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THE VOLUME CUT OFF A SIMPLEX BY A HALF-SPACE

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A formula for the volume cut off an n-dimensional simplex by a half-space has immediate application in probability theory. This note presents a derivation of such a formula in a short and completely elementary way and also yields the moments of this volume about the coordinate hyperplanes, and the volumes cut off the lower dimensional faces of the simplex and their moments.

Before stating our results we introduce some notation. The "minus function" $(x)_-^k$ equals x^k if $x < 0$ and is zero otherwise. The nth divided difference of a real-valued function $f(x)$ is the symmetric function of $n+1$ arguments defined inductively by

$$D\{f(x)\colon x_0, x_1\} = \frac{f(x_0) - f(x_1)}{x_0 - x_1},$$

$$\begin{aligned} &D\{f(x)\colon x_0, x_1, \cdots, x_n\} \\ &\quad = \frac{D\{f(x)\colon x_0, x_1, \cdots, x_{n-1}\} - D\{f(x)\colon x_n, x_1, \cdots, x_{n-1}\}}{x_0 - x_n}. \end{aligned}$$

If $x_0 = x_n$ we define

$$D\{f(x)\colon x_0, x_1, \cdots, x_{n-1}, x_0\} = \frac{\partial}{\partial x_0} D\{f(x)\colon x_0, x_1, \cdots, x_{n-1}\}$$

and similarly for all repeated arguments. The algorithm of Varsi [1, p 317] enables the calculation of such differences for $f(x) = (x)_-^k$ without the use of limiting processes. A proof of the symmetry and other elementary properties of divided differences may be found in [4].

The unit n-simplex with vertices $\boldsymbol{a}_0 = (0, \cdots 0)$, $\boldsymbol{a}_1 = (1, 0, \cdots, 0)$, $\cdots$, $\boldsymbol{a}_n = (0, \cdots, 0, 1)$ will be denoted by $\mathscr{A}_n$, and $\mathscr{B}_n$ will denote an n-simplex with vertices $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n$ where $\boldsymbol{b}_i$ has cartesian coordinates $(b_{i1}, \cdots, b_{in})$. We denote by $\mathscr{B}_k$ the k-face of $\mathscr{B}_n$ with vertices $\boldsymbol{b}_0$, $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_k$. The n-volume (n-dimensional volume) of $\mathscr{B}_n$ and the k-volume of $\mathscr{B}_k$, $k = 1, \cdots, n$ are given by

$$v_n(\mathscr{B}_n) = \frac{1}{n!}\begin{vmatrix} 1 & b_{01} \cdots b_{0n} \\ & \cdots \\ 1 & b_{n1} \cdots b_{nn} \end{vmatrix},$$

$$v_k(\mathscr{B}_k) = \frac{1}{k!}\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & \boldsymbol{b}_0\cdot\boldsymbol{b}_0 & \boldsymbol{b}_0\cdot\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_0\cdot\boldsymbol{b}_k \\ & & \cdots & & \\ 1 & \boldsymbol{b}_k\cdot\boldsymbol{b}_0 & \boldsymbol{b}_k\cdot\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_k\cdot\boldsymbol{b}_k \end{vmatrix}^{1/2} .$$

We are now ready for our main result.

THEOREM. *Consider a linear function* $L(\boldsymbol{x}) = a_0 + a_1x_1 + \cdots + a_nx_n$ *and the half-space* $\mathscr{H}$ *defined by* $L(\boldsymbol{x}) \leqq 0$. *Let* p *be any non-negative integer. Then for* $k = 1, 2, \cdots, n$:

1. $$\int_{\mathscr{B}_k} [L(\boldsymbol{x})]^p dv_k = \frac{k!\,p!}{(k+p)!} v_k(\mathscr{B}_k) D\{x^{k+p}: L(\boldsymbol{b}_0), L(\boldsymbol{b}_1), \cdots, L(\boldsymbol{b}_k)\} .$$

2. *The k-volume cut off the k-simplex* $\mathscr{B}_k$ *by* $\mathscr{H}$ *is given by*

$$v_k(\mathscr{B}_k \cap \mathscr{H}) = v_k(\mathscr{B}_k) D\{(x)_-^k: L(\boldsymbol{b}_0), L(\boldsymbol{b}_1), \cdots, L(\boldsymbol{b}_k)\} .$$

3. *If* $p = p_1 + \cdots + p_n$, *then the* pth *mixed moment of* $\mathscr{B}_k \cap \mathscr{H}$ *about the coordinate hyperplanes is given by*

$$\int_{\mathscr{B}_k \cap \mathscr{H}} x_1^{p_1} \cdots x_n^{p_n} dv_k = \frac{k!}{(k+p)!} v_k(\mathscr{B}_k) \frac{\partial^p}{\partial a_1^{p_1} \cdots \partial a_n^{p_n}} D\{(x)_-^{k+p}: L(\boldsymbol{b}_0), L(\boldsymbol{b}_1), \cdots, L(\boldsymbol{b}_k)\} .$$

Proof. Our first observation is that for all p

$$\int_{\mathscr{A}_n} [c_0 + (c_1 - c_0)y_1 + \cdots + (c_n - c_0)y_n]^p d\boldsymbol{y} = \frac{p!}{(n+p)!}\{x^{n+p}: c_0, c_1, \cdots, c_n\}$$

where $\int_{\mathscr{A}_n} f(\boldsymbol{y})d\boldsymbol{y}$ means $\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_n=0}^{1-y_1-\cdots-y_{n-1}} f(\boldsymbol{y})\,dy_n \cdots dy_1$. The The proof is by a straightforward induction on n when the c_i are distinct, and by continuity otherwise.

