PACIFIC JOURNAL OF MATHEMATICS
Vol. 94, No. 2, 1981

INVARIANT SUBSPACE LATTICES FOR A CALSS OF OPERATORS

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We study the invariant subspace lattices for a one parameter family of operators $\{T_\alpha\}_\alpha$ on $L^p(0,1)$, α a complex number, where

$$T_\alpha f(x) = xf(x) + \alpha \int_0^x f(t)dt ,$$

and their adjoints T_α^*,

$$T_\alpha^* f(x) = xf(x) + \alpha \int_x^1 f(t)dt .$$

The closed invariant subspaces for T_α are in one-to-one correspondence with certain closed ideals of $\tilde{\mathscr{R}}_\alpha$, where $\tilde{\mathscr{R}}_\alpha$ is a Silov algebra with unit and in which the range $\mathscr{R}_\alpha$ of the Riemann Liouville operator J_α

$$\left(J_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \right)$$

is embedded as a closed ideal. When n is a positive integer, there is a complete lattice isomorphism between the closed ideals of $\tilde{\mathscr{R}}_n$ and the n-tuples $(E_0, E_1, \cdots, E_{n-1})$ of closed subsets of $[0, 1]$ where $E_0 \supseteq E_1 \supseteq \cdots \supseteq E_{n-1} \supseteq$ derived set of E_0. Every closed ideal of $\tilde{\mathscr{R}}_n$ is the intersection of closed primary ideals. Similar results carry over to α where the real part of α is an integer and also to the adjoint operators.

1. Introduction. Not many operators have had their invariants subspace lattices completely described. To name but a few, the invariant subspace lattice for the (simple) Volterra operator on $L^p(0, 1)$ was completely determined by Donoghue [4] and a more general result by Kalisch [7], that for the (simple) shift operator on l^2 by Beurling [1]. Further investigation of the invariant subspaces for the weighted shift operators have been made by Donoghue [4], Korenbljum [12], Nikol'skii ([14], [15]) and many others.

The main object of this paper is to characterize the invariant subspace lattices for a one parameter family of operators $\{T_\alpha\}_\alpha$ on $L^p(0, 1)$ (in general $1 < p < \infty$, but in some cases $1 \leqq p < \infty$) where

$$T_\alpha f(x) = xf(x) + \alpha \int_0^x f(t)dt ,$$

$f \in L^p(0, 1)$, $x \in [0, 1]$ and α is any complex number with integer real part. (And hence for their adjoints, namely $\{T_\alpha^*\}$ where $T_\alpha^* f(x) =$

$xf(x) + \alpha\int_x^1 f(t)dt.$)

We convert the invariant subspace problem into an equivalent problem where we characterize the closed ideals of certain Silov algebras. Sarason [18] had employed this approach to characterize all the closed invariant subspaces of T_1. Our result is a generalization of his. The equivalence of the two problems is established by using analysis of Kantorovitz [10] on the functional calculus of the operators T_α.

We prove that there is a complete lattice isomorphism between the closed invariant subspaces of $T_n (n \in N$, the natural numbers) and certain closed ideals of a Silov algebra $\tilde{\mathscr{R}}_n$ where $\tilde{\mathscr{R}}_n = \{f : f^{(n-1)}$ is absolutely continuous, $f^{(n)} \in L^p(0, 1)\}$ with norm $|f|_n = \|f^{(n)}\|_p + \sum_{i=0}^{n-1} |f^{(i)}(0)|$ (Theorem 4.1). This correspondence is induced by the Riemann Liouville operator J_n on $L^p(0, 1)$ where

$$J_\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x - t)^{\alpha-1} f(t)dt, \ \mathrm{Re}\,\alpha \text{ (the real part of } \alpha) > 0 ,$$

Γ is the gamma function.

There is a complete lattice isomorphism between the closed ideals of $\tilde{\mathscr{R}}_n$ and the n-tuples $(E_0, E_1, \cdots, E_{n-1})$ of closed subsets of $[0, 1]$ with $E_0 \supseteqq E_1 \supseteqq \cdots \supseteqq E_{n-1} \supseteqq$ derived set of E_0 (Theorems 3.19 and 4.3). Every closed ideal of $\tilde{\mathscr{R}}_n$ is found to be the intersection of closed primary ideals. Several other algebras were known to have this property. Stone [21] proved it for the algebra $C[0, 1]$, Silov [19] for $C^1[0, 1]$, Whitney [24] for $C^n[0, 1]$, Snol [20] for some algebras lying between $C[0, 1]$ and $C^1[0, 1]$, Osadchii [16] for the algebra of functions on the unit circle for which the nth derivatives are square summable, Daly and Downum [3] for a subalgebra of $C^{n-1}[0, 1]$ consisting of functions whose $(n - 1)$th derivatives satisfy a bounded Lipschitz condition.

Similar results carry over easily to the more general parameter α, where $\mathrm{Re}\,\alpha$ is an integer, and also the adjoints of these operators.

When $\mathrm{Re}\,\alpha$ is not integral, the situation is more complex. It is not apparent that $\mathscr{R}_\alpha$, the range of J_α is an algebra. Via functional calculus, we show that indeed it is, for $\mathrm{Re}\,\alpha \geqq 1$. Moreover it can be embedded as a closed ideal of a Silov algebra with unit, $\tilde{\mathscr{R}}_\alpha$, which is a natural generalization of $\tilde{\mathscr{R}}_n$. As is in the case of T_n, there is a one-to-one correspondence between the closed T_α-invariant subspaces and certain closed ideals of $\mathscr{R}_\alpha$ (and hence of $\tilde{\mathscr{R}}_\alpha$). We conjecture that all the closed ideals of $\tilde{\mathscr{R}}_\alpha (n < \mathrm{Re}\,\alpha < n + 1)$ are completely determined as in the case of $\tilde{\mathscr{R}}_n$, by n-tuples of closed subsets of $[0, 1]$ satisfying certain conditions. We have not succeeded in proving this and we hope to return to the problem on a later occasion. Kantorovitz ([10], [11]) showed that the operator T_α has C^n-functional calculus if and only if $|\operatorname{Re}\alpha| \leqq n$. Imitating his argument, we found that for $\alpha, \beta \in \boldsymbol{C}$ (the complex numbers) with $\operatorname{Re}\alpha \geqq 1$ and $0 \leqq \operatorname{Re}\beta \leqq \operatorname{Re}\alpha$, T_β has $\tilde{\mathscr{R}}_\alpha$-functional calculus.

Finally it should be remarked that Erdös [5] and Waterman [23] had independently found all the invariant subspaces for the operator T_f on $L^p(0, 1)$ where T_f is defined as

$$T_f g(x) = f(x)g(x) - \int_0^x f'(t)g(t)dt ,$$

$g \in L^p(0, 1)$, $x \in [0, 1]$ and f is a function with some suitable conditions. The particular case when $f(t) = t$ gives $T_f = T_{-1}$. Furthermore, Waterman, in [23], claims to have found all the closed invariant subspaces for the operators $\{T_{-n}\}$, n a positive integer, by using recent results on L^p-approximation by splines. However, our work is conducted independently of his work which was not available to us[1].

2. The Silov algebra $\mathscr{R}_n$. The range $\mathscr{R}_n$ of the Riemann Liouville operator J_n,

$$J_n f(x) = \frac{1}{\Gamma(n)} \int_0^x (x - t)^{n-1} f(t)dt,\ f \in L^p(0, 1),\ x \in [0, 1] ,$$

where $n \in \boldsymbol{N}$, $1 \leqq p < \infty$, is given by

$$\mathscr{R}_n = \{g\colon g^{(n-1)} \text{ is absolutely continuous, } g^{(n)} \in L^p(0, 1) ,\ g^{(i)}(0) = 0,\ 0 \leqq i \leqq n - 1\} .$$

Clearly $\mathscr{R}_n$ is a subalgebra of the well known Banach algebra $C^{n-1}[0, 1]$, the space of all complex valued functions with $(n - 1)$ continuous derivatives. But we will endow $\mathscr{R}_n$ with its own norm. As $\mathscr{R}_n$ has no unit, it is convenient to embed it in a larger algebra $\tilde{\mathscr{R}}_n$ with unit, namely

$$\tilde{\mathscr{R}}_n = \mathscr{R}_n \oplus \boldsymbol{C}x^{n-1} \oplus \boldsymbol{C}x^{n-2} \oplus \cdots \oplus \boldsymbol{C}x \oplus \boldsymbol{C}$$

where the sum is direct. Thus

$$\tilde{\mathscr{R}}_n = \{g\colon g^{(n-1)} \text{ is absolutely continuous, } g^{(n)} \in L^p(0, 1)\} .$$

Define a norm $|\cdot|_n$ on $\tilde{\mathscr{R}}_n$ as follows: for $g \in \tilde{\mathscr{R}}_n$,

$$|g|_n = \|g^{(n)}\|_p + \sum_{i=0}^{n-1} |g^{(i)}(0)| ,$$

[1] Still unable to locate this reference.

where $\|\cdot\|_p$ denotes the L^p-norm and $|\cdot|$ the absolute value. It is easily seen that $(\tilde{\mathscr{R}}_n, |\cdot|_n)$ is complete and $\mathscr{R}_n$ is a closed ideal of $\tilde{\mathscr{R}}_n$. The norm $|||\cdot|||_n$ defined as

$$|||g|||_n = \|g^{(n)}\|_p + \sum_{i=1}^{n-1} \|g^{(i)}\|_\infty , \quad g \in \tilde{\mathscr{R}}_n ,$$

(where $\|\cdot\|_\infty$ is the supremum norm) is equivalent to $|\cdot|_n$.

Since for any $g, h \in \tilde{\mathscr{R}}_n$, $|hg|_n \leqq c|h|_n|g|_n$ for some constant c independent of g, h and n, $\tilde{\mathscr{R}}_n$ can be made into a Banach algebra by the equivalent norm $|\cdot|'_n$ where $|g|'_n = c|g|_n$. Moreover $\tilde{\mathscr{R}}_n$ is a Silov algebra. (A commutative semisimple Banach algebra $\mathfrak{A}$ is a Silov algebra if for any closed subset F of the maximal ideal space Φ_a of $\mathfrak{A}$ and $x \in \Phi_a$, $x \notin F$, there is an element $h \in \mathfrak{A}$ such that $h(x) = 1$ and $h(F) = \{0\}$.) Henceforth we shall use the norm $|\cdot|_n$ on $\tilde{\mathscr{R}}_n$, but the arguments used work for both norms.

Observe that for any fixed i and a, $0 \leqq i \leqq n-1$, $a \in [0, 1]$, the evaluation map $E_{i,a}\colon \tilde{\mathscr{R}}_n \to C$ defined as $E_{i,a}(g) = g^{(i)}(a)$, $g \in \tilde{\mathscr{R}}_n$, is continuous.

It will be seen that the collection $\mathscr{C}_0^n$ of all the closed ideals of $\mathscr{R}_n$ which are closed under multiplication by the function x are in one-to-one correspondence with the closed invariant subspaces of T_n. $\mathscr{C}_0^n$ consists of precisely those closed ideals of $\tilde{\mathscr{R}}_n$ which lie in $\mathscr{R}_n$. The collection $\mathscr{C}^n$ of all the closed ideals of $\tilde{\mathscr{R}}_n$ (and hence $\mathscr{C}_0^n$) can be neatly characterized and every closed ideal of $\tilde{\mathscr{R}}_n$ is the intersection of closed primary ideals.

3. THE MAIN THEOREM. *Characterization of the closed ideals of* $\tilde{\mathscr{R}}_n$.

For any closed ideal $\mathscr{I}$ in $\tilde{\mathscr{R}}_n$, we define

$$Z_0(\mathscr{I}) = \operatorname{Hull}\mathscr{I}, \text{ and for } 0 \leqq i \leqq n-1 ,$$
$$Z_i(\mathscr{I}) = \{x \in [0, 1]\colon f^{(j)}(x) = 0,\ \forall 0 \leqq j \leqq i,\ \forall f \in \mathscr{I}\} .$$

Whenever there is no confusion, we abbreviate $Z_i(\mathscr{I})$ as Z_i, $0 \leqq i \leqq n-1$.

REMARK 3.1. For any closed ideal $\mathscr{I}$ in $\tilde{\mathscr{R}}_n$, deriv $Z_0 \subseteqq Z_{n-1} \subseteqq Z_{n-2} \subseteqq \cdots \subseteqq Z_0$, where deriv Z_0 is the derived set of Z_0.

Each closed ideal $\mathscr{I}$ not only determines an n-tuple of sets Z_0, $Z_1, \cdots, Z_{n-1}$, but in fact is completely determined by these sets. We shall now state the main theorem.

MAIN THEOREM 3.2. *For any n-tuple of closed subsets* $(E_0, E_1, \cdots,$

E_{n-1}) *with* deriv $E_0 \subseteqq E_{n-1} \subseteqq E_{n-2} \subseteqq \cdots \subseteqq E_0 \subseteqq [0, 1]$, *there exists a unique closed ideal* $\mathscr{I}$ *in* $\tilde{\mathscr{R}}_n$ *such that* $Z_i = E_i$, $0 \leqq i \leqq n-1$.

The following remark will be useful in establishing the uniqueness part of the main theorem.

REMARK 3.3. Let E, F be any closed subset of $[0, 1]$ such that deriv $F \subseteqq E \subseteqq F$. Then $F \backslash E$ consists of at most countably many isolated points.

The existence of a closed ideal of $\tilde{\mathscr{R}}_n$ satisfying the required properties is easily established. There is an obvious candidate.

THEOREM 3.4. *Given* $E_0 \supseteqq E_1 \supseteqq \cdots \supseteqq E_{n-1}$, *closed subsets of* $[0, 1]$ *such that* $E_{n-1} \supseteqq$ deriv E_0, *there exists a closed ideal* $\mathscr{I}$ *in* $\tilde{\mathscr{R}}_n$ *such that* $Z_i = E_i$, $0 \leqq i \leqq n-1$.

Proof. $\mathscr{I} = \{f \in \tilde{\mathscr{R}}_n : f = f^{(1)} = \cdots = f^{(i)} = 0$ on E_i, $0 \leqq i \leqq n-1\}$, has all the required properties.

There are some special ideals which play an important role in the proof of uniqueness. For any n-tuple $(E_0, E_1, \cdots, E_{n-1})$ of closed subsets of $[0, 1]$ such that deriv $E_0 \subseteqq E_{n-1} \subseteqq E_{n-2} \subseteqq \cdots \subseteqq E_0$, define

$$\mathscr{M}(E_0, E_1, \cdots, E_{n-1}) = \{f \in \tilde{\mathscr{R}}_n : f = f^{(1)} = \cdots = f^{(i)} = 0 \text{ on } E_i, 0 \leqq i \leqq n-1\} .$$

We shall abbreviate $\mathscr{M}(E, E, \cdots, E)$ by $\mathscr{M}(E)$. Let

$$\mathscr{M}_0(E_0, E_1, \cdots, E_{n-1}) = \{f \in \mathscr{M}(E_0, E_1, \cdots, E_{n-1}) : f = 0 \text{ in some neighborhood of } E_{n-1}\} ,$$

and for any closed set F in $[0, 1]$,

$$\mathscr{J}(F) = \{f \in \tilde{\mathscr{R}}_n : f = 0 \text{ in some neighborhood of } F\} .$$

$\mathscr{M}(E_0, E_1, \cdots, E_{n-1})$ is clearly a closed ideal but $\mathscr{M}_0(E_0, E_1, \cdots, E_{n-1})$ and $\mathscr{J}(F)$ are ideals which may not be closed.

Two general results quoted below will be useful.

THEOREM 3.5. ([13], p. 225, Thm. 4). *Let* $\mathfrak{A}$ *be a Silov algebra and* F *a closed subset of the maximal ideal space of* $\mathfrak{A}$. *Let* $\mathscr{I}$ *be an ideal such that hull* $\mathscr{I} = F$. *Then* $\mathscr{J}(F) \subseteqq \mathscr{I}$ *and hull* $\mathscr{J}(F) = F$.

In other words, $\mathscr{J}(F)$ is the smallest ideal with hull F.

PROPOSITION 3.6. *Let X be a Banach space and Y a dense subspace of X. If M is a closed subspace of X which is of finite codimension, then $M \cap Y$ is dense in M.*

Briefly, the process of establishing that every closed ideal of $\tilde{\mathscr{R}}_n$ is uniquely determined by the closed subset Z_i, $0 \leqq i \leqq n-1$, consists of three major steps. First we analyze the structure of the closure $\overline{\mathscr{J}(F)}$ of $\mathscr{J}(F)$ where F is any closed subset of $[0, 1]$. We then prove that $\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I})) \subseteqq \mathscr{I}$ and lastly we prove $\overline{\mathscr{M}_0(E_0, \cdots, E_{n-1})} = \mathscr{M}(E_0, \cdots, E_{n-1})$. It is immediate from the last two steps that $\mathscr{I} = \overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))} = \mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))$.

PROPOSITION 3.7. *In $\tilde{\mathscr{R}}_n$, $\overline{\mathscr{J}(\{\lambda\})} = \mathscr{M}(\{\lambda\})$, $\lambda \in [0, 1]$.*

Proof. Clearly $\overline{\mathscr{J}(\{\lambda\})} \subseteqq \mathscr{M}(\{\lambda\})$, so it suffices to show $\mathscr{M}(\{\lambda\}) \subseteqq \overline{\mathscr{J}(\{\lambda\})}$.

Let $f \in \mathscr{M}(\{\lambda\})$, then $f^{(i)}(\lambda) = 0$, $0 \leqq i \leqq n-1$. Define

$$K_m = [0, 1] \Big\backslash \left[\lambda - \frac{1}{m}, \lambda + \frac{1}{m}\right]$$

and

$$f_m(x) = \int_\lambda^x \int_\lambda^{r_1} \cdots \int_\lambda^{r_{n-1}} f^{(n)}(t) \chi_{K_m}(t) dt\, dr_{n-1} \cdots dr_1 ,$$

where m is a positive integer. Then $f_m \in \mathscr{J}(\{\lambda\})$ and $f_m \to f$ in $\tilde{\mathscr{R}}_n$, since $f_m^{(j)}(t) \to f^{(j)}(t)$ uniformly for $0 \leqq j \leqq n-1$, and $f_m^{(n)} \to f^{(n)}$ in $L^p(0, 1)$. Thus $f \in \overline{\mathscr{J}(\{\lambda\})}$. This completes the proof.

As a consequence, the closed primary ideals of $\tilde{\mathscr{R}}_n$ are easily identified. (An ideal is primary if it is contained in a unique maximal ideal.) Indeed, they have simple structures.

COROLLARY 3.8. *Any closed primary ideal $\mathscr{I}$ of $\tilde{\mathscr{R}}_n$ is of the form $\mathscr{M}(\{\lambda\}, \{\lambda\}, \cdots, \{\lambda\}, \phi, \cdots, \phi)$ where $\{\lambda\} = \text{Hull } \mathscr{I}$ and the multiplicity of λ whithin the parenthesis is $i+1$ for $0 \leqq i \leqq n-1$.*

Proof. By Theorem 3.5, $\overline{\mathscr{J}(\{\lambda\})} \subseteqq \mathscr{I}$. Since $\overline{\mathscr{J}(\{\lambda\})} = \mathscr{M}(\{\lambda\})$ and $\mathscr{M}(\{\lambda\})$ has finite codimension, it follows that $\mathscr{I}$ also has finite codimension. Thus $\mathscr{P} \cap \mathscr{I}$ is dense in $\mathscr{I}$, since the set $\mathscr{P}$ of all polynomials is dense in $\tilde{\mathscr{R}}_n$. This implies that Hull $(\mathscr{P} \cap \mathscr{I}) = \text{Hull } \mathscr{I}$. Further- more $\mathscr{P} \cap \mathscr{I}$ is an ideal in $\mathscr{P}$ and thus is a principal ideal whose generator must be of the form $(x-\lambda)^{i+1}$ for some $0 \leqq i \leqq n-1$.

Hence $\mathscr{I} = \{f \in \tilde{\mathscr{R}}_n: f(\lambda) = \cdots = f^{(i)}(\lambda) = 0\}$, that is $\mathscr{I} = \mathscr{M}(\{\lambda\}, \cdots, \{\lambda\}, \phi, \cdots, \phi)$.

REMARK 3.9. The argument used in Proposition 3.7 still holds when the point λ is replaced by an interval $[a, b] \subseteq [0, 1]$.

LEMMA 3.10. *Let F_1, F_2 be disjoint closed subsets of the maximal ideal space of a Silov algebra $\mathfrak{A}$. Let $\mathscr{I}_1, \mathscr{I}_2$ be closed ideals with $\overline{\mathscr{J}(F_i)} = \mathscr{I}_i$, $i = 1, 2$. Then $\overline{\mathscr{J}(F_1 \cup F_2)} = \mathscr{I}_1 \cap \mathscr{I}_2$.*

Proof. Clearly $\overline{\mathscr{J}(F_1 \cup F_2)} \subseteq \overline{\mathscr{J}(F_1)} \cap \overline{\mathscr{J}(F_2)} = \mathscr{I}_1 \cap \mathscr{I}_2$. To show the reverse inclusion, let $f \in \mathscr{I}_1 \cap \mathscr{I}_2$. Choose $g_i \in \mathfrak{A}$, $i = 1, 2$ such that

$$g_1 = \begin{cases} f & \text{in a neighborhood of } F_1 \\ 1 & \text{in a neighborhood of } F_2 \end{cases}$$

and

$$g_2 = \begin{cases} f & \text{in a neighborhood of } F_2 \\ 1 & \text{in a neighborhood of } F_1 . \end{cases}$$

Then $f = g_1 g_2 + h$ where $h \in \mathscr{J}(F_1 \cup F_2)$. Observe that $g_i \in \mathscr{I}_i$ (by a property of Silov algebras ([13], p. 224, Thm. 3″)), so we can choose $U_{k,i} \in \mathscr{J}(F_i)$ such that $U_{k,i} \to g_i$ as $k \to \infty$. Then $U_{k,1}U_{k,2} + h \to f$. Moreover, $U_{k,1}U_{k,2} + h \in \mathscr{J}(F_1 \cup F_2)$, thus $f \in \overline{\mathscr{J}(F_1 \cup F_2)}$.

REMARK 3.11. The above result, which holds for a general Silov algebra, enables us to conclude that, in $\tilde{\mathscr{R}}_n$, $\overline{\mathscr{J}(F)} = \mathscr{M}(F)$ for any closed F which is a finite disjoint union of points and closed intervals.

The following observation will be used.

Observation 3.12. Let $\{[a_i, b_i]\}$ be a countable disjoint collection of intervals in $[0, 1]$ such that $f, f^{(1)}, \cdots, f^{(n-1)}$ vanish at all the a_i's and b_i's. Then clearly the truncated function $f\chi_{[0,1]\setminus\cup_i[a_i,b_i]} \in \tilde{\mathscr{R}}_n$, where χ_E is the characteristic function on E.

We can now describe $\overline{\mathscr{J}(F)}$ for any closed subset $F \subseteq [0, 1]$, for this general case can be reduced to the situation of Remark 3.11.

THEOREM 3.13. *In $\tilde{\mathscr{R}}_n$, $\overline{\mathscr{J}(F)} = \mathscr{M}(F)$ for any closed $F \subseteq [0, 1]$.*

Proof. We only need to show $\mathscr{M}(F) \subseteq \overline{\mathscr{J}(F)}$. Let $f \in \mathscr{M}(F)$. The set F^c, the complement of F in $[0, 1]$, being open, is a disjoint union of a countable number of open intervals (with the possible exception that the interval with end point 0 or 1 might be closed at that end). Let $F^c = \bigcup_{i=1}^{\infty} (a_i, b_i)$ (the union may be finite). There exists $N > 0$ such that $\|f^{(n)}\chi_{\bigcup_{i>N}(a_i,b_i)}\|_p$ is very small. Let $G = \bigcup_{i=1}^{N} (a_i, b_i)$. (Here G includes the interval with end point 0 if there is one.) Define $\tilde{f} = f\chi_G$; then by Remark 3.11 $\tilde{f} \in \overline{\mathscr{J}(G^c)} \subseteqq \overline{\mathscr{J}(F)}$ and it approximates f, therefore $f \in \overline{\mathscr{J}(F)}$ and we have the result.

COROLLARY 3.14. *Let $\mathscr{I}$ be a closed ideal of $\tilde{\mathscr{R}}_n$, then $\mathscr{I} \supseteqq \mathscr{M}(Z_0(\mathscr{I}))$.*

Proof. $\mathscr{M}(Z_0(\mathscr{I})) = \overline{\mathscr{J}(Z_0(\mathscr{I}))} \subseteqq \mathscr{I}$.

LEMMA 3.15. *Let $\mathscr{I}$ be a closed ideal of $\tilde{\mathscr{R}}_n$ and let $h \in \mathscr{I}$ be such that $h^{(i)}(a) \neq 0$ for some $0 \leqq i < n$ and some $a \in [0, 1]$. Then for any neighborhood N_a of a, there exists $H \in \mathscr{I}$ such that support $(H) \subseteqq N_a$ and $H^{(l)}(a) = 0$ for $0 \leqq l < n$, $l \neq i$, but $H^{(i)}(a) \neq 0$.*

Proof. Let k be the smallest positive integer $\leqq i$ such that $h^{(k)}(a) \neq 0$. Define

$$Q(x) = (x - a)^{i-k} \cdot h(x)$$

and $g(x) = Q(x) \cdot (1 + \sum_{n=1}^{(n-1)-i} C_j \cdot (x - a)^j)$, where the C_j's are constants yet to be determined. For $l < i$, $g^{(l)}(a) = 0$ while $g^{(i)}(a) \neq 0$. The fact that $Q^{(i)}(a) \neq 0$ and $Q^m(a) = 0$ for $m < i$, enables the C_j's to be suitably chosen successively so as to make $g^{(l)}(a) = 0$ for $i < l < n$ while $g^{(l)}(a)$ remains unchanged for $l \leqq i$. Let f be a C^∞ function on $[0, 1]$ such that support $(f) \subseteqq N_a$ and $f = 1$ in some neighborhood of a. Evidently $H = gf$ has all the required properties.

For any arbitrary f in any ideal $\mathscr{I}$ of $\tilde{\mathscr{R}}_n$, the above lemma annihilate its derivative (of any order $\leqq n - 1$) at any point $a \in [0, 1]$ by the addition of some appropriate function in $\mathscr{I}$ but simultaneously leaving all the other derivatives at a undisturbed.

LEMMA 3.16. *For any closed ideal $\mathscr{I}$ of $\tilde{\mathscr{R}}_n$, $\mathscr{I} \supseteqq \mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_n(\mathscr{I}))$.*

Proof. Let $f \in \mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))$; then $f = 0$ in some neighborhood $U \supseteqq Z_{n-1}(\mathscr{I})$. Since $Z_{n-1}(\mathscr{I}) \supseteqq$ deriv $Z_0(\mathscr{I})$, the closed sets $Z_j(\mathscr{I}) \backslash U$ must be finite for all $0 \leqq j \leqq n - 2$. Using Lemma 3.15, repeatedly if necessary, we can find $G \in \mathscr{I}$ such that the function $K = f + G$ has the property that $K^{(i)}$ vanishes on

$Z_0(\mathscr{I})$, $0 \leqq i \leqq n-1$. Thus $K \in \mathscr{I}$ by Corollary 3.14, and hence $f \in \mathscr{I}$.

The simple well behaved function given below is worth noting, for it makes the calculation on annihilation work very neatly.

LEMMA 3.17. *Let*

$$Q(x) = \int_0^x \int_0^{r_1} \cdots \int_0^{r_{n-1}} \chi_{[\alpha,\beta]} dt\, dr_{n-1} \cdots dr_1 ,$$

where $[\alpha, \beta] \subseteqq [0, 1]$. *Then*

$$Q(x) = 0 \quad for \quad x \leqq \alpha$$

and

$$Q^{(i)}(x) = \frac{(x-\alpha)^{n-i}}{(n-i)!} \quad for \quad \alpha \leqq x \leqq \beta , \quad 0 \leqq i \leqq n-1 .$$

In the process of establishing $\overline{\mathscr{M}_0(E_0, \cdots, E_{n-1})} = \mathscr{M}(E_0, \cdots, E_{n-1})$, several reductions occur but the chief and final burden is shouldered by the miniature case described in the main lemma.

MAIN LEMMA 3.18. *Let*

$$K(x) = \int_0^x \int_0^{r_1} \cdots \int_0^{r_{n-1}} f(t) dt\, dr_{n-1} \cdots dr_1 ,$$

where $f \in L^p(0, 1)$. *Suppose* $\{x_j\}_1^\infty$ *is a sequence of distinct isolated zeros of* K *such that* $x_1 < x_2 < \cdots$ *and* $x_j \to 1$. *Given* $\varepsilon > 0$, *there exist consecutive points* $a, b \in \{x_j\}_1^\infty$ *and a function* G *such that*

$$G(x) = K(x) \quad for \quad x \leqq a$$
$$G(x) = 0 \quad for \quad b \leqq x \leqq 1$$

and

$$|K - G|_n < \varepsilon .$$

Proof. We may assume K to be a real function (for otherwise we can write K as $K_1 + iK_2$ with K_1, K_2 real). Given $\varepsilon > 0$, there exist $a_1, a_2, \cdots, a_n \in \{x_j\}_1^\infty$ such that $0 < a_1 < a_2 < \cdots < a_n < 1$, the a_i's are consecutive points in $\{x_j\}_1^\infty$, and $\|K^{(n)}\chi_{[a_1,1]}\|_p < \varepsilon$.

Denote the largest interval of $\{[a_1, a_2], [a_2, a_3], \cdots, [a_{n-1}, a_n]\}$ by $[a, b]$. Since $K(a_1) = K(a_2) = \cdots = K(a_n) = 0$, there exists $t_j \in (a_1, a_n)$ such that $K^i(t_j) = 0$, $1 \leqq j \leqq n-1$.

$$|K^{(n-1)}(b)| = \left| \int_{t_{n-1}}^b K^{(n)}(s) ds \right|$$
$$\leqq \varepsilon[(n-1)(b-a)]^{1/q} \quad (\text{for } p > 1) .$$

Similarly $|K^{(j)}(b)| \leqq \varepsilon[(n-1)(b-a)]^{(n-1)-j+1/q}$, $0 \leqq j \leqq n-2$. When $p = 1$, the estimates are

$$|K^{(j)}(b)| \leqq \varepsilon[(n-1)(b-a)]^{(n-1)-j}, \quad 0 \leqq j \leqq n-1 .$$

We shall confine ourselves to the case $p > 1$, for the same argument works for the case $p = 1$. Denote the length of the interval $[a, b]$ by $n\delta$, $\delta > 0$. Let $s_1, s_2, \cdots, s_n$ be n equally spaced points in $[a, b]$ such that $s_1 > s_2 > \cdots > s_n = a$ and $b - s_j = j\delta$, $1 \leqq j \leqq n$. Let

$L(x) = K(x) - \sum_{j=1}^{n} Q_j(x)$, where

$$Q_j(x) = \int_0^x \int_0^{r_1} \cdots \int_0^{r_{n-1}} C_j \chi_{[s_j, b]}(t) dt dr_{n-1} \cdots dr_1 ,$$

and where the C_j's are constants yet to be determined.

Let $G = L\chi_{[0,b]}$. We need $L(b) = L^{(1)}(b) = \cdots = L^{(n-1)}(b) = 0$ so that $G \in \tilde{\mathscr{R}}_n$ and we want $|K - G|_n < \varepsilon$. Now

$$|K - G|_n \leqq \| K^{(n)} \chi_{[a_1, 1]} \|_p + \sum_{j=1}^{n} |Q_j|_n$$

and

$$|Q_j|_n = |C_j|(b - s_j)^{1/p} = |C_j|(j\delta)^{1/p} .$$

Observe that $L(x) = K(x)$ for $x \leqq a$,

$$L(b) = -\sum_{j=1}^{n} \frac{C_j(j\delta)^n}{n!}$$

$$L^{(i)}(b) = K^{(i)}(b) - \sum_{j=1}^{n} \frac{C_j(j\delta)^{n-i}}{(n-i)!}, \quad 1 \leqq i \leqq n-1 .$$

Hence we require

$$(*) \qquad \begin{cases} \sum_{j=1}^{n} C_j(j\delta) = K^{(n-1)}(b) \\ \sum_{j=1}^{n} C_j(j\delta)^2 = 2K^{(n-2)}(b) \\ \vdots \\ \sum_{j=1}^{n} C_j(j\delta)^{n-1} = (n-1)!\, K^{(1)}(b) \\ \sum_{j=1}^{n} C_j(j\delta)^n = 0 \end{cases}$$

and $|C_j|(j\delta)^{1/p} < \varepsilon$, $1 \leqq j \leqq n$. Treating the C_j's as unknowns, (*) is a system of simultaneous linear equations which can be solved by Cramer's rule. The determinant

$$D = \begin{vmatrix} \delta & 2\delta & & n\delta \\ \delta^2 & (2\delta)^2 & \cdots & (n\delta)^2 \\ \vdots & \vdots & & \vdots \\ \delta^n & (2\delta)^n & & (n\delta)^n \end{vmatrix} = \delta(2\delta)\cdots(n\delta)\prod_{1\leqq k<j\leqq n}(k-j)\delta$$

and

$$|C_1| = \frac{1}{D}\begin{vmatrix} K^{(n-1)}(b) & 2\delta & \cdots & n\delta \\ 2K^{(n-2)}(b) & (2\delta)^2 & & (n\delta)^2 \\ \vdots & \vdots & & \vdots \\ (n-1)!\,K^{(1)}(b) & & & \\ 0 & (2\delta)^n & \cdots & (n\delta)^n \end{vmatrix}.$$

Since $K^{(j)}(b) \leqq \varepsilon[n(n-1)\delta]^{n-1-j+1/q}$, $0 \leqq j \leqq n-1$, it is clear that,

$$|C_1| \leqq \frac{d_1\varepsilon\delta^{n(n+1)/2-1+1/q}}{\delta^{n(n+1)/2}}$$

where d_1 is a positive constant which depends only on n. Therefore $|C_1|\delta^{1/p} \leqq d_1\varepsilon$. Similarly there exist constants $d_j > 0$ such that

$$|C_j|(j\delta)^{1/p} \leqq d_j\varepsilon\ , \quad 2 \leqq j \leqq n\ ,$$

where the d_j's depend only on n and are independent of the choice of b. Thus $|K-G|_n < B\varepsilon$, where B is a constant dependent only on n.

We are now ready to prove that $\mathscr{I}$ is completely determined by the sets $Z_0(\mathscr{I}), Z_1(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I})$.

THEOREM 3.19. *For any closed ideal* $\mathscr{I}$ *of* $\tilde{\mathscr{R}}_n$, $\mathscr{I} = \mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I})) = \overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))}$.

