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REPRESENTATIONS OF GAUSSIAN PROCESSES BY WIENER PROCESSES

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Let $\{X(t),\ a \leqq t \leqq b\}$ and $\{W(t),\ 0 \leqq t < \infty\}$ be a Gaussian process and the standard Wiener process, respectively. Investigating covariance structure of $X(t)$, the paper gives various representations of $X(t)$ in terms of $W(t)$, including stochastic integral representations. Some of these representations are useful in finding hitherto unknown barrier-crossing probabilities of $X(t)$.

1. Introduction. Let $X(t)$ be a Gaussian process on some interval I with covariance function

$$R(s, t) = E\{X(s) - \mu(s)\}\{X(t) - \mu(t)\} ,$$

where $\mu(t)$ is the mean function, $\mu(t) = EX(t)$.

It is well-known that a Gaussian process is uniquely determined, up to the mean function, by the covariance function $R(s, t)$.

The Gaussian process which has been studied most extensively is, of course, the Wiener process $\{W(t);\ t \geqq 0\}$. Therefore, it is natural to seek representations of Gaussian processes in terms of Wiener processes.

(i) A classical result by Doob [3, pp. 401–402] shows that: If a Gaussian process $X(t)$ has mean zero and the covariance function in the form

$$R(s, t) = u(s)v(t),\ s \leqq t$$

for s, t in some interval, and if the ratio $u(t)/v(t) \equiv a(t)$ is continuous and increasing with its inverse function $a_1(t)$, then

$$X(a_1(t))/v(a_1(t)) = W(t) ,$$

where $W(t)$ is the standard Wiener process (or the Brownian motion process).

Another well-known result is the following:

(ii) If a Gaussian process $X(t)$ with zero mean has a factorable covariance function on $[0, T]^2$, i.e.,

$$R(s, t) = \int_0^T r(s, u)r(t, u)du ,$$

where for each $t \in [0, T]$ $r(t, \cdot) \in L_2[0, T]$, then $X(t)$ has a stochastic integral representation

$$X(t) = \int_0^T r(t, u)dW(u), \ 0 \leqq t \leqq T .$$

(iii) More recently Berman [1, p. 32] gives stochastic integral representations of Gaussian processes with biconvex covariances:

$$X(t) = \sqrt{R(a, b)}Z + \int_a^t \sqrt{R_1(u, b)}dW_1(u) + \int_t^b \sqrt{-R_2(a, u)}dW_2(u)$$
$$+ \int_a^t \int_t^b \sqrt{-R_{12}(u, v)}dW(u, v) \text{ for } a \leqq t \leqq b ,$$

where $Z \sim N(0, 1)$.

$W_1(t)$ and $W_2(t)$ are standard Wiener processes on $[a, b]$ $W(s, t)$ is a standard Wiener process on $[a, b] \times [a, b]$ with Z, $W_1(t)$, $W_2(t)$, and $W(s, t)$ all mutually independent. Biconvexity guarantees that the partial derivatives satisfy: $R(a, b) \geqq 0$, $R_1(u, b) \geqq 0$, $-R_2(a, u) \geqq 0$, $-R_{12}(u, v) \geqq 0$ on respective domain.

By definition a covariance function $R(s, t)$ is symmetric. It is also nonnegative definite, for

$$\sum_{j=1}^n \sum_{i=1}^n x_i R(s_i, t_j) x_j = \sum_{j=1}^n \sum_{i=1}^n x_i E[X(s_i) - \mu(s_i)][X(t_j) - \mu(t_j)]x_j$$
$$= \left\{ \sum_{i=1}^n x_i E[X(s_i) - \mu(s_i)] \right\}^2 \geqq 0 .$$

It is also known that for each symmetric nonnegative definite function $R(s, t)$, there exists a Gaussian process whose covariance function equals $R(s, t)$. (See Doob [2, p. 72, Theorem 3.1] for reference.)

Investigating covariance structures of Gaussian processes, the paper gives numerous representations of Gaussian processes, in terms of standard Wiener processes, including stochastic integral representations. Some of these representations are demonstrated to be useful in finding hitherto unknown barrier-crossing probabilities of some Gaussian processes.