To prove part 1 for $k = n$ consider the transformation defined by

$$x_1 = b_{01} + (b_{11} - b_{01})y_1 + \cdots + (b_{n1} - b_{01})y_n$$
$$\cdots$$
$$x_n = b_{0n} + (b_{1n} - b_{0n})y_1 + \cdots + (b_{nn} - b_{0n})y_n ,$$

so when $\boldsymbol{y} = \boldsymbol{a}_i$ then $\boldsymbol{x} = \boldsymbol{b}_i$, $i = 0, 1, \cdots, n$. Note that the Jacobian of this transformation is

$$\begin{vmatrix} b_{11}-b_{01} & \cdots & b_{n1}-b_{01} \\ & \cdots & \\ b_{1n}-b_{0n} & \cdots & b_{nn}-b_{0n} \end{vmatrix} = \begin{vmatrix} 1 & 1 \cdots & 1 \\ b_{01} & b_{11} \cdots & b_{n1} \\ & \cdots & \\ b_{0n} & b_{1n} \cdots & b_{nn} \end{vmatrix} = n!\; v_n(\mathscr{B}_n) .$$

Then

$$\int_{\mathscr{B}_n} [L(x_1, \cdots, x_n)]^p dx_1 \cdots dx_n/n!\; v_n(\mathscr{B}_n)$$
$$= \int_{\mathscr{A}_n} \{L(\boldsymbol{b}_0) + [[L(\boldsymbol{b}_1) - L(\boldsymbol{b}_0)]y_1 + \cdots + [L(\boldsymbol{b}_n) - L(\boldsymbol{b}_0)y_n]\}^p d\boldsymbol{y}$$
$$= \frac{p!}{(n+p)!} D\{x^{n+p}: L(\boldsymbol{b}_0), L(\boldsymbol{b}_1), \cdots, L(\boldsymbol{b}_n)\} .$$

For the case $1 \leqq k \leqq n-1$ we may assume $\mathscr{B}_k$ is not orthogonal to the coordinate k-flat $\mathscr{F}$ defined by $x_{k+1} = \cdots = x_n = 0$ (since $\mathscr{B}_k$ cannot be orthogonal to all $\binom{n}{k}$ such k-flats) so $\mathscr{B}_k$ can be defined by linear equations in the form

$$x_j = M_j(x_1, \cdots, x_k) , \quad j = k+1, \cdots, n .$$

Let $\boldsymbol{x}^*$ denote the orthogonal projection of a point $\boldsymbol{x}$ on $\mathscr{F}$ and let

$$L^*(x_1, \cdots, x_k) = L(x_1, \cdots, x_k, M_{k+1}(x_1, \cdots, x_k), \cdots, M_n(x_1 \cdots, x_k))$$

so for any point $\boldsymbol{x}$ of $\mathscr{B}_k$ we have $L^*(\boldsymbol{x}^*) = L(\boldsymbol{x})$, and in particular, $L^*(\boldsymbol{b}_i^*) = L(\boldsymbol{b}_i)$, $i = 0, 1, \cdots, k$. Since orthogonal projection shrinks dv_k and $v_k(\mathscr{B}_k)$ by the same factor, we have

$$\int_{\mathscr{B}_k} [L(\boldsymbol{x})]^p dv_k/v_k(\mathscr{B}_k) = \int_{\mathscr{B}_k^*} [L^*(x_1, \cdots, x_k)]^p dx_1 \cdots dx_k/v_k(\mathscr{B}_k{}^*)$$
$$= \frac{k!\,p!}{(k+p)!} D\{x^{k+p}: L^*(\boldsymbol{b}_0^*), L^*(\boldsymbol{b}_1^*), \cdots, L^*(\boldsymbol{b}_k^*)\}$$

which completes the proof of the first result.

Now it is easy to verify that if $a \neq 0$ then

$$\int (ax+b)_-^p dx = \frac{ax+b)_-^{p+1)}}{a(p+1)} + c .$$

This means that the minus function integrates formally like a polynomial and we may conclude that

$$\int_{\mathscr{B}_k} [L(\boldsymbol{x})]_-^p dv_k = \frac{k!\,p!}{(k+p)!} v_k(\mathscr{B}_k) D\{(x)_-^{p+k}: L(\boldsymbol{b}_0), L(\boldsymbol{b}_1), \cdots, L(\boldsymbol{b}_k)\} .$$

In particular, since $\boldsymbol{x} \in \mathscr{H}$ if and only if $L(\boldsymbol{x}) \leqq 0$ we have

$$v_k(\mathscr{B}_k \cap \mathscr{H}) = \int_{\mathscr{B}_k} [L(\boldsymbol{x})]_-^0 dv_k = v_k(\mathscr{B}_k) D\{(x)_-^k: L(\boldsymbