denote a Poincaré imbedding of M^m into V^{m+2} , where M and V are compact manifolds and θ is induced by an actual imbedding that carries the boundary of M transversely to that of V ;

2. $\mathfrak{F}: \mathbf{Z}\pi_1(E) \rightarrow \mathbf{Z}\pi_1(E \mathbf{U}_{S(\xi)} T(\xi)) = \mathbf{Z}\pi_1(V)$ is the map induced by inclusion;

3. θ is regular—see the remarks following 7.1;

4. the underlying map of θ induces an isomorphism of fundamental groups and a surjection of π_2 —this implies that $\pi_1(E) = G_\theta$ as defined in remark 7 following 7.1, and that the kernel of the homomorphism of fundamental groups inducing \mathfrak{F} above is the cyclic subgroup C_θ .

DEFINITION 8.2. A realization of θ will be called *simple* if its complementary map is $[(m + 1)/2]$ -connected. □

REMARK. Lemma 5.1 implies that every realization of θ is concordant to a simple realization if $m \geq 3$ and $\partial V = \emptyset$ or $m \geq 4$.

PROPOSITION 8.3. Let $c: E' \rightarrow E$ be the complementary map of a normal realization of θ and suppose $m \geq 3$ if $\partial V = \emptyset$, ≥ 4 otherwise. Then c is normally cobordant (though not necessarily rel $S(\xi)$) to the identity map of E .

Proof. Consider the map $c \cup 1: E' \mathbf{U}_{S(\xi)} T(\xi) \rightarrow E \mathbf{U}_{S(\xi)} T(\xi) = V$. By the definition of a *normal realization* of θ , this map is *s-cobordant* to the identity map of V . Let $G: (W; V, E' \mathbf{U}_{S(\xi)} T(\xi)) \rightarrow V$ be an *s-cobordism*. The *s-cobordism theorem* implies that there exists a homeomorphism $F: V \times I \rightarrow W$ whose restriction to $V \times 0$ is the identity map and if $f = F|V \times 1$, we get a homotopy $G \circ F: V \times I \rightarrow V$ between $(c \cup 1) \circ f$ and the identity map of V . This map can be made transversal to $M \subset V$ (this is the imbedding that defines θ) without altering it on $V \times \partial I$ —assume this done and let $Q = (G \circ F)^{-1}(M)$. This will be a (locally flat) submanifold of $V \times I$. If R is a regular neighborhood of Q , we may assume that $R \cap (V \times \partial I) =$

$F^{-1}(T(\xi) \times \partial I)$, where $T(\xi) \times 0$ is a tubular neighborhood of the imbedding of M in V that defines θ . Then $G: \overline{W - F(R)} \rightarrow E$ is a normal cobordism between c and the identity map of E . \square

Let Ω denote the following diagram:

$$\begin{array}{ccc} Z\pi_1(E) & \xrightarrow{1} & Z\pi_1(E) \\ 1 \downarrow & & \downarrow \mathfrak{F} \\ Z\pi_1(E) & \xrightarrow{\mathfrak{F}} & Z\pi_1(V) . \end{array}$$

REMARK. Note that, under the assumptions in effect in this section, the inclusion of $S(\xi)$ in E induces an *isomorphism* of fundamental groups, by 1.1 in [33].

The following result relates the methods of Cappell and Shaneson in [7] (particularly Theorem 3.3) to dual surgery theory:

PROPOSITION 8.4. *Suppose that $m \geq 5$ if $\partial V = \phi$, or ≥ 4 otherwise. Let $c: (E', S(\xi)) \rightarrow (E, S(\xi))$ be the complementary map of a normal realization of θ and let $F_c: (W; E, E') \rightarrow E \times I$ be a normal cobordism of c rel ∂V to the identity map of E (use a Morse function on the cobordism constructed in 8.3 to get a mapping to $E \times I$). Then:*

1. *the homology surgery obstruction of F_c rel $\partial V \times I \cup E \cup E'$ is the same as the homology surgery obstruction rel E' ;*
2. *the homology surgery obstruction of F_c rel E' is the image of the dual surgery obstruction of c under the isomorphism $h: D_{m+2}^s(\mathfrak{F}) \rightarrow \Gamma_{m+3}^s(\Omega)$ defined in the discussion preceding 4.3 and in Appendix A.*

Proof. Statement 1 is an immediate consequence of the algebraic description of relative homology surgery theory given in Appendix A and the fact that $F_c|_{\partial V \times I \cup E}$ is a *simple homotopy equivalence* (i.e., taking the obstruction rel $\partial V \times I \cup E \cup E'$ merely means that we have to *collapse* the quadratic invariant of this part of the boundary—but the quadratic invariant of $\partial V \times I \cup E$ is *simply contractible* so collapsing *it* does not alter the simple homotopy type of the final quadratic pair that represents the obstruction in $\Gamma_{m+3}^s(\Omega)$).

Statement 2 follows from the discussion preceding 4.3 and a bordism-theoretic description of homology surgery obstructions like that given in Chapter 9 of [40]. In this description two surgery problems have the *same* surgery obstruction if there exist *simultaneous normal cobordisms* of their *domain* and *range* (see Theorem 9.3 in [40]). The conclusion follows upon observing that, for Δ

sufficiently small, $F_c^{-1}([1 - \Delta, 1] \times E)$ is a product neighborhood, $E' \times [1 - \Delta, 1]$, of E' in W . Furthermore $E \times [1 - \Delta, 1]$ and $E' \times [1 - \Delta, 1]$ are simultaneously normally cobordant rel E and E' to $E \times I$ and W , respectively (here we use an argument similar to that used on p. 12 of [40] to show that surgery on the boundary of a manifold gives a result that is normally cobordant to the manifold). \square

At this point we can complete the argument used by Cappell and Shaneson in § 10 of [7] to obtain:

THEOREM 8.5. *Suppose that $m \geq 3$ if $\partial V = \phi$, or ≥ 4 otherwise, and that the image of $L_{m+3}^s(G_\theta)$ in $L_{m+3}^s(\pi_1(V))$ acts trivially on V^{m+2} —see p. 111 of [40]. Then a class $x \in D_{m+2}^s(\mathfrak{F})$ is the dual surgery obstruction of the complementary map of a normal realization of θ if and only if there exists $y \in L_{m+1}^s(\pi_1(M))$, that acts trivially on M (see p. 111 of [40]), such that $q_\xi(y) = p(x)$ where:*

1. $p: D_{m+2}^s(\mathfrak{F}) \rightarrow L_{m+3}^s(\mathfrak{F})$ is the homomorphism defined in Remark 3 following 4.3;
2. $q_\xi: L_{m+1}^s(\pi_1(M)) \rightarrow L_{m+3}^s(\mathfrak{F})$ is the homomorphism defined by inducing degree-1 normal maps over the 2-disk bundle, $T(\xi)$. \square

REMARKS. 1. The homomorphism, q_ξ , may be described as follows: If $y \in L_{m+1}^s(\pi_1(M))$ is the surgery obstruction (rel boundary) of $f: W \rightarrow M \times I$, then $q_\xi(y)$ is the relative surgery obstruction of $f': (T(f^*(\xi \times I)), S(f^*(\xi \times I))) \rightarrow (T(\xi \times I), S(\xi \times I))$, where f^* denotes the pullback.