2. Main results and proofs.

In application it is often more convenient to restate the Doob's theorem in the following form:

THEOREM 1 (Doob). *If $X(t)$ is a Gaussian process with covariance function*

(2.1) $$R(s, t) = u(s)v(t) \quad (s \leqq t)$$

for s, t in some interval, and if the ratio $u(t)/v(t)$ is nondecreasing, then

$$X(t) = v(t)W[u(t)/v(t)] + \mu(t) ,$$

where $\mu(t)$ *is the mean function of* $X(t)$.

The theorem follows immediately by checking the covariance function. Another version of Theorem 1 can be stated as follows:

THEOREM 2. *If the covariance function of a Gaussian process* $X(t)$ *satisfies* (2.1) *and if the ratio* $v(t)/u(t)$ *is nonincreasing, then*

$$X(t) = u(t) W[v(t)/u(t)] + \mu(t) .$$

While Theorems 1 and 2 are very useful, they have some drawback in assuming the monotoneness of $u(t)/v(t)$. To remedy this situation, we give the following:

THEOREM 3. *If* $X(t)$ *is a Gaussian process with covariance function*

$$R(s, t) = u(s)v(t),\ u(s)/v(s) \leqq u(t)/v(t) , \tag{2.2}$$

for s, t *in some interval, then*

$$\begin{aligned} X(t) &= v(t) W[u(t)/v(t)] + \mu(t) \\ &= u(t) W[v(t)/u(t)] + \mu(t) . \end{aligned}$$

Proof. Since $R(s, t)$ is symmetric, the condition (2.2) is equivalent to:

$$\begin{aligned} R(s, t) &= u(s)v(t),\ u(s)/v(s) \leqq u(t)/v(t) \\ &= u(t)v(s),\ u(s)/v(s) > u(t)/v(t) . \end{aligned}$$

Now

$$\begin{aligned} &E\{v(s) W[u(s)/v(s)]v(t) W[u(t)/v(t)]\} \\ &\qquad = v(s)v(t) \min\{u(s)/v(s),\ u(t)/v(t)\} , \end{aligned}$$

which agrees with the $R(s, t)$. The second expression of $X(t)$ gives rise to the same covariance function.

It should be noted that if $u(t)$ or $v(t)$ vanishes at some point t_0, then $R(t_0, t_0) = E[X(t_0) - \mu(t_0)]^2 = 0$. Thus $X(t_0) = \mu(t_0)$ with probability one. Hence we regard $v(t_0) W[u(t_0)/v(t_0)] = u(t_0) W[v(t_0)/u(t_0)] = 0$.

THEOREM 4. *Let* $X(t)$ *be a Gaussian process with covariance function*

$$R(s, t) = \sum_1^\infty u_k^*(s)v_k^*(t) ,$$

where

$$u_k^*(s)v_k^*(t) = u_k(s)v_k(t),\ u_k(s)/v_k(s) \leqq u_k(t)/v_k(t)$$
$$= u_k(t)v_k(s),\ u_k(s)/v_k(s) > u_k(t)/v_k(t)$$

for each $k = 1, 2, \cdots$. *Then*

$$X(t) = \sum_1^\infty v_k(t) W_k[u_k(t)/v_k(t)] + \mu(t)$$
$$= \sum_1^\infty u_k(t) W_k[v_k(t)/u_k(t)] + \mu(t)\ ,$$

where $\{W_k(t)\}$ *is a sequence of independent standard Brownian motion processes.*

Proof. First of all, let us establish the L_2-convergence of the series $\sum_1^\infty v_k(t) W_k[u_k(t)/v_k(t)]$ with respect to the probability measure:

$$E\left\{\sum_m^n v_k(t) W_k[u_k(t)/v_k(t)]\right\}^2$$
$$= \sum_m^n v_k^2(t)u_k(t)/v_k(t) = \sum_m^n u_k(t)v_k(t)$$

which converges to 0 pointwise as $m, n \to \infty$. Therefore,

$$E\left\{\sum_1^\infty v_j(s) W_j[u_j(s)/v_j(s)]\right\}\left\{\sum_1^\infty v_k(t) W_k[u_k(t)/v_k(t)]\right\}$$
$$= \sum_1^\infty v_k(s)v_k(t) \min\{u_k(s)/v_k(s),\ u_k(t)/v_k(t)\}$$
$$= \sum_1^\infty u_k^*(s)v_k^*(t) = R(s, t)\ .$$

Similarly the second expression of $X(t)$ also holds.