Proof. Clearly $\overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))} \subseteqq \mathscr{I} \subseteqq \mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))$. So it suffices to show $\mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I})) \subseteqq \overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))}$.

Case A. $Z_{n-1}(\mathscr{I}) = \phi$.

Since $Z_{n-1}(\mathscr{I}) = \phi$, $\mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I})) = \mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))$ by definition. Hence $\overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))} = \mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))$.

Case B. $Z_{n-1}(\mathscr{I}) \neq \emptyset$.

Let $f \in \mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))$. Being open, $(Z_{n-1}(\mathscr{I}))^c = \bigcup_{i=1}^{\infty}(a_i, b_i)$, a countable (possibly finite) disjoint union of open intervals. Given $\varepsilon > 0$, there exists $N > 0$ such that $\|f^{(n-1)}\chi_{\bigcup_{i>N}(a_i,b_i)}\|_p < \varepsilon$. Also $f\chi_{\bigcup_{1 \leq i \leq N}(a_i,b_i)} \in \tilde{\mathscr{R}}_n$. Let $f_k = f\chi_{[a_k,a_k]}$, $1 \leqq k \leqq N$. It suffices to show that each of the functions f_k lies in $\overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))}$. Note that each interval (a_k, b_k) contains at most a countable subset of $Z_0(\mathscr{I})$ whose only possible limit points are end points a_k, b_k (by Remark 3.3). By a linear transformation, we can assume without loss of generality that $a_k = 0$ and $b_k = 1$. Now there exists points $s, t \in [0, 1]$ such that $0 < s < t < 1$ and $[s, t] \cap Z_0(\mathscr{I}) = \phi$. We can find C^∞-functions y_1, y_2 such that $y_1 + y_2 = 1$ and y_1 is supported in $[0, t]$ while y_2 is supported in $[s, 1]$. Let $\phi_1 = f_k y_1$ and $\phi_2 = f_k y_2$; trivially $f_k = \phi_1 + \phi_2$. It suffices to show $\phi_2 \in \overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))}$ for ϕ_1 can be similarly dealt with by applying the affine transformation $x \to 1 - x$. It should however be noted that for any $H \in \tilde{\mathscr{R}}_n$, the composition function defined as $G(x) = H(1 - x)$ also belongs to $\tilde{\mathscr{R}}_n$ since $G^{(i)}(x) = (-1)^i H^{(i)}(1 - x)$, $1 \leqq i \leqq n$.

For ϕ_2, there are two subcases:

Subcase 1. $F_0 = (Z_0(\mathscr{I}) \backslash Z_{n-1}(\mathscr{I})) \cap [t, 1]$ is finite. Let $F_j = (Z_j(\mathscr{I}) \backslash Z_{n-1}(\mathscr{I})) \cap [t, 1]$, $1 \leqq j \leqq n - 2$, and $F_{n-1} = Z_{n-1}(\mathscr{I}) \cap [t, 1]$, then $F_j \subseteqq F_0$, $1 \leqq j \leqq n - 2$ and $F_{n-1} \subseteqq \{1\}$. Recall that by Theorem 3.13, $\mathscr{M}(Z) = \overline{\mathscr{J}(Z)}$ for any closed $Z \subseteqq [0, 1]$. Take $Z = [0, s] \cup F_{n-1}$. Since $\mathscr{M}(F_0, F_1, \cdots, F_{n-2}, \phi)$ is of finite codimension in $\tilde{\mathscr{R}}_n$ and is closed, then by Proposition 3.6, $\mathscr{J}(Z) \cap \mathscr{M}(F_0, \cdots, F_{n-2}, \phi)$ is dense in $\mathscr{M}(Z \cup F_0, \cdots, Z \cup F_{n-2}, Z)$. Thus $\phi_2 \in \overline{\mathscr{M}_0(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))}$.

Subcase 2. $F_0 = (Z_0(\mathscr{I}) \backslash Z_{n-1}(\mathscr{I})) \cap [t, 1) = \{x_1, x_2, \cdots\}$, where $x_i \to 1$. Note that $Z_{n-1}(\mathscr{I}) \cap [t, 1] = \{1\}$. $\phi_2(0) = \phi_2^{(1)}(0) = \cdots = \phi_2^{(n-1)}(0) = 0$, so $\phi_2 \in \mathscr{R}_n$ and hence can be written in the form $\phi_2(x) = \int_0^x \int_0^{r_1} \cdots \int_0^{r_{n-1}} f(t)dr_{n-1} \cdots dr_1$, for some $f \in L^p(0, 1)$. The Main Lemma 3.18 implies that $\phi_2 \in \overline{\mathscr{M}_0(Z^0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))}$. This completes the proof.

COROLLARY 3.20. *Every closed ideal $\mathscr{I}$ in $\tilde{\mathscr{R}}_n$ is the intersection of closed primary ideals of $\mathscr{R}_n$.*

Proof.

$$\mathscr{I} = \mathscr{M}(Z_0(\mathscr{I}), \cdots, Z_{n-1}(\mathscr{I}))$$
$$= \bigcap_{i=0}^{n-1} \bigcap_{\lambda \in Z_i(\mathscr{I})} \mathscr{M}(\{\lambda\}, \{\lambda\}, \cdots, \{\lambda\}, \phi, \cdots, \phi)$$

where the multiplicity of λ within the parenthesis is $i + 1$.

REMARK 3.21. All the ideals in $\mathscr{C}_0^n$ are of the form $\mathscr{M}(E_0, \cdots, E_{n-1})$ with $0 \in E_i$, $0 \leqq i \leqq n-1$ and deriv $E_0 \subseteqq E_{n-1} \subseteqq \cdots \subseteqq E_0$, where the E_i's are closed subsets of $[0, 1]$.

4. Invariant subspaces of T_n and closed ideals of $\tilde{\mathscr{R}}_n$: Their correspondence and lattice structures.

J_n is a homeomorphism from $L^p(0, 1)$ onto $\mathscr{R}_n$. Its bounded inverse, J_n^{-1}, is naturally the differentiation operator of order n on $\mathscr{R}_n$, namely

$$J_n^{-1}g(x) = \frac{d^n}{dx^n}g(x),\ g \in \mathscr{R}_n,\ x \in [0, 1] .$$

Moreover, by Leibnitz's rule and Fubini's theorem,

$$J_n^{-1}MJ_nf = T_nf \text{ for } f \in L^p(0, 1), \text{ where } Mf(x) = xf(x),\ x \in [0, 1] .$$

This similarity relation achieves the following correspondance:

$$\mathscr{S} \text{ is a closed } T_n\text{-invariant subspace of } L^p(0, 1) \Leftrightarrow J_n\mathscr{S} \in \mathscr{C}_0^n .$$

(From here onwards $\mathscr{C}_0^n$ will include the improper ideal $\mathscr{R}_n$ and the trivial ideal {zero function}, similarly for the collection $\mathscr{C}^n$.) We have thus proved the following result.

THEOREM 4.1. *The map J_n establishes a one-to-one correspondence between all the closed T_n-invariant subspaces and all the ideals in $\mathscr{C}_0^n$ via the relation $J_n^{-1}MJ_n = T_n$. Thus the closed T_n-invariant subspaces are in one-to-one correspondence with the n-tuples $(E_0, E_1, \cdots, E_{n-1})$ of closed subsets of $[0, 1]$ where $E_0 \supseteqq E_1 \supseteqq \cdots \supseteqq E_{n-1} \supseteqq$ deriv E_0 and $0 \in E_i$, $0 \leqq i \leqq n-1$.*

Observation 4.2. We note an interesting observation that falls out immediately of our above discussion without further effort.

Let $T_1^k = T_1T_1 \cdots T_1$ (k times, $k \in N$), then $T_1^k = J_1^{-1}M(x^k)J_1$, where $M(\phi)f(x) = \phi(x)\cdot f(x)$ (since $T_1 = J_1^{-1}MJ_1$). So

$$T_1^k(x) = x^kf(x) + kx^{k-1}\int_0^x f(t)dt,\ f \in L^p(0, 1) .$$

The linear span $\{x^{k-1}, x^{2k-1}, x^{3k-1}, \cdots\}$ is dense in $L^p(0, 1)$, therefore the linear span of $\{x^k, x^{2k}, x^{3k}, \cdots\}$ is dense in the range of $\mathscr{R}_1$ of J_1. Thus

$\mathscr{S}$ is a closed T_1^k-invariant subspace $\Leftrightarrow J_1\mathscr{S}$ is a closed ideal of $\mathscr{R}_1$.

Hence all the operators in $\{T_1^k\}_{k=1}^{\infty}$ have exactly the same closed invariant subspaces.