2. The theorem above, coupled with the results of §§ 5 and 6 makes it possible to characterize the complementary homology of a simple realization of a Poincaré imbedding:

COROLLARY 8.6. *Suppose C_θ (see Remark 7 following 7.1) is infinite cyclic and $\pi_1(V)$ is a finite extension of a polycyclic group. Let $\bar{A} = \mathbf{Z}G_\theta[S^{-1}]$ denote the localization defined in the beginning of § 6. Suppose $m \geq 3$ if $\partial V = \emptyset$ or ≥ 4 otherwise. If $\{A_i\}$ is a sequence of $\mathbf{Z}G_\theta$ -modules, $1 \leq i \leq m + 2$ then the $\{A_i\}$ are the complementary homology modules of a simple realization of θ if and only if either of the following two conditions are satisfied:*

1. $m = 2k + 1$, $A_i \cong H_i(E; \mathbf{Z}G_\theta)$ for $i \neq k$, and $A_k \cong H_k(E; \mathbf{Z}G_\theta) \oplus A$, where A is a finitely generated \bar{A} -torsion module that possesses linking and self-linking forms b and q , respectively, such that the t -form (A, b, q) represents an element of $D_{2k+1}^s(\mathfrak{F})$ (see § 6) that satisfies the conclusions of the previous theorem;
2. $m = 2k$, $A_i \cong H_i(E; \mathbf{Z}G_\theta)$ for $i \neq k - 1, k$, and $A_{k-1} \cong H_{k-1}(E; \mathbf{Z}G_\theta) \oplus A$, $A_k \cong H_k(E; \mathbf{Z}G_\theta) \oplus B$ where A and B are finitely

generated \bar{A} -torsion modules such that:

- (a) A possesses a free resolution of the form

$$0 \longrightarrow F' \longrightarrow F \xrightarrow{\text{ad } \lambda} F^* \longrightarrow A \longrightarrow 0$$

where (F, λ, μ) is a special $(-1)^k$ -hermitian form with pre-subkernel (see p. 286 of [7]) equal to $\text{im } F'$ (when regarding it as defining an element of $\Gamma_{2k}^s(\mathfrak{F})$) and such that the element of $D_{2k}^s(\mathfrak{F})$ represented by the quadruple $(F, \lambda, \mu, \text{im } F' \subset F \otimes_{\mathbf{Z}G_\theta} A')$ (see the end of §5) satisfies the conclusions of the preceding theorem;

- (b) $B \cong \text{Hom}_{\mathbf{Z}G_\theta}(A, \bar{A}/\mathbf{Z}G_\theta)$.

Proof. This follows from 8.5, the torsion-theoretic description of odd-dimensional dual surgery theory in § 6 (especially 6.14), the description of even-dimensional dual surgery theory at the end of § 5 (and the discussion of the obstructions following 5.11) and 2.14 (and the discussion following it). Statement 2b follows from Poincaré duality and an argument like that used in the proof of 6.5, which shows that $\text{Hom}_{\mathbf{Z}G_\theta}(A, \bar{A}/\mathbf{Z}G_\theta) \cong \text{Ext}_{\mathbf{Z}G_\theta}^1(A, \mathbf{Z}G_\theta)$. □

Now we will show that, in many cases, the dual surgery obstruction of the complementary map of a realization of θ determines the cobordism class of the realization. First note the following proposition:

PROPOSITION 8.7. *Let \mathcal{S} denote the identity map of $\pi_i(S(\xi))$. Then relativization induces a natural isomorphism*

$$D_i^s(\mathfrak{F}) \longrightarrow D_i^s(\mathcal{S} \longrightarrow \mathfrak{F})$$

for all i , where $\mathcal{S} \rightarrow \mathfrak{F}$ is induced by the inclusion of $S(\xi)$ in E and V .

Proof. This is a direct consequence of the fact that $D_i^s(\mathcal{S})=0$ for all i (see 3.2) and the existence of a long exact sequence

$$\dots \longrightarrow D_i^s(\mathcal{S}) \longrightarrow D_i^s(\mathfrak{F}) \longrightarrow D_i^s(\mathcal{S} \longrightarrow \mathfrak{F}) \longrightarrow D_{i-1}^s(\mathcal{S}) \longrightarrow \dots$$

(which is implied by 4.5). □

This implies that:

PROPOSITION 8.8. *Taking the dual surgery obstruction of complementary maps of normal realizations of θ gives rise to a well-defined map $t: C(\theta) \longrightarrow D_{m+2}^s(\mathfrak{F})$.*

Proof. This follows from the fact that complementary maps of θ -cobordant normal realizations of θ are homology s -cobordant in

such a way that the induced cobordism of $S(\xi)$ is an *actual s-cobordism*. □

THEOREM 8.9. *Let $m \geq 4$, or if $S(\xi) = S^m \times S^1$ and $\partial V = \emptyset$ let $m \geq 3$. Then the map t defined in the preceding proposition, is injective—i.e., two normal realizations of θ are θ -cobordant if and only if their complementary maps have the same dual surgery obstruction.*

REMARKS. 1. This implies that, for many classes of codimension-two imbeddings, the set of cobordism classes of imbeddings has a natural group structure.

2. Suppose that $\pi_1(V)$ is a finite extension of a polycyclic group and C_θ is infinite cyclic and that the homology of E , the complement of θ , is \bar{A} -torsion free (with local coefficients in $\mathbb{Z}G_\theta$). If $c: E' \rightarrow E$ is the complementary map of a normal realization of θ it is not difficult to see that the homomorphism, c_* , induced in homology by c will be that induced by factoring out the \bar{A} -torsion submodules of the homology modules of E' and that the kernel modules of c_* can be defined *without reference to θ* .

It follows that we can define *simple realizations*, in this case, without reference to θ and if M and V are *odd-dimensional* we can extract from the homology of E' an *intrinsic and complete θ -cobordism invariant*, using the results of § 6.

Proof. We will assume, at first, that $m \geq 4$. Let $c_i: E_i \rightarrow E$, $i = 1, 2$ be complementary maps of normal realizations of θ that have the same dual surgery obstruction. Proposition 8.4 implies that there exist cobordisms $F_i: (W_i; E, E_i) \rightarrow E \times [0, (-1)^i]$, such that the relative homology surgery obstructions of the F_i in $\Gamma_{m+3}^s(\Omega)$ are equal. If we form the union along E we get a cobordism $F_1 \cup F_2: (W_1 \cup W_2; E_1, E_2) \rightarrow E \times [-1, +1]$ whose homology surgery obstruction in $\Gamma_{m+3}^s(\Omega)$ is zero. It follows that we can perform surgery on $W_1 \cup W_2$ rel $\partial V \times I \cup E_1 \cup E_2$ to get a homology s -cobordism $F': (W'; E_1, E_2) \rightarrow E \times [-1, +1]$ such that the induced cobordism of $S(\xi)$ is an *actual s-cobordism*. The conclusion follows from the s -cobordism theorem, and in the case where $m = 3$ and $S(\xi) = S^3 \times S^1$, from Shaneson's results in [29]. □

9. **Knotted lens spaces.** In this section we will apply the methods developed in this paper to codimension-two imbeddings of homotopy lens spaces. Such imbeddings have been studied before as invariant knots under a free \mathbb{Z}_n -action on spheres. Cappell and Shaneson obtained a cobordism classification of such imbeddings in

[7] and Stoltzfus showed, in [37], that the cobordism theory of such imbeddings could be formulated in a manner similar to Levine's classification of high-dimensional knots in [15] and [16]. Using a somewhat different approach, we will determine the cobordism classes of knotted lens spaces as well as their relation to the complementary homology. The results in this section will apply in the *PL-category* though many of them can be rephrased in the smooth and topological categories.