COROLLARY 4.1. *Let* $X(t)$ *be a Gaussian process with covariance function*

$$R(s, t) = \sum_1^n u_k^*(s)v_k^*(t)\ ,$$

with $\{u_k(s)\}$ *and* $\{v_k(s)\}$ *satisfying the same conditions in Theorem 4 for* $k = 1, 2, \cdots, n$. *Then*

$$X(t) = \sum_1^n v_k(t) W_k[u_k(t)/v_k(t)] + \mu(t)$$
$$= \sum_1^n u_k(t) W_k[v_k(t)/u_k(t)] + \mu(t)\ .$$

COROLLARY 4.2. *Let* $X(t)$ *be a Gaussian process with covariance function*

$$R(s, t) = \sum_1^\infty \lambda_k u_k(s) u_k(t), \ \lambda_k \geqq 0 \quad (k = 1, 2, \cdots) .$$

Then

$$X(t) = \sum_1^\infty \sqrt{\lambda_k}\, u_k(t) W_k(1) + \mu(t) .$$

THEOREM 5. *Let a Gaussian process $\{X(t), \ 0 \leqq t \leqq T\}$ have a square integrable covariance function $R(s, t)$ on $[0, T]^2$, and let $R(s, t) = \text{l.i.m.}_{n\to\infty} \sum_1^n \lambda_k \phi_k(s)\phi_k(t)$ be the Mercer's expansion (see [12]) with $\sum_1^\infty \lambda_k \phi_k^2(t) = R(t, t)$ on $[0, T]$. Then $X(t)$ has a stochastic integral representation*

$$X(t) = \int_0^T h(t, u) dW(u) + \mu(t) \ \text{a.e. } on \ [0, T] ,$$

where

$$h(t, s) = \sum_{k=1}^\infty \sqrt{\lambda_k}\, \phi_k(t)\phi_k(s) .$$

Proof. As covariance functions $R(s, t)$ are always nonnegative definite, the eigenvalues λ_k of $R(s, t)$ are nonnegative. Observe now that

$$E\Big[\sum_1^\infty \sqrt{\lambda_k}\, \phi_k(s) W_k(1)\Big]\Big[\sum_1^\infty \sqrt{\lambda_k}\, \phi_k(t) W_k(1)\Big]$$
$$= \sum_1^\infty \lambda_k \phi_k(s)\phi_k(t) ,$$

which converges in the mean to $R(s, t)$, we conclude that

$$X(t) = \sum_1^\infty \sqrt{\lambda_k}\, \phi_k(t) W_k(1) + \mu(t) \ \text{a.e. on } [0, T] . \tag{2.3}$$

Since the $\phi_k(t)$ are eigenfunctions of $R(s, t)$, they are orthonormal on $[0, T]$. Therefore, we may write

$$W_k(1) = \int_0^T \phi_k(s) dW(s), \ k = 1, 2, \cdots .$$

Therefore, (2.3) may be rewritten as

$$X(t) = \sum_1^\infty \sqrt{\lambda_k}\, \phi_k(t) \int_0^T \phi_k(s) dW(s) + \mu(t) .$$

But the last expression, other than $\mu(t)$, is exactly the Paley-Wiener-Zygmund stochastic integral (see [6] and [7])

$$\int_0^T h(t, u) dW(u) ,$$

because $h(t, s) \equiv \sum_1^\infty \sqrt{\lambda_k}\, \phi_k(t)\phi_k(s)$ converges in s in the mean-square sense on $[0, T]$.

Although Paley-Wiener-Zygmund stochastic integral and the Ito's stochastic integral are defined quite differently, they agree almost surely for functions $h(t, \cdot) \in L_2[0, T]$.