The set Lat T_n of all the closed invariant subspaces of T_n is a complete lattice under $\leqq$ where $\mathscr{S}_1 \leqq \mathscr{S}_2$ if $\mathscr{S}_1 \subseteqq \mathscr{S}_2$. Theorem 3.19 puts $\mathscr{C}^n$ in a one-to-one correspondence with the collection of all n-tuples $\{(E_0, E_1, \cdots, E_{n-1})$: E_j's are closed subsets of $[0, 1]$ such that $E_0 \supseteqq E_1 \supseteqq \cdots \supseteqq E_{n-1} \supseteqq \text{deriv } E_0\}$. Define an ordering on $\mathscr{C}^n$ by $(E_0, E_1, \cdots, E_{n-1}) < (F_0, F_1, \cdots, F_{n-1})$ if $E_j \supseteqq F_j$, $0 \leqq j \leqq n-1$. This makes $(\mathscr{C}^n, <)$ into a lattice.

To show completeness of the lattice, consider any subset of $\mathscr{C}^n$, $\{(E_0^\alpha, E_1^\alpha, \cdots, E_{n-1}^\alpha)\colon \alpha \in I \text{ an index set}\}$. It is clear that $\bigvee_\alpha \{(E_0^\alpha, E_1^\alpha, \cdots, E_{n-1}^\alpha)\} = (\bigcap_\alpha E_0^\alpha, \cdots, \bigcap_\alpha E_{n-1}^\alpha) \in \mathscr{C}^n$, and since the smallest element $([0, 1], \cdots, [0, 1]) \in \mathscr{C}^n$, $(\mathscr{C}^n, <, \wedge, \vee)$ is a complete lattice ([2], p. 49). However, it can be checked that

$$\bigwedge_\alpha \{(E_0^\alpha, E_1^\alpha, \cdots, , E_{n-1}^\alpha)\}$$
$$= \left(\overline{\bigcup_\alpha E_0^\alpha}, \overline{\bigcup_\alpha E_1^\alpha} \cup \text{deriv}\left(\overline{\bigcup_\alpha E_0^\alpha}\right), \cdots, \overline{\bigcup_\alpha E_{n-1}^\alpha} \cup \text{deriv}\left(\overline{\bigcup_\alpha E_0^\alpha}\right)\right).$$

Similarly the subcollection $\mathscr{C}_0^n$ is also a complete sublattice of $\mathscr{C}^n$.

THEOREM 4.3. *The map J_n induces a complete lattice isomorphism ψ: Lat $T_n \to \mathscr{C}_0^n$ which is defined as*

$$\psi(\mathscr{S}) = (Z_0(\tilde{\mathscr{S}}), Z_1(\tilde{\mathscr{S}}), \cdots, Z_{n-1}(\tilde{\mathscr{S}})),$$

where $\mathscr{S} \in \text{Lat } T_n$ and $\tilde{\mathscr{S}} = J_n(\mathscr{S})$.

The proof is left to the reader.

5. The operators $M \pm \alpha J_1$ and $M \pm \alpha J_1^*$ with $\text{Re}\,\alpha = n$. We extend our results on T_n to the operator $U_n\colon L^p(0, 1) \to L^p(0, 1)$ where

$$U_n f(x) = xf(x) - n\int_0^x f(t)dt, \; f \in L^p(0, 1), \; n \in N,$$

and their adjoint operators T_n^* and U_n^* on $L^q(0, 1)(1/p + 1/q = 1)$ where

$$T_n^* f(x) = xf(x) + n\int_x^1 f(t)dt, \quad f \in L^q(0, 1),$$
$$U_n^* f(x) = xf(x) - n\int_x^1 f(t)dt, \quad f \in L^q(0, 1).$$

Furthermore, the parameter can be allowed to be complex with integral real part.

Let us first deal with the operator U_n^*. We shall work with $1 \leqq q < \infty$ instead of $1 < q \leqq \infty$.

Define an operator W_n on $L^q(0, 1)$, which is analogous to J_n, as

$$W_n f(x) = \frac{1}{\Gamma(n)} \int_x^1 (x - t)^{n-1} f(t) dt,\ f \in L^q(0, 1) .$$

The same argument used in §3 can be applied with slight modification and we have the following result which is analogous to Theorems 4.1 and 4.3.

THEOREM 5.1. ($1 \leqq q < \infty$). *Via the similarity relation* $W_n^{-1} M W_n = U_n^*$, *there is a complete lattice isomorphism between* $\mathrm{Lat}(U_n^*)$ *and the lattice of all n-tuples* $(E_0, E_1, \cdots, E_{n-1})$ *of closed subsets of* $[0, 1]$ *where* $E_0 \supseteqq E_1 \supseteqq \cdots \supseteqq E_{n-1} \supseteqq \operatorname{deriv} E_0$ *and* $1 \in E_i$, $0 \leqq i \leqq n - 1$.

REMARK 5.2. The closed invariant subspaces of T_n^* (respectively U_n) on $L^p(0, 1)(1 < p < \infty)$ are $\{\mathscr{S}^\perp : \mathscr{S}$ is a closed invariant subspace of T_n (respectively U_n^*)$\}$, where

$$\mathscr{S}^\perp = \left\{\phi \in L^p(0, 1) : \int_0^1 \phi(x) f(x) dx = 0 , \quad \forall f \in \mathscr{S}\right\} .$$

Now we turn to complex parameters with integral real parts. The resulting operators are not more complex than those we have investigated. In fact the real part of the parameter is the similarity invariant of these operators. Kantorovitz ([10], [11]) and Kalisch ([8], [9]) had investigated the similarity invariants of the operators $M + \alpha J_1$ and $M + \alpha J_1^*$ where $\alpha \in \boldsymbol{C}$. We shall quote their result.

THEOREM 5.3. *For* $\beta, \gamma \in \boldsymbol{R}$ *(the real numbers), the operators* $M + \beta J_1$ *and* $M + (\beta + i\gamma) J_1$ *(respectively* $M + \beta J_1^*$ *and* $M + (\beta + i\gamma) J_1^*$*) acting on* $L^p(0, 1)(1 < p < \infty)$ *are similar.*

Through Theorems 4.1, 5.1, 5.3 and Remark 5.2, we have now obtained a complete characterization of all closed invariant subspaces for the operators $M \pm \alpha J_1$ and $M \pm \alpha J_1^*$ in the spaces $L^p(0, 1), 1 < p < \infty$, for those complex values of α where $\operatorname{Re} \alpha$ is a positive integer.

6. The case of the parameter with nonintegral real part. When the parameter has nonintegral rerl part, the functions in the range $\mathscr{R}_\alpha$ of J_α cannot be easily identified. Using results on functional calculus, we establish that $\mathscr{R}_\alpha$, for $\operatorname{Re} \alpha \geqq 1$, is an algebra without unit, and that it can indeed be embedded as a closed ideal of a Silov algebra with unit in the same manner as have done for $\mathscr{R}_n$.

For $n < \alpha < n + 1$, $n \in \boldsymbol{N}$, it is clear that $x^j \notin \mathscr{R}_\alpha$ for $j \leqq n - 1$, but the situation of x^n is governed by the values of p of the L^p spaces.

PROPOSITION 6.1. *For* $n < \alpha < n + 1$, $n \in N$,

(a) *if* $1 \leqq p < 1/(\alpha - n)$, *then* $x^n \in \mathscr{R}_\alpha$,

(b) *if* $p \geqq 1/(\alpha - n)$, *then* $x^n \notin \mathscr{R}_\alpha$.

Proof. Observe that $J_\alpha x^{-(\alpha-n)} = cx^n$, for some constant c.

(a) is trivial since $x^{-(\alpha-n)} \in L^p(0, 1)$.

When $p \geqq 1/(\alpha-n)$, then $x^{-(\alpha-n)} \notin L^p(0, 1)$, but for any $q < 1/(\alpha-n)$, $x^{-(\alpha-n)} \in L^q(0, 1)$. Hence the fact that $J_\alpha: L^q(0, 1) \rightarrow L^q(0, 1)$ is injective implies that $x^n \notin \mathscr{R}_\alpha$.

We shall now define $\tilde{\mathscr{R}}_\alpha$ accordingly: If $1 \leqq p < 1/(\alpha - n)$, define $\tilde{\mathscr{R}}_\alpha = \mathscr{R}_\alpha \oplus Cx^{n-1} \oplus Cx^{n-2} \oplus \cdots \oplus C$; for $p \geqq 1/(\alpha - n)$, define $\tilde{\mathscr{R}}_\alpha = \mathscr{R}_\alpha \oplus Cx^n \oplus Cx^{n-1} \oplus \cdots \oplus C$. We will show that for some natural norm, $\tilde{\mathscr{R}}_\alpha$ is a Silov algebra with $\mathscr{R}_\alpha$ as a closed ideal.

There are some known properties of T_α and the Riemann Liouville holomorphic semigroups $\{J_\alpha: \alpha \in C \text{ and } \operatorname{Re} \alpha > 0\}$ which are useful to us. We shall quote them without proof and refer the reader to Hille and Phillips [6], Kalisch [8], [9] and Kantorovitz [10], [11].