Before we can proceed we must recall some of the invariants of homotopy lens spaces and their properties. See Chapter 14.E of [40] for proofs. Throughout this section n will denote an *odd* integer and T will denote a *preferred* generator of Z_n .

9.1. A homotopy lens space L^{2k+1} of index n (i.e., S^{2k+1}/Z_n), and of dimension ≥ 5 is determined up to *PL-homeomorphism* by the following invariants (other than n and k):

1. $\Delta(L)$ —the Reidemeister torsion—see 7.8 and the discussion preceding it;

2. $\rho(L)$ —this invariant is defined on p. 175 of [40]. It takes its values in $\mathbf{Q}\hat{R}_n = \mathbf{Q}[\hat{Z}_n]/(\hat{z})$, where \hat{Z}_n is the Pontryagin dual of Z_n and (\hat{z}) is the ideal generated by the sum of the elements of \hat{Z}_n . $\mathbf{Q}\hat{R}_n$ is isomorphic, as a ring, to $\mathbf{Q}(\tau)$.

REMARKS. 1. Throughout this section $\chi \in \hat{Z}_n$ will denote the representation of Z_n in S^1 that maps the preferred generator, T , to $\exp(2\pi i/n)$.

2. Recall, from the discussion preceding 7.8, that $\Delta(L^{2k-1})$ determines the *simple* homotopy type of L and its image in $I_n^k/I_n^{k+1} = Z_n$ determines the *homotopy* type of L .

9.2. An element, x , of the group $L_{2k+2}^h(Z_n) = L_{2k+2}(0) \oplus \tilde{L}_{2k+2}^h(Z_n)$ is completely characterized by:

1. $D(x)$ —this is the Whitehead torsion of the adjoint map of a hermitian form representing x and is regarded as being a unit in R_n (see 7.8 and the discussion preceding it);

2. $M(x)$ —this is the multisignature of x —see p. 165 of [40] for a definition. It takes its values in $4\hat{R}_n \subset \mathbf{Q}\hat{R}_n$, where $\hat{R}_n = \mathbf{Z}[\hat{Z}_n]/(\hat{z})$;

3. $c(x)$ (if k is odd)—this is the Arf invariant, or the signature (if k is even).

REMARKS. 1. See [1], [2] or [41] for proofs of these statements.

2. Our definition of $M(x)$ is slightly different from that of

Wall given in Chapter 13A of [40]. In our definition $M(x)$ is the restriction of Wall's multisignature (which is the character of a representation of Z_n) to $Z_n - \{0\}$. In the Pontryagin dual \hat{Z}_n , this corresponds to dividing out by the regular representation of Z_n so our multisignature takes its values in $Z[\hat{Z}]/(\hat{z}) = \hat{R}_n$ (and, in fact, in $4\hat{R}_n$). The definition used in Chapter 14E of [40], however, corresponds to that given here.

Recall the definition of a suspension of a homotopy lens space in Chapter 14A of [40]—also see the proof of 1.8 in [34]. The following result describes the way the invariants described above interact with each other:

9.3. *Let L_1 and L_2 be homotopy lens spaces of index n and of dimension ≥ 3 . Then:*

1. *If $\dim(L_1) = \dim(L_2) = 2k + 1$, there exists a normal cobordism W between L_1 and L_2 if and only if*

(a) $\Delta(L_1) \equiv \Delta(L_2) \pmod{I_n^{k+2}}$ —see the discussion preceding 7.8;

(b) $\rho(L_2) - \rho(L_1) = t \in 4\hat{R}_n \subset Q\hat{R}_n$;

in which case $\Delta(L_2) \cdot \Delta(L_1)^{-1}$ and t are the Whitehead torsion and multisignature, respectively, of the surgery obstruction of any normal cobordism between them that maps to L_1 .

2. *If L is a suspension of L_1 via a free action of Z_n on S^1 with the preferred generator acting as χ^e (see the remark following 9.1) then:*

(a) $\rho(L) = \rho(L_1)(1 + \chi^e)/(1 - \chi^e)$;

(b) $\Delta(L) = \Delta(L_1)(\tau^d - 1)$, where $d \cdot e \equiv 1 \pmod{n}$ and τ is the image of T under the isomorphism $Z[Z_n]/(z) \rightarrow R_n$.

At this point, all that remains to be done before we can state our main results on knotted lens spaces is to describe the criteria developed by Cappell and Shaneson in [7] for the existence of codimension-two imbeddings of homotopy lens spaces:

THEOREM 9.4. *Let L_1^{2k-1}, L_2^{2k+1} , $k \geq 2$, be homotopy lens spaces of index n . Then there exists a locally-flat imbedding of L_1 in L_2 if and only if L_2 is normally cobordant to a suspension of L_1 , i.e., if and only if $\rho(L_2) - \rho(L_1)(1 + \chi^e)/(1 - \chi^e) = t \in 4\hat{R}_n$, where $e \cdot d(L_2) \equiv d(L_1) \pmod{n}$ —see the discussion preceding 7.8.*

Proof. This is a direct consequence of Theorem 9.4 in [7]. The condition in that theorem is state slightly differently—it says that L_1 must be normally cobordant to a *desuspension* of L_2 . The latter condition is equivalent to the condition that $\rho(L_2)(1 - \chi^e)/(1 + \chi^e) - \rho(L_1) \in 4\hat{R}_n$, since normal cobordism implies homotopy

equivalence and χ^e defines the only suspension of L_1 that can be homotopy equivalent to L_2 (see the proof of 1.9 in [34]). The statement of the theorem then follows from the fact that $(1+\chi^e)/(1-\chi^e)$ is a unit in \widehat{R}_n . \square

Let $D_n^h(\mathfrak{F})$ be the dual surgery obstruction groups defined like the $D_n^s(\mathfrak{F})$, except that no attention is paid to preferred bases and Whitehead torsion. In this case we also define a map $p: D_n^h(\mathfrak{F}) \rightarrow L_{h+1}^h(\mathfrak{F})$ —it has essentially the same properties as the map described in Appendix A.

Throughout the remainder of this section $\mathfrak{F}: Z[Z] \rightarrow Z[Z_n]$ will denote the homomorphism induced by reduction of elements of the group of integers mod n .

DEFINITION 9.5. If $x \in D_{2k+1}^h(\mathfrak{F})$, its *signature* is defined to be the image of x in $L_{2k+1}^h(Z) = L_{2k}^h(0)$. \square

REMARKS. 1. If k is even the signature is an element of Z and if k is odd, an element of Z_2 .

2. Using the results of [24] it is possible to give an explicit algorithm for the computation of the signature. If x is represented by the split formation $(F, \left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix} \right), \theta)G$, where γ and μ are $m \times m$ matrices then choose $m \times m$ matrices r' and s' such that the matrix

$$\begin{pmatrix} \gamma & r' \\ \mu & s' \end{pmatrix}$$

is invertible. The proof of Theorem 1.1 in [23] implies that the matrix

$$\Delta = \begin{pmatrix} \gamma & r \\ \mu & s \end{pmatrix}$$

defines an element of $SU_{2m}(Z[Z])$, where $r = r'\alpha - \gamma\alpha^*r'^*s'\alpha$ and $s = s'\alpha - \mu\alpha^*r'^*s'\alpha$ and $\alpha = (\gamma^*s' + \varepsilon\mu^*r')^{-1}$ (the proof of the theorem cited above in [23] also implies the invertibility of $\gamma^*s' + \varepsilon\mu^*r'$).

Now Δ is a $2m \times 2m$ matrix that defines an automorphism of the standard kernel on $F \oplus F^*$ and its entries are Laurent polynomials in t (after identifying $Z[Z]$ with $Z[t, t^{-1}]$). Let l denote the largest exponent of t occurring in the entries of Δ . Since the inverse of Δ is

$$\begin{pmatrix} s^* & \varepsilon r^* \\ \varepsilon \mu^* & \gamma^* \end{pmatrix}$$

it follows that l is \geq the parameter N in the definition of \bar{B} in

[24]. Let F_0 denote the free abelian group on a basis that is the standard basis of F (over $Z[Z]$) and let F_0^* denote its image under the adjoint of the bilinear form on $F \oplus F^*$. Now define the finitely generated free abelian group (of rank $2ml$)

$$K = \bigoplus_{j=0}^{l-1} t^j(F_0 \oplus F_0^*)$$

and, for $x, y \in K$, set $r(x, y) = [(\mathcal{J}_i(\Delta x))^* \cdot \mathcal{J}_i(\tilde{\Delta} y)]_0$, where:

1. $\tilde{\Delta}$ denotes the matrix $\begin{pmatrix} \mu & s \\ 0 & 0 \end{pmatrix}$;
2. $\mathcal{J}_i: F \oplus F^* \rightarrow F \oplus F^*$ is the Z -linear map whose restriction to $K \subset F \oplus F^*$ is the identity and which maps $t^i(F_0 \oplus F_0^*) \subset F \oplus F^*$ to zero if $i \geq l$;
3. $[\]_0$ denotes the constant term of a polynomial in t . Then r is a sesquilinear form on K and:
 - A. If k is *even* the signature of x is $1/8$ of the signature of the symmetrization of r ;
 - B. if k is *odd* the signature of x is the Arf invariant of r (calculated with respect to a maximal symplectic set for the skew-symmetrization of r).

If the signature of a formation *vanishes*, the exact sequence 4.1 implies that it will be stably isomorphic to a *split graph formation*—see Remark 3 following 5.5.

PROPOSITION 9.6. A *split formation* $x = \left(E, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta \right) G \right)$ is stably isomorphic to a *split graph formation* if and only if there exists a $(-\varepsilon)$ -Hermitian kernel (see [40], p. 47) (K_{2j}, α, β) and a matrix Ω such that $\det \{ \gamma \oplus I_{2j} + (\Omega - \varepsilon\Omega^*)(\mu \oplus \alpha) \} = \pm t^i$ for some i .

If this condition is satisfied then x is stably isomorphic to the *split graph formation*:

$$\left(F, \left(\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}, \theta \right) F \right)$$

where $\varepsilon = \gamma^* \mu \oplus \alpha^* + (\mu^* \oplus \alpha^*)(\Omega - \varepsilon\Omega^*)(\mu \oplus \alpha)$ and

$$\theta = \theta \oplus \beta + (\mu^* \oplus \alpha^*)\Omega(\mu \oplus \alpha).$$

REMARKS. 1. I_{2j} denotes the identity matrix with $2j$ rows and β actually denotes a $2j \times 2j$ matrix such that $\alpha = \beta - \varepsilon\beta^*$.

2. If $x \in D_{2k+1}^h(\mathfrak{F})$ has signature zero then x will be represented by a split formation that satisfies the hypothesis of the proposition above. In this case we will define the *multisignature* of x to be the multisignature of the $(-1)^{k+1}$ -Hermitian bilinear form $(F, \varepsilon) \otimes_{Z[Z]} Z[Z_n]$ as defined on p. 165 of [40].

Proof. This is a direct consequence of the definition of an isomorphism of split formations (5.3) with $\gamma' = 1$. □

We are now in a position to state the main result of this section:

THEOREM 9.7. *Let L_1^{2k-1}, L_2^{2k+1} be homotopy lens spaces, $k \geq 3$, of index n and suppose there exists an imbedding of L_1 in L_2 . Then an element $x \in D_{2k+1}^h(\mathfrak{F})$ is the dual surgery obstruction of the complementary map of a normal realization of the Poincaré imbedding θ_i defined in 7.8 if and only if:*

1. x has a signature equal to $j \cdot n$ for some integer j ;
2. the multisignature of $x - j \cdot \mathcal{N}_{2k+1,n}$ is equal to

$$\rho(L_2) - \rho(L_1)(1 + \chi^e)/(1 - \chi^e),$$

where $e \cdot d(L_2) \equiv d(L_1) \pmod{n}$;

3. the Reidemeister torsion of x is $t = \Delta(L_1)(\tau^d - 1)\Delta(L_2)^{-1}$, where $e \cdot d \equiv 1 \pmod{n}$.

Furthermore, two such imbeddings are cobordant if and only if the dual surgery obstructions of their complementary maps are equal.

REMARKS. 1. In the case $k = 2$, Rokhlin's theorem implies that the signature of x must be a multiple of $2n$.

2. See Appendix B for a definition of the elements $\mathcal{N}_{2k+1,n} \in D_{2k+1}^h(\mathfrak{F})$.

3. The Reidemeister torsion of an element of $D_{2k}^h(\mathfrak{F})$ represented by a split formation $\left(F, \left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta \right) G \right)$ is defined to be $\det(\mu)(\tau)$.

Proof. The proof of this theorem is very similar to that of 8.5 so that we will only indicate the differences. First, instead of L_2 being s -cobordant to the identity map of the suspension of L_1 with invariant χ^e we only know that L_2 is normally cobordant to it (by 9.4) and that the surgery obstruction of any normal cobordism has multisignature $\rho(L_2) - \rho(L_1)(1 + \chi^e)/(1 - \chi^e)$ and Whitehead torsion t . The remainder of the proof of 8.5 implies that x can be the dual surgery obstruction of the complementary map of a normal realization of θ_i if and only if $p(x) = q_\varepsilon(y) + B$, where $y \in L_{2k}^h(\mathbf{Z}_n)$ acts trivially on L_1 and $B \in L_{2k+2}^h(\mathfrak{F})$ is the image of the surgery obstruction of a normal cobordism between L_2 and the suspension of L_1 with invariant χ^e . The theorem now follows from the fact that the subgroup of $L_{2k}^h(\mathbf{Z}_n)$ that acts trivially on L_1 is precisely $L_{2k}(0)$ and from the results in Appendix B. □

Let $A = \mathbf{Z}[\mathbf{Z}]$ and $\bar{A} = A[S^{-1}]$, where S is the multiplicatively closed set of all Laurent polynomials with augmentation ± 1 whose exponents are all multiples of n .

DEFINITION 9.8. Let $u = \sum_0^{(n-1)/2} 2\alpha_i(\chi^i + \varepsilon\chi^{-i}) \in \hat{R}_n$ and $\Delta \in R_n$ be such that $\text{sign}(\chi^i(\Delta)) = (-1)^{\alpha_i}$. Then $\mathcal{L}_\varepsilon(u, \Delta)$ denotes a special ε -Hermitian form (F, λ, μ) with multisignature u and Reidemeister torsion Δ , and $\mathcal{G}_\varepsilon(u, \Delta)$ denotes the associated graph form—i.e., a t -form representing the image of $\mathcal{L}_\varepsilon(u, \Delta)$ in $L_{2k+2}^h(\mathfrak{F})$ (see B.1 in Appendix B). □

REMARK. The existence of $\mathcal{L}_\varepsilon(u, \Delta)$ follows from 13A.5 in [40]. Its class as an element of $L_{2k}^h(\mathbf{Z}_n)$ is clearly uniquely determined by u and Δ .

THEOREM 9.9. Let L_1^{2k-1}, L_2^{2k+1} be homotopy lens spaces $k \geq 3$, of index n and suppose there exists a locally-flat imbedding of L_1 in L_2 . Then a A -module, T , can be the middle-dimensional homology module of the complement of a simple imbedding of L_1 in L_2 if and only if T is a finitely generated \bar{A} -torsion module with a short free resolution and such that there exists an isomorphism $b: T \rightarrow \text{Hom}_1(T, \bar{A}/A)$ with the following properties:

1. there exists a self-linking form $q: T \rightarrow \bar{A}_k$ (see 6.7) compatible with b so that the triple (T, b, q) constitutes a t -form defining an element of $D_{2k+1}^h(\mathfrak{F})$;
2. there exist \mathfrak{F} -trivial graph forms $G_i, i = 1, 2$, (see B.1 in Appendix B) such that $(T, b, q) \oplus G_1$ is isometric to $G_2 \oplus \mathcal{G}_\varepsilon(u, \Delta) \oplus j \cdot \mathcal{B}_{2k+1, n}$ where:
 - (a) j is some integer and $\varepsilon = (-1)^k$;
 - (b) $u = \rho(L_2) - \rho(L_1)(1 + \chi^e)/(1 - \chi^e), e \cdot d(L_2) \equiv d(L_1) \pmod{n}$;
 - (c) $\Delta = \Delta(L_1)(\tau^d - 1)\Delta(L_2)^{-1}, e \cdot d \equiv 1 \pmod{n}$. □

REMARKS 1. See Appendix B for a definition of the t -form $\mathcal{B}_{2k+1, n}$.

2. Statement 2c is equivalent to the statement that the Alexander polynomial (see [34]) of T evaluated at a primitive n th root of unity is Δ (given above) times some n th root of unity.

3. If L_2 is a suspension of L_1 the statement of the theorem above is valid with $\mathcal{G}_\varepsilon(u, \Delta)$ and statements 2(b) and (c) omitted.

4. The \mathfrak{F} -trivial graph forms $G_i, i = 1, 2$, must define elements of $D_{2k+1}^s(\mathfrak{F})$ —i.e., they must be simple.

APPENDIX A. The homomorphism $p: D_n^s(\mathfrak{F}) \rightarrow L_{n+1}^s(\mathfrak{F})$. In this appendix we will give an algebraic description of relative homology

surgery obstructions and the map $p: D_n^s(\mathfrak{F}) \rightarrow \Gamma_{n+1}^s(\Phi)$ defined in § 4, where \mathfrak{F} and Φ are the diagrams:

$$\begin{array}{ccc} \mathfrak{F}: A & \longrightarrow & A' \\ & \searrow \scriptstyle 1 & \downarrow \scriptstyle \mathfrak{F} \\ & A & \longrightarrow & A' \\ \phi: 1 \downarrow & & & \downarrow \scriptstyle \mathfrak{F} \\ & A & \xrightarrow{\mathfrak{F}} & A' \end{array}$$

and all of the conventions at the beginning of Chapter I are in effect.

We begin by recalling Ranicki's definition of a quadratic triad in § 10 of [25]:

DEFINITION A.1. Let \mathcal{E} denote the following commutative square of A -chain complexes:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ f' \downarrow & & \downarrow g \\ D' & \xrightarrow{g'} & C' \end{array}$$

and define a A -chain complex, $C(\mathcal{E})$, by

$$d_{C(\mathcal{E})} = \begin{pmatrix} d_{C'} & (-1)^{i-1}g & (-1)^i g' & 0 \\ 0 & d_D & 0 & (-1)^i f \\ 0 & 0 & d_{D'} & (-1)^i f' \\ 0 & 0 & 0 & d_C \end{pmatrix}$$

$$: C(\mathcal{E})_i = C'_i \oplus D_{i-1} \oplus D'_{i-1} \oplus C_{i-2} \rightarrow C(\mathcal{E})_{i-1} = C'_{i-1} \oplus D_{i-2} \oplus D'_{i-2} \oplus C_{i-3} .$$

Then the homology groups of the diagram, \mathcal{E} , are defined by $H_i(\mathcal{E}) = H_i(C(\mathcal{E}))$. □

DEFINITION A.2. Given a commutative square, \mathcal{E} , of chain complexes let $\mathcal{E}^t \otimes_A \mathcal{E}$ denote the commutative square of $\mathbf{Z}[\mathbf{Z}_2]$ -chain complexes

$$\begin{array}{ccc} C^s \otimes_A C & \xrightarrow{f^t \otimes f} & D^t \otimes_A D \\ f'^t \otimes f' \downarrow & & \downarrow g^t \otimes g \\ D'^t \otimes_A D' & \xrightarrow{g'^t \otimes g'} & C'^t \otimes_A C' \end{array}$$

with $T \in \mathbf{Z}_2$ acting by the ε -transposition T_ε , and define the hyperhomology groups $Q_n(\mathcal{E}, \varepsilon) = H_n(\mathbf{Z}_2, C(\mathcal{E}^t \otimes_A \mathcal{E}))$. □

REMARK. An element $(\delta \mathcal{E}, \delta \psi, \delta \psi', \psi) \in Q_{n+2}(\mathcal{E}, \varepsilon)$ is represented by a collection of chains

$$(\delta \mathcal{E}_s, \delta \psi_s, \delta \psi'_s, \psi_s) \in (C'^t \otimes_A C')_{n-s+2} \oplus (D^t \otimes_A D)_{n-s+1} \oplus (D'^t \otimes_A D')_{n-s+1} \\ \oplus (C^t \otimes_A C)_{n-s}, \quad s \geq 0$$

such that

$$d_{(C'^t \otimes C)}(\delta \mathcal{E}_s) + (-1)^{n-s+1} \{ \delta \mathcal{E}_{s+1} + (-1)^{s+1} T_\varepsilon \delta \mathcal{E}_{s+1} \} \\ + (-1)^{n+1} \{ g^t \otimes g \} (\delta \psi_s) - (g'^t \otimes g') (\delta \psi'_s) = 0 \\ d_{(D^t \otimes D)}(\delta \psi_s) + (-1)^{n-s} \{ \delta \psi_{s+1} + (-1)^{s+1} T_\varepsilon \delta \psi_{s+1} \} + (-1)^n (f^t \otimes f) (\psi_s) = 0 \\ d_{(D'^t \otimes D')}(\delta \psi'_s) + (-1)^{n-s} \{ \delta \psi'_{s+1} + (-1)^{s+1} T_\varepsilon \delta \psi'_{s+1} \} + (-1)^n (f'^t \otimes f') (\psi'_s) = 0 \\ d_{(C^t \otimes C)}(\psi_s) + (-1)^{n-s-1} \{ \psi_{s+1} + (-1)^{s+1} T_\varepsilon \psi_{s+1} \} = 0$$

for all $s \geq 0$.

(Compare this with the definitions at the beginning of § 2.)