COROLLARY 5.1. *If a Gaussian process $\{X(t),\ 0 \leqq t \leqq T\}$ has continuous covariance function $R(s, t)$ on $[0, T]^2$, then*

$$X(t) = \int_0^T h(t, u)dW(u) \quad \text{on} \quad [0, T]\ ,$$

where $h(t, u)$ has the same expression as in Theorem 5.

THEOREM 6. *Let $\{X(t), 0 \leqq t \leqq T\}$ be a Gaussian process, and let μ_X and μ_W be the probability measures generated by $\{X(t), 0 \leqq t \leqq T\}$ and $\{W(t),\ 0 \leqq t \leqq T\}$, respectively. Then $u_X \sim u_W$ (mutually absolutely continuous)* iff *there exists a complete orthonormal system $\{\alpha_k(t)\}$ on $[0, T]$ and a positive sequence $\{a_k\}$ with $\sum (1 - a_k)^2 < \infty$ such that*

$$X(t) = \int_0^T h(t, u)dW(u) + \mu(t)\ ,$$

where

$$h(t, u) = \sum \sqrt{a_k}\, \alpha_k(u) \int_0^t \alpha_k(v)dv\ .$$

Proof. According to Shepp [12, p. 322], $\mu_X \sim \mu_W$ iff $X(t)$ has covariance function $R(s, t) = \min(s, t) - \int_0^s\int_0^t K(u, v)dvdu$ for a symmetric kernel $K(s, t) \in L_2[0, T]^2$ and the spectrum

$$\sigma(K) \equiv \left\{\lambda\colon \int_0^T K(t, u)\beta(u)du = \lambda\beta(t),\ \beta \in L^2[0, T]\right\} \subset (-\infty, 1)\ .$$

Let $\sum \lambda_k\beta_k(u)\beta_k(v)$ be the Mercer's expansion of $K(u, v)$. Then

$$\begin{aligned} R(s, t) &= \min(s, t) - \int_0^s\int_0^t K(u, v)dudv \\ &= \sum \int_0^s\int_0^t \beta_k(u)\beta_k(v)dvdu - \sum \int_0^s\int_0^t \lambda_k\beta_k(u)\beta_k(v)dvdu \\ &= \sum a_k \int_0^s \beta_k(u)du \int_0^t \beta_k(v)dv,\ a_k = 1 - \lambda_k\ . \end{aligned}$$

Hence by Corollary 4.2,

$$X(t) = \sum \sqrt{a_k} \int_0^t \beta_k(v)dv\, W_k(1) + \mu(t)\ .$$

This expression is identical to (2.3) with $\phi_k(t) = \int_0^t \beta_k(v)dv$.

$$\therefore \quad X(t) = \int_0^t h(t, u)dW(u) + \mu(t) ,$$

where

$$h(t, u) = \sum \sqrt{a_k}\, \beta_k(u) \int_0^t \beta_k(v)dv .$$

Now, since $\lambda_k < 1$, we have $a_k \equiv 1 - \lambda_k > 0$, and $\sum (1 - a_k)^2 = \sum \lambda_k^2 = \int_0^T \int_0^T K(u, v)^2 dudv < \infty$. Thus the conditions are necessary. The converse follows by reversing the steps.

3. Examples and applications.

A. The standard Wiener process $X(t)$ has covariance function $R(s, t) = \min(s, t)$. Therefore $R(s, t) = s$, $s \leqq t$. Thus, by Theorem 1, $W(t) = tW(1/t)$, $t > 0$. Now, consider the tied-down Brownian motion: $\{X(t), 0 \leqq t \leqq 1\} = \{W(t), 0 \leqq t \leqq 1 \mid W(1) = x_0\}$. Using the above expression, one can express:

$$\{X(t), 0 \leqq t \leqq 1\} = \{t\{W(1/t) - W(1)\} + tx_0, 0 \leqq t \leqq 1\} ,$$

where we use the convention that $tW(1/t) = 0$ at $t = 0$. Thus, the covariance function of $X(t)$ is $R(s, t) = s(1 - t)$, $s \leqq t$. Hence, Theorem 1 gives

$$\begin{aligned} X(t) &= (1 - t)W\left(\frac{t}{1 - t}\right) + tx_0, \quad 0 \leqq t < 1 \\ &= x_0, \qquad\qquad\qquad\qquad\qquad t = 1 , \end{aligned}$$

while Theorem 2 gives

$$\begin{aligned} X(t) &= tW\left(\frac{1 - t}{t}\right) + tx_0, \quad 0 < t \leqq 1 , \\ &= 0, \qquad\qquad\qquad\qquad t = 0 . \end{aligned}$$