Recall that $J_\alpha f(x) = 1/\Gamma(\alpha)\int_0^x (x - t)^{\alpha-1} f(t)dt$, $f \in L^p(0, 1)$. These operators are injective and bounded on $L^p(0, 1) (1 \leqq p < \infty)$ with $\|J_\alpha\| \leqq 1/(\beta|\Gamma(\alpha)|)$, where $\alpha = \beta + i\gamma$, $\gamma \in R$ and $\beta > 0$. The inverse J_α^{-1} (which will also be denoted by $J_{-\alpha}$), with domain $\mathscr{R}_\alpha$, is thus a closed operator. We also have the identity $(d/dx)J_{\alpha+1}f(x) = J_\alpha f(x)$. $\mathscr{R}_\alpha$ is invariant under M; more precisely, $MJ_\alpha = J_\alpha T_\alpha$.

For $1 < p < \infty$, the semigroup $\{J_\alpha: \alpha \in C.\ \operatorname{Re} \alpha > 0\}$ admits a boundary group of bounded operators $\{J_{i\gamma}: \gamma \in R\}$ on $L^p(0, 1)$ with purely imaginary parameters and $\| J_{i\gamma} \| \leqq e^{\pi|\gamma|/2}$. For $\beta, \gamma \in R$, $J_{i\gamma} T_{\beta+i\gamma} J_{-i\gamma} = T_\beta$; and for $\beta > 0$, $J_{\beta+i\gamma} = J_\beta J_{i\gamma}$ which implies that $\mathscr{R}_{\beta+i\gamma} = \mathscr{R}_\beta$.

In the papers of Kantorovitz [10], [11], it was shown that the operator T_α is of class C^n if and only if $|\operatorname{Re} \alpha| \leqq n$. We say that T_α is of class C^n if there is a continuous representation $\tau: C^n[0, 1] \rightarrow \mathscr{B}(L^p(0, 1))$ where $\mathscr{B}(L^p(0, 1))$ is the set of bounded operators in $L^p(0, 1)$, such that $\tau(1) = I$ (the identity operator) for the function $1(t) = 1$, $t \in [0, 1]$, and $\tau(x) = T_\alpha$. Imitating the argument of Kantorovitz for the C^n-functional calculus, we shall establish the $\tilde{\mathscr{R}}_n$-functional calculus for T_α, denoting the map by $\tau_{\alpha,n}$. (When there is no confusion, we shall just abbreviate $\tau_{\alpha,n}$ by τ). We will then use this $\tilde{\mathscr{R}}_n$-functional calculus to prove that $\tilde{\mathscr{R}}_\alpha$ is a Banach algebra under a norm $|\cdot|_\alpha$ which is a natural generalization of the norm $|\cdot|_n$ of $\tilde{\mathscr{R}}_n$.

For $1 \leqq p < \infty$, $n \in N$, T_n has $\tilde{\mathscr{R}}_n$-functional calculus.

PROPOSITION 6.2. *The map* $\tau: \tilde{\mathscr{R}}_n \rightarrow \mathscr{B}(L^p(0, 1))$ *defined by* $\tau(\phi) =$

$J_n^{-1}M(\phi)J_n$, $\phi \in \tilde{\mathscr{R}}_n$, *is a continuous representation such that* $\tau(1) = I$ *and* $\tau(x) = T_n$, *where* $M(\phi)$ *is the multiplication operator on* $L^p(0, 1)$ *by* ϕ.

Proof. That τ is a representation is trivial.

$\tau(\phi) = \sum_{j=0}^{n} \binom{n}{j} M(\phi^{(j)}) J_j$ (by Leibnitz's rule), and therefore $\|\tau(\phi)f\|_p \leqq c\|f\|_p|\phi|_n$,

where $\binom{n}{j} = n!/j!(n-j)!$ and c is some positive constant depending only on n. Thus τ is continuous and $\mathscr{R}_n$-functional calculus is established.

REMARK 6.3. It is trivial but useful to note that the operator T_0 which is just M also has an $\tilde{\mathscr{R}}_n$-functional calculus for $n \in N$, $1 \leqq p < \infty$, namely $\tau(\phi)f = \phi \cdot f$, $\phi \in \tilde{\mathscr{R}}_n$, $f \in L^p(0, 1)$.

The next lemma is an imitation of an argument of Kantorovitz [10].

LEMMA 6.4. $(1 < p < \infty)$. *Suppose that for some integer* $n \geqq 1$ *and some* $\alpha_0 \in C$, $\operatorname{Re}\alpha_0 \geqq 0$, T_{α_0} *is of class* $\tilde{\mathscr{R}}_n (1 < p < \infty)$, *then* T_α *is of class* $\tilde{\mathscr{R}}_n$ *for all* α *in the strip* $0 \leqq \operatorname{Re}\alpha \leqq \operatorname{Re}\alpha_0$.

Proof. Let $\operatorname{Re}\alpha_0 = \beta_0 \geqq 0$ and $\alpha = \beta + i\gamma$, $\beta, \gamma \in \boldsymbol{R}$. For any fixed polynomial ϕ and vectors $f \in L^p(0, 1)$, $g \in L^q(0, 1)(1/p + 1/q = 1)$, define

$$\Phi(\alpha) = \langle e^{\pi\alpha^2}\phi(T_\alpha)f, g\rangle\ , \quad \alpha \in C\ ,$$

where $\phi(T_\alpha)$ is the polynomial ϕ in T_α. Observe that for $\operatorname{Re}\alpha \leqq \operatorname{Re}\alpha_0 = \beta_0$, $\|T_\alpha\| \leqq 1 + (\beta_0^2 + \gamma^2)^{1/2}$ and $|e^{\pi\alpha^2}| \leqq e^{\pi\beta_0^2}$. Since $\phi(T_\alpha)$ is a polynomial in α, with operator coefficients, we thus have for any $\varepsilon > 0$,

$$|\Phi(\alpha)| = 0(e^{\varepsilon|\gamma|}) \quad \text{as} \quad |\gamma| \longrightarrow \infty$$

in the strip $0 \leqq \operatorname{Re}\alpha \leqq \beta_0$. Furthermore,

$$\|e^{\pi\alpha^2}\cdot\phi(T_\alpha)f\|_p = \|e^{\pi\alpha^2}\cdot(J_{-i\gamma}\phi(T_\beta)J_{i\gamma})f\|_p\ ,$$

therefore

$$\begin{aligned}|\Phi(\alpha)| &\leqq e^{\pi(\beta^2-\gamma^2+|\gamma|)}\|f\|_p\|g\|_p\|\phi(T_\beta)\| \\ &\leqq e^{\pi(\beta^2+1/4)}\|f\|_p\|g\|_q\|\phi(T_\beta)\|\ .\end{aligned}$$

Now T_0 and T_{β_0} are of class $\tilde{\mathscr{R}}_n$, so there is some constant $K > 0$

such that

$$\|\phi(T_0)\| \leqq K|\phi|_n \quad \text{and} \quad \|\phi(T_{\beta_0})\| \leqq K|\phi|_n .$$

Thus

$$|\Phi(0+i\gamma)| \leqq A\|f\|_p\|g\|_q|\phi|_n$$
$$|\Phi(\beta_0+i\gamma)| \leqq A\|f\|_p\|g\|_q|\phi|_n ,$$

where $A = Ke^{(\beta_0^2+1/4)}$. By the Paragman-Lindelof Principle ([22], p. 180),

$$|\Phi(\alpha)| \leqq A\|f\|_p\|g\|_p|\phi|_n , \quad \text{for all} \quad 0 \leqq \operatorname{Re}\alpha \leqq \beta_0 .$$

Therefore for $0 \leqq \operatorname{Re}\alpha \leqq \beta_0$,

$$\|\phi(T_\alpha)\| \leqq Ae^{\pi(\gamma^2-\beta^2)}|\phi|_n .$$

Since the polynomials are dense in $\tilde{\mathscr{R}}_n$, the homomorphism $\phi \to \phi(T_\alpha)$ can be extended continuously to a homomorphism τ on $\tilde{\mathscr{R}}_n$. Thus T_α is class $\tilde{\mathscr{R}}_n$.

An immediate consequence of Proposition 6.2 and Lemma 6.4 is:

COROLLARY 6.5. *For* $n \in N$, $0 \leqq \operatorname{Re}\alpha \leqq n (1 < p < \infty)$, T_α *has* $\tilde{\mathscr{R}}_n$-*functional calculus.*

The $\tilde{\mathscr{R}}_n$-functional calculus of T_α is explicitly determined below.

PROPOSITION 6.6. $(1<p<\infty)$. *For* $0 \leqq \operatorname{Re}\alpha \leqq n$, $\mathscr{R}_\alpha$ *is invariant under* $M(\phi)$, *where* $\phi \in \tilde{\mathscr{R}}_n$, *and the* $\tilde{\mathscr{R}}_n$-*functional calculus* $\tau: \tilde{\mathscr{R}}_n \to \mathscr{B}(L^p(0,1)$ *for* T_α *is given by*

$$\tau(\phi) = J_{-\alpha}M(\phi)J_\alpha, \; \phi \in \tilde{\mathscr{R}}_n .$$

Proof. First let ϕ be any polynomial. Clearly $M(\phi)J_\alpha = J_\alpha\phi(T_\alpha)$; therefore $\phi(T_\alpha) = J_{-\alpha}M(\phi)J_\alpha$. Since ϕ is a polynomial and τ defines the functional calculus, $\tau(\phi) = \phi(T_\alpha)$, and hence $\tau(\phi) = J_{-\alpha}M(\phi)J_\alpha$.

Now let $\phi \in \mathscr{R}_n$. There exists polynomials $\{\phi_k\}$ such that $\phi_k \to \phi$ in $\mathscr{R}_n$; in particular, $\phi_k \to \phi$ uniformly in $[0,1]$. Therefore $\tau(\phi_k) \to \tau(\phi)$ and for any $g \in L^p(0,1)$,

$$(1) \qquad \phi_k \cdot J_\alpha g \longrightarrow \phi \cdot J_\alpha g \quad \text{in} \quad L^p(0,1) ,$$

$$(2) \qquad J_{-\alpha}\phi_k J_\alpha g = \phi_k(T_\alpha)g \longrightarrow \tau(\phi)g \quad \text{in} \quad L^p(0,1) .$$

The graph of $J_{-\alpha}$ is closed in $\mathscr{R}_\alpha \times L^p(0,1)$. Thus (1) and (2) imply that $\phi \cdot J_\alpha g \in \mathscr{R}_\alpha$ and $J_{-\alpha}\phi \cdot J_\alpha g = \tau(\phi)g$.

This proves that $\tau(\phi) = J_{-\alpha}M(\phi)J_\alpha$, for all $\phi \in \tilde{\mathscr{R}}_n$.

COROLLARY 6.7. *For* $\operatorname{Re}\alpha \geqq 1 (1 < p < \infty) \tilde{\mathscr{R}}_\alpha$ *is an algebra.*

Proof. Without loss of generality, we may assume α to be real, since $\mathscr{R}_\alpha = \mathscr{R}_{\operatorname{Re}\alpha}$. It suffices to consider only non-integral values of α and we only need to show that $\mathscr{R}_\alpha$ is closed under multiplication, since $\mathscr{R}_\alpha$ is invariant under M.

Write $\alpha = n + r$ where n is an integer $\geqq 1$, $0 < r < 1$. Let $f, g \in L^p(0, 1)$.

$$J_n^{-1}(J_\alpha(f)\cdot J_\alpha(g)) = \sum_{j=0}^{n}\binom{n}{j} J_{\alpha-j}(f) J_{\alpha-(n-j)}(g) .$$

Proposition 6.6 implies that for $0 \leqq j \leqq n$,

$$J_{\alpha-j}(f)\cdot J_{\alpha-(n-j)}(g) \in J_r(L^p(0, 1)) .$$

Thus $J_\alpha(f)\cdot J_\alpha(g) \in J_{n+r}(L^p(0, 1)) = J_\alpha(L^p(0, 1))$ and hence the result.

On $\mathscr{R}_\alpha$, $\operatorname{Re}\alpha > 0$, let us define $|J_\alpha f|_\alpha = \|f\|_p$, $f \in L_p(0, 1)$. Then J_α is an isometry from $L^p(0, 1)$ onto $\mathscr{R}_\alpha$.

PROPOSITION 6.8. $(1 < p < \infty)$. *For* $\operatorname{Re}\alpha \geqq 1$, *there exists a constant* $c_\alpha > 0$ *such that for all* $f, g \in L^p(0, 1)$,

$$|J_\alpha(f)\cdot J_\alpha(g)|_\alpha \leqq c_\alpha |J_\alpha(f)|_\alpha |J_\alpha(g)|_\alpha .$$

Proof. We need to consider only real nonintegral α. Write $\alpha = n + r$ where n is an integer $\geqq 1$ and $0 < r < 1$. By Leibnitz's rule,

$$J_n^{-1}(J_{n+r}(f)\cdot J_{n+r}(g)) = \sum_{j=0}^{n}\binom{n}{j} J_{n+r-j}(f) J_{n+r-(n-j)}(g) .$$

Therefore $|J_{n+r}(f)\cdot J_{n+r}(g)|_\alpha \leqq \sum_{j=0}^{n}\binom{n}{j}|J_{n+r-j}(f)\cdot J_{r+j}(g)|_r$. For $1 \leqq j \leqq n$, by Proposition 6.6,

$$J_{r+j}(g)\cdot J_r(J_{n-j}(f)) = J_r(\tau_{r,1}(J_{r+j}(g)))(J_{n-j}(f)) .$$

Now

$$|J_{r+j}(g)|_1 \leqq \frac{1}{\Gamma(r+j)}\|g\|_p ,$$

therefore

$$\begin{aligned} |J_{n+r-j}(f)\cdot J_{r+j}(g)|_r &= \|(\tau_{r,1}(J_{r+j}(g)))J_{n-j}(f)\|_p \\ &\leqq \|\tau_{r,1}\|\frac{1}{\Gamma(r+j)}\|g\|_p \frac{1}{(n-j)!}\|f\|_p . \end{aligned}$$

Similarly there exists a constant $K_\alpha > 0$, dependent only on α, such that

$$|J_{n+r}(f)\cdot J_r(g)| \leqq K_\alpha \|f\|_p \|g\|_p .$$

Thus there exists a constant $c_\alpha > 0$ which depends only on α, such that

$$|J_{n+r}(f)\cdot J_{n+r}(g)|_\alpha \leqq c_\alpha |J_{n+r}(f)|_\alpha |J_{n+r}(g)|_\alpha , \quad \text{for all} \quad f, g \in L^p(0, 1) .$$

Now we can norm $\tilde{\mathscr{R}}_\alpha$ a Banach algebra. Our attention will be restricted to the case when $1 \leqq p < 1/(\alpha - n)$ since the same argument holds for the complementary case.

Every element F of $\tilde{\mathscr{R}}_\alpha$ is of the form $J_\alpha f + \sum_{i=0}^{n-1} c_i x_i$, $f \in L^p(0, 1)$, $c_i \in C$. We extend the norm $|\cdot|_\alpha$ on $\mathscr{R}_\alpha$ to $\tilde{\mathscr{R}}_\alpha$, namely,

$$|F|_\alpha = \|f\|_p + \sum_{n=0}^{n-1} |c_i| ,$$

then $\tilde{\mathscr{R}}_\alpha$ is a Banach space with $\mathscr{R}_\alpha$ as a closed ideal. Furthermore, $|\cdot|_\alpha$ is equivalent to some Banach algebra norm, i.e., for any $F, G \in \tilde{\mathscr{R}}_\alpha$, there exists a constant K_α, dependent only on α and p, such that

$$|FG|_\alpha \leqq K_\alpha |F|_\alpha |G|_\alpha .$$

The map $F \to F^{(j)}(a)$, $0 \leqq j \leqq \alpha - 1$, is continuous on $\tilde{\mathscr{R}}_\alpha$.

As it would have been expected, the maximal ideal space of $\tilde{\mathscr{R}}_\alpha$ is [0, 1] and it is then evident that $\tilde{\mathscr{R}}_\alpha$ is semisimple. With C^∞ being contained in $\tilde{\mathscr{R}}_\alpha$, it is clear that $\tilde{\mathscr{R}}_\alpha$ a Silov algebra.

By carrying out the same argument used previously for $\tilde{\mathscr{R}}_n$ functional calculus, we can establish the $\tilde{\mathscr{R}}_\alpha$-functional calculus, $\operatorname{Re}\alpha \geqq 1$.

THEOREM 6.10. ($1<p<\infty$). *For* $\alpha, \beta \in C$, $\operatorname{Re}\alpha \geqq 1$ *and* $0 \leqq \operatorname{Re}\beta \leqq \operatorname{Re}\alpha$, T_β *has* $\tilde{\mathscr{R}}_\alpha$*-functional calculus* $\tau\colon \tilde{\mathscr{R}}_\alpha \to \mathscr{B}(L^p(0, 1))$,

$$\tau(\phi) = J_{-\beta} M(\phi) J_\beta, \ \phi \in \tilde{\mathscr{R}}_\alpha .$$

We conclude our discussion with the following remark.

REMARK 6.11. By virtue of the identity $J_{-\alpha} M J_\alpha = T_\alpha$ and the fact that the polynomials are dense in $\mathscr{R}_\alpha$, $\operatorname{Re}\alpha > 1$, the closed T_α-invariant subspaces are in one-to-one correspondence with the closed ideals of $\mathscr{R}_\alpha$ which are closed under multiplication by the function x.

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Received August 9, 1980. This paper is condensed from the author's Ph. D. dissertation at University of California, Berkeley, June 1978. The author wishes to express her deep gratitude to her thesis adviser Professor W. G. Badé and Professor D. Sarason.

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