DEFINITION A.3. An $(n + 2)$ -dimensional quadratic triad, (\mathcal{E}, P) is a commutative square

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \mathcal{E}: f' \downarrow & & \downarrow g \\ D' & \xrightarrow{g'} & C' \end{array}$$

of A -chain complexes such that C is n -dimensional, D and D' are $(n + 1)$ -dimensional and C' is $(n + 2)$ -dimensional, together with an element $P = (\delta \mathcal{E}, \delta \psi, \delta \psi', \psi) \in Q_{n+2}(\mathcal{E}, \varepsilon)$. Such a triad is Poincaré if the $(n + 1)$ -dimensional quadratic pairs $(f: C \rightarrow D, (\delta \psi, \psi))$ and $(f': C \rightarrow D', (\delta \psi', \psi'))$ are Poincaré (in which case, (C, ψ) is a Poincaré complex) and such that the chain map $(1 + T_\varepsilon)P_0: C'^{n+2-*} \rightarrow C_*(\mathcal{E})$ defined by

$$(1 + T_\varepsilon)P_0 = \begin{pmatrix} (1 + T_\varepsilon)\delta \mathcal{E}_0 \\ (-1)^{n-i}(1 + T_\varepsilon)\delta \psi_0 g^* \\ (-1)^{n-i}(1 + T_\varepsilon)\delta \psi'_0 g'^* \\ 0 \end{pmatrix}: C'^{n+2-i} \longrightarrow C(\mathcal{E}) \\ = C'_i \oplus D_{i-1} \oplus D'_{i-1} \oplus D_{i-2}$$

is a simple chain equivalence. □

Now we are in a position to give an algebraic-surgery-theoretic

definition of the groups $\Gamma_{n+1}^s(\Phi)$:

PROPOSITION A.4. *The relative homology surgery groups, $\Gamma_{n+1}^s(\Phi)$, are grothendiek groups of equivalence classes of $(n + 1)$ -dimensional quadratic pairs $x = (f: C \rightarrow D, (\delta\psi, \psi))$ such that the n -dimensional quadratic complex, (C, ψ) is Poincaré and the tensor product of x with A' satisfies relative Poincaré duality and two such pairs $x_i = (f_i: C_i \rightarrow D_i, (\delta\psi_i, \psi_i))$, $i = 1, 2$, are equivalent if and only if there exists an $(n + 2)$ -dimensional quadratic triad, (E, P) with:*

$$\begin{array}{ccc} \text{\textit{E the diagram}} & C_1 \oplus C_2 & \xrightarrow{r_1 \oplus r_2} W \\ & \downarrow f_1 \oplus f_2 & \downarrow \\ & D_1 \oplus D_2 & \longrightarrow V \end{array}$$

and $P = (\delta E, \delta\psi_W, \delta\psi_1 \oplus -\delta\psi_2, \psi_1 \oplus -\psi_2)$, and such that $(r_1 \oplus r_2: C_1 \oplus C_2 \rightarrow W, (\delta\psi_W, \psi_1 \oplus -\psi_2))$ is Poincaré and the tensor product of (E, P) with A' is a Poincaré triad over A' . □

REMARKS. 1. The fact that every quadratic pair is equivalent to itself implies that the inverse of $(f: (C \rightarrow D, (\delta\psi, \psi)))$ is $(f: C \rightarrow D, (-\delta\psi, -\psi))$ so that every element of $\Gamma_{n+1}^s(\Phi)$ has a representative that is an *actual quadratic pair* rather than just a *formal difference* of two such pairs.

2. Repeated application of Proposition 2.15 implies that a quadratic pair $(f: C \rightarrow D, (\delta\psi, \psi))$ represents the zero element of $\Gamma_{n+1}^s(\Phi)$ if and only if it is possible to perform surgery on C to render it *acyclic* and on D to render it *relatively acyclic*.

3. Suppose $F: (N^{n+1}; M_-, M_+) \rightarrow (Y; X_-, X_+)$ is a relative homology surgery problem with $n \geq 5$, and with coefficients in Φ , and suppose $F|M_-$ is a simple A' -homology equivalence. In order to determine the relative homology surgery obstruction of $F \text{ rel } M_-$ in $\Gamma_{n+1}^s(\Phi)$ we must first compute the relative quadratic signature of F —suppose this is $(C_1 \oplus C_2 \rightarrow D, (\delta\psi, \psi_1 \oplus \psi_2))$, where (C_1, ψ_1) is the quadratic signature of $F|M_+$ and (C_2, ψ_2) is the quadratic signature of $F|M_-$ —then *collapse* the relatively acyclic components of the boundary that we want to remain *fixed* throughout surgery—in this case (C_2, ψ_2) —and then regard the resulting \mathfrak{F} -Poincaré pair as defining an element of $\Gamma_{n+1}^s(\Phi)$. This follows upon comparing A.4 with the bordism-theoretic definition of $\Gamma_{n+1}^s(\Phi)$ given in [7] and using Remark 2 above.

One consequence of this is:

COROLLARY A.5. *The isomorphism $i: D_n^s(\mathfrak{F}) \rightarrow \Gamma_{n+1}^s(\Phi)$ carries the*

element represented by the relatively acyclic n -dimensional Poincaré complex (C, ψ) to the class of the $(n + 1)$ -dimensional \mathfrak{F} -Poincaré pair $(0: C \rightarrow 0, (0, \psi))$ in $\Gamma_{n+1}^s(\Phi)$.

Proof. This follows from the geometric definition of the map i —i.e., if the quadratic signature of $f: M^m \rightarrow X$ is (C, ψ) we form the product with a unit interval and measure the homology surgery obstruction $\text{rel } M \times 0$. The algebraic analogue of this is forming the algebraic mapping cylinder of the identity map of (C, ψ) in the boundary. The result is clearly simple homotopy equivalent (as a pair) to $(0: C \rightarrow 0, (0, \psi))$. □

REMARK. At this point we could give a purely algebraic proof of Theorem 4.3.

In order to study the map $p: D_n^s(\mathfrak{F}) \rightarrow L_{n+1}^s(\mathfrak{F})$, defined in the third remark following Theorem 4.3 we must recall Ranicki's description of relative Wall groups:

DEFINITION A.6. The *relative Wall group*, $L_{n+1}^s(\mathfrak{F})$, is the Grothendieck group of the semigroup of equivalence classes of triples $((C, \psi), (D, \psi'), h)$ where (C, ψ) is an n -dimensional Poincaré complex over A and h is a simple homotopy equivalence $h: (C, \psi) \otimes_A A' \rightarrow \partial(D, \psi')$ and two such triples, $((C_i, \psi_i), (D_i, \psi'_i), h_i)$, $i = 1, 2$, are equivalent if and only if

1. there exists an $(n + 1)$ -dimensional quadratic complex (E, ψ'') over A and a simple homotopy equivalence $s: \partial(E, \psi'') \rightarrow (C_1, -\psi_1) \oplus (C_2, \psi_2)$;
2. there exists an $(n + 2)$ -dimensional quadratic complex (F, ψ'') over A' and a simple homotopy equivalence

$$t: \{(E, \psi'') \otimes_A A'\} \bigcup_{h_1 \oplus h_2} \{(D_1, \psi_1) \oplus (D_2, \psi_2)\} \longrightarrow \partial(F, \psi''). \quad \square$$

(See § 10 of [25] for a proof that this is equivalent to the usual definition.)

A comparison of this description of relative Wall groups with the usual bordism-theoretic one in Chapter 9 of [40] implies that:

COROLLARY A.7. 1. The homomorphism $p: \Gamma_{n+1}^s(\Phi) \rightarrow L_{n+1}^s(\mathfrak{F})$, defined in [7] carries the element of $\Gamma_{n+1}^s(\Phi)$ represented by the \mathfrak{F} -Poincaré pair $x = (f: C \rightarrow D, (\partial\psi, \psi))$ to the class of the triple $((C, \psi), (\mathbb{C}(x \otimes A')), h)$ —see the discussion following 2.12.

2. The homomorphism $p: D_n^s(\mathfrak{F}) \rightarrow L_{n+1}^s(\mathfrak{F})$ carries the element represented by the relatively acyclic n -dimensional Poincaré (C, ψ) to the class of the triple $((C, \psi), (0, 0), 0)$. □

APPENDIX B. The composite $L_{2k}(0) \rightarrow L_{2k}(\mathbb{Z}_n) \xrightarrow{q_{\mathfrak{F}}} L_{2k+2}(\mathfrak{F})$. Throughout this appendix n will denote an odd integer and $\mathfrak{F}: \mathbb{Z}[\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Z}_n]$ will denote the map induced by reduction of group elements of \mathbb{Z} mod n . In this appendix a t -form will be determined that represents the image of the generator of $L_{2k}(0) \subset L_{2k}(\mathbb{Z}_n)$ in $L_{2k+2}(\mathfrak{F})$ under the map defined in § 8.

PROPOSITION B.1. *The relative Wall group $L_{1k+2}^e(\mathfrak{F})$, where $e = s$ or h , is isomorphic to the quotient $D_{2k+1}^e(\mathfrak{F})/J$ where J is the subgroup generated by t -forms (M, b, q) satisfying the conditions:*

(a) *there exists a special $(k + 1)$ -Hermitian form (F, λ, μ) defining an element of $\Gamma_{2k+2}^e(\mathfrak{F})$ that maps to 0 under the map $p: \Gamma_{2k+2}^e(\mathfrak{F}) \rightarrow L_{2k+2}^e(\mathbb{Z}_n)$;*

(b) $M = \text{coker}(\text{ad } \lambda: F \rightarrow F^*)$;

(c) $b: M \rightarrow M^d$ maps $[y]$ to $([x] \rightarrow x(z)/s)$;

(d) $q: M \rightarrow \bar{A}_k$ maps $[y]$ to $(1/\bar{s})\mu(z)(1/s)$

where $x, y \in F^*$, and $s \in S, z \in F$ are such that $ys = (\text{ad } \lambda)(z)$.

REMARKS. 1. A t -form representing an element of the subgroup J will be called an \mathfrak{F} -trivial graph form.

2. The map $p: D_{2k+1}^e(\mathfrak{F}) \rightarrow L_{2k+2}^e(\mathfrak{F})$ is just projection to the quotient, with the interpretation of $L_{2k+2}^e(\mathfrak{F})$ given above.

Proof. This follows from the diagram in Remark 3 following 4.3, the fact that $p: \Gamma_n^e(\mathfrak{F}) \rightarrow L_n^e(\mathbb{Z}_n)$ is surjective in even dimensions and injective in odd dimensions, the 5-Lemma and the relation between formations on t -forms following 6.9. □

We are now ready to proceed. We will consider the case where k is even. The generator $1 \in L_s(0)$ is represented by the sesquilinear form

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $S + S^{tr}$ is the Milnor matrix. This bilinear form is represented geometrically by the Milnor manifold \mathcal{M}^8 .

Consider the surgery problem

$$F: L^7 \times I \# \mathcal{M}^8 \longrightarrow L^7 \times I \# D^8$$

where L^7 is some 7-dimensional lens space of index n (i.e., $\pi_1(L^7) = \mathbf{Z}_n$) and F is the identity map on $L^7 \times I$ and the map on \mathcal{M}^8 whose surgery obstruction is $1 \in L_8(\mathbf{Z}_n)$ (the connected sum is taken on $L^7 \times (0, 1)$). Let ∂B denote the total space of the nontrivial S^1 -bundle, η , on L^7 and B denote the corresponding D^2 -bundle. Then pulling this bundle back over F gives rise to the surgery problem:

$$\mathcal{F}: W \longrightarrow (B, \partial B) \times I.$$

Let V denote the quadratic signature of the surgery problem $g: \mathcal{M}^8 \rightarrow D^8: \sigma(g) = (V_*, \psi)$ where $V_i = 0$ unless $i = 4$ in which case $V_4 = \mathbf{Z}^8$ and $\psi_0 = S: V^4 \rightarrow V_4$. Since the restriction of the bundle η to \mathcal{M}^8 is *trivial* (because \mathcal{M}^8 is 4-connected) it follows that the quadratic signature of $\mathcal{F}|_{\mathcal{M}^8 \times S^1}$ is $(V, \psi) \otimes \sigma^*(S^1)$, where $\sigma^*(S^1)$ denotes the symmetric signature of S^1 given in § 14 of [25] by (C_*, φ) where:

$$\begin{aligned} C_i &= 0 \text{ unless } i = 0, 1 \\ C_0 &= C_1 = \mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[z, z^{-1}] \end{aligned}$$

and

$$d = 1 - z: C_1 \longrightarrow C_0, \quad d^* = 1 - z^{-1}: C^0 \longrightarrow C^1.$$

$$\text{B.2. } \varphi_0 = \begin{cases} 1: C^1 \longrightarrow C_0 \\ z^{-1}: C^0 \longrightarrow C_1 \end{cases} \quad \varphi_1 = 1: C^1 \longrightarrow C_1.$$

The tensor product formulae in § 11 of [25], the fact that the only nontrivial contribution to the surgery obstruction of \mathcal{F} comes from the restriction of η to \mathcal{M}^8 , the fact that the generator of the fundamental group of a *fiber* of η maps to $n \times$ (a generator of the fundamental group of ∂B) imply that:

THEOREM B.3. $q_i(1) \in L_{10}(\mathbf{Z} \rightarrow \mathbf{Z}_n)$ is represented by $((\hat{C}, \hat{\psi}), (\hat{D}, 0), h)$ (in the notation of A.6) where:

- (a) $\hat{C}_i = 0$ unless $i = 4, 5$;
 - (b) $\hat{C}_4 = \hat{C}_5 = \mathbf{Z}[\mathbf{Z}]^8$,
- $$d = 1 - t^n: \hat{C}_5 \rightarrow \hat{C}_4, \quad d^* = 1 - t^{-n}: \hat{C}^4 \rightarrow \hat{C}^5;$$
- (c) $\hat{\psi}_0 = \begin{cases} S: \hat{C}^5 \rightarrow \hat{C}_4 \\ t^{-n}S: \hat{C}^4 \rightarrow \hat{C}_5 \end{cases} \quad \hat{\psi}_1 = 0;$

- (d) $\hat{D}_i = 0$ unless $i = 4$;
- (e) $\hat{D}_4 = \mathbf{Z}[\mathbf{Z}_n]^s$;
- (f) $h = 1: \hat{C}_4 \otimes_{\mathbf{Z}[\mathbf{Z}]} \mathbf{Z}[\mathbf{Z}_n] \rightarrow \hat{D}_4.$ □

COROLLARY B.4. *The element $q_\varepsilon(1) \in L_{10}(\mathbf{Z} \rightarrow \mathbf{Z}_n)$, constructed in B.3 is represented by $((E, \mu), (0, 0), 0)$ where:*

- (a) $E_i = 0$ unless $i = 4, 5$;
- (b) $E_4 = E_5 = \mathbf{Z}[\mathbf{Z}]^s, d = S + t^n S^{tr}: E_5 \rightarrow E_4$;
- (c) $\mu_0 = \begin{cases} 0: E^s \rightarrow E_4 \\ 1 - t^{-n}: E^4 \rightarrow E_5, \end{cases} \mu_1 = S^{tr}(1 - t^n): E^4 \rightarrow E_4.$

REMARK. Note that $E \otimes_{\mathbf{Z}[\mathbf{Z}]} \mathbf{Z}[\mathbf{Z}_n]$ is acyclic so that (E, μ) defines an element of $D_{2k+1}^s(\mathfrak{S})$ that maps to $q_\varepsilon(1)$ under the map $p: D_{2k+1}^s(\mathfrak{S}) \rightarrow L_{2k+1}^s(\mathfrak{S})$.

Proof. We begin by performing algebraic surgery on (\hat{C}, ψ) via

- (a) $f_5 = 1: \hat{C}_5 \rightarrow F = \mathbf{Z}[\mathbf{Z}]^s,$
- (b) $f_i = 0: \hat{C}_i \rightarrow 0, i \neq 5.$

The result is (G, ψ') where:

- (a) $G_i = 0, i \neq 4, 5$;
- (b) $G_4 = \hat{C}_4 \oplus F, G_5 = \hat{C}_5 \oplus F^*$

$$d = \begin{bmatrix} 1 - t^n & S + t^n S^{tr} \\ -1 & 0 \end{bmatrix}: G_5 \longrightarrow G_4$$

$$(c) \quad \psi'_0 = \begin{cases} \begin{bmatrix} t^{-n} S & 0 \\ 0 & 1 \end{bmatrix}: G^4 \longrightarrow G_5 \\ \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}: G^5 \longrightarrow G_4 \end{cases}$$

$$\psi'_1 = \begin{bmatrix} 0 & -t^n S \\ 0 & 0 \end{bmatrix}: G^4 \longrightarrow G_4$$

(see 2.14) and this can easily be seen to be chain-homotopy equivalent to (E, μ) via the chain-map:

$$m_5 = (0 \ 1): G_5 \longrightarrow E_5 = \mathbf{Z}[\mathbf{Z}]^s$$

$$m_4 = (1 \ 1 - t^n): G_4 \longrightarrow E_4 = \mathbf{Z}[\mathbf{Z}]^s$$

and $m_i = 0$ if $i \neq 4, 5$.

In order to show that $((E, \mu), (0, 0), 0)$ defines the same element of $L_{2k+2}^k(\mathfrak{S})$ as $((\hat{C}, \hat{\psi}), (\hat{D}, 0), h)$ it is necessary to show that the Poincaré complex

B.5. $\{(T, 0) \otimes_{\mathbf{Z}[\mathbf{Z}]} \mathbf{Z}[\mathbf{Z}_n]\} \cup_h (\hat{D}, 0)$

is null-cobordant, i.e., is the boundary of a higher-dimensional quadratic complex (see A.6), where $(T, 0)$ is the *trace* of the algebraic surgery performed above on $(\hat{C}, \hat{\psi})$. This is given by the proof of Proposition 7.1 in [25] and is:

$$\begin{aligned} T_i &= 0 \quad i \neq 4, 5 \\ T_4 &= \hat{C}_4 \\ T_5 &= \hat{C}_5 \oplus F^* \\ d &= (1 - t^n S): T_5 \longrightarrow T_4 \end{aligned}$$

and the inclusions of the boundaries $(C, \hat{\psi}), (E, \mu)$ are given by

$$\begin{aligned} \text{(a)} \quad \hat{C}_4 &\xrightarrow{1} \hat{C}_4 = T_4 \\ \hat{C}_5 &\xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \hat{C}_5 \oplus F^* = T_5; \\ \text{(b)} \quad E_4 &\xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} \hat{C}_4 = T_4, \\ E_5 &\xrightarrow{1} \hat{C}_5 \oplus F^* = T_5. \end{aligned}$$

Direct computation, using the *union construction* (2.5) shows that the union, B.5, is *acyclic* and so, certainly null-cobordant. \square

The case where k is *odd* proceeds via a similar argument except that we begin with the sesquilinear form $(\mathbf{Z} \oplus \mathbf{Z}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ rather than (\mathbf{Z}^s, S) and the process of taking the *dual* of a map is different since the parameter, ε , is now equal to -1 . The result, in this case, is $((H, \beta), (0, 0), 0)$ where

$$\begin{aligned} \text{(a)} \quad H_i &= 0 \quad i \neq 3, 4; \\ H_3 &= H_4 = \mathbf{Z}[\mathbf{Z}]^2, \quad d = \begin{pmatrix} 1 - t^n & 1 \\ -t^n & 1 - t^n \end{pmatrix}: H_4 \longrightarrow H_3; \\ \text{(b)} \quad \beta_0 &= \begin{cases} t^{-n} - 1: H^3 \longrightarrow H_4 \\ 0: H^4 \longrightarrow H_3 \end{cases} \\ \beta_1 &= (1 - t^n) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}: H^3 \longrightarrow H_3. \end{aligned}$$

The results of this appendix can be summarized as follows:

THEOREM B.6. *Elements of $D_{2k+1}^h(\mathfrak{F})$ mapping to $q_\varepsilon(1)$ under $p: D_{2k+1}^h(\mathfrak{F}) \rightarrow L_{2k+2}^h(\mathfrak{F})$ are given by (A) the split formations $\mathcal{A}_{2k+1,n}$:*
 (i) (k odd)

$$\left(\mathbf{Z}[\mathbf{Z}]^2, \begin{bmatrix} t^{-n} - 1 & 0 \\ 0 & t^{-n} - 1 \\ 1 - t^{-n} & -t^{-n} \\ 1 & 1 - t^{-n} \end{bmatrix}, \begin{bmatrix} t^{-n} - 1 & t^{-n} - 1 \\ 0 & t^{-n} - 1 \end{bmatrix} \mathbf{Z}[\mathbf{Z}]^2 \right);$$

(ii) (*k even*)

$$\left(\mathbf{Z}[\mathbf{Z}]^8, \left(\begin{bmatrix} 1 - t^{-n} & \\ & t^{-n}S + S^{tr} \end{bmatrix}, -(1 + t^n S) \right) \mathbf{Z}[\mathbf{Z}]^8 \right);$$

(B) *the t-forms* $\mathcal{B}_{2k+1,n} = (M, b, q)$ *where:*

(i) *if k is odd*

$$M = \text{coker} \begin{bmatrix} 1 - t^n & -t^n \\ 1 & 1 - t^n \end{bmatrix} : \mathbf{Z}[\mathbf{Z}]^2 \longrightarrow \mathbf{Z}[\mathbf{Z}]^2$$

$$b([x], [y]) = -\frac{1 - t^n}{t^n + t^{-n} - 1} x^* \begin{bmatrix} t^{-n} - 1 & -t^{-n} \\ 1 & t^{-n} - 1 \end{bmatrix} y$$

$$q([x]) = \frac{(1 - t^{-n})}{(t^n + t^{-n} - 1)^2} (3 - t^n - t^{-n}) x^* \cdot x$$

(ii) *and if k is even*

$$M = \text{coker} (S + t^n S^{tr}) : \mathbf{Z}[\mathbf{Z}]^8 \longrightarrow \mathbf{Z}[\mathbf{Z}]^8$$

$$b([x], [y]) = (1 - t^{-n}) x^* (t^{-n} S + S^{tr})^{-1} y$$

$$q([x]) = -(1 + t^n) x^* \{ S S^{tr} + S^{tr} S + t^n (S^2)^{tr} + t^{-n} S^2 \}^{-1} x . \quad \square$$

REMARK. Notice that the *Reidemeister torsion* of these *t-forms* (calculated by setting *t* equal to a primitive *n*th root of unity in the matrices defining *M* and computing the determinant) is *zero* (i.e., a trivial unit in the group of units of $\mathbf{Z}[\tau]$, where τ is a primitive *n*th root of unity). This implies that the *t-forms* constructed above are lifts of $q_s(1) \in L_{2k+2}^e(\mathfrak{F})$ to $D_{2k+1}^e(\mathfrak{F})$ for *e* equal to either *s* or *h*.

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