Other well-known representations are:

$$X(t) = W(t) - tW(1) + tx_0 ,$$

and

$$X(t) = W^*(t, 1 - t) + tx_0 ,$$

where $W^*(s, t)$ stands for the standard two-parameter Yeh-Wiener process (see [11] and [14]). Malmquist [5] and Park and others ([8], [9], [10]) used these representations extensively to obtain their results.

B. The Ornstein-Uhlenbeck process $X(t)$ has covariance function $R(s, t) = \sigma^2 \exp\{-\beta(t - s)\}$ for $s < t$ and $\beta > 0$ with $\mu(t) \equiv 0$. Therefore by Theorems 1 and 2,

$$\begin{aligned} X(t) &= \sigma e^{-\beta t} W(e^{2\beta t}) \\ &= \sigma e^{\beta t} W(e^{-2\beta t}) . \end{aligned}$$

These representations enable us to evaluate the barrier-crossing probabilities of the type

$$P\{\sup_{0 \leqq t \leqq T} X(t) - f(t) \geqq 0\}$$

for sectionally continuous functions $f(t)$. (See [9] for reference.)

C. Consider the Gaussian process $\{X(t),\ 0 \leqq t \leqq 1\}$ with zero mean and covariance function

$$R(s, t) = (-3s^3 + 4s^2)t \text{ if } -3s^2 + 4s \leqq -3t^2 + 4t .$$

Then by Theorem 3,

$$X(t) = tW(-3t^2 + 4t), \quad 0 \leqq t \leqq 1 .$$

Suppose we want to find the probability

$$P_1 = P\{\sup_{0 \leqq t \leqq 1} X(t) - f(t) < 0\} \tag{3.1}$$

for a sectionally continuous function $f(t)$ on $[0, 1]$ with $f(0) > 0$. Then by the representation, we get

$$\begin{aligned} P_1 &= P\{\sup_{0 \leqq t \leqq 1} tW(-3t^2 + 4t) - f(t) < 0\} \\ &= P\Big\{\sup_{0 < t \leqq 2/3} W(-3t^2 + 4t) - \frac{1}{t} f(t) < 0 , \\ &\qquad \sup_{2/3 \leqq t \leqq 1} W(-3t^2 + 4t) - \frac{1}{t} f(t) < 0\Big\} \\ &= P\Big\{\sup_{0 < s \leqq 4/3} W(s) - \frac{3}{2 - \sqrt{4 - 3s}} f\Big(\frac{2 - \sqrt{4 - 3s}}{3}\Big) < 0 , \\ &\qquad \sup_{1 \leqq s \leqq 4/3} W(s) - \frac{3}{2 + \sqrt{4 - 3s}} f\Big(\frac{2 + \sqrt{4 - 3s}}{3}\Big) < 0\Big\} . \end{aligned}$$

Let

$$\begin{aligned} g(s) &= \frac{3}{2 - \sqrt{4 - 3s}} f\Big(\frac{2 - \sqrt{4 - 3s}}{3}\Big) , \quad 0 < s \leqq 1 \\ &= \min\Big\{\frac{3}{2 - \sqrt{4 - 3s}} f\Big(\frac{2 - \sqrt{4 - 3s}}{3}\Big) , \end{aligned}$$

$$\frac{3}{2+\sqrt{4-3s}} f\left(\frac{2+\sqrt{4-3s}}{3}\right)\Big\}\,, \quad 1 \leqq s \leqq \frac{4}{3}\,.$$

Then

$$P_1 = P\{\sup_{0<s\leqq 4/3} W(s) - g(s) < 0\}\,.$$

This can be evaluated by the method in [9]. Until now we were unable to find probabilities of the type (3.1) mainly because the covariance function of $X(t)$ does not satisfy the Doob's condition in Theorem 1.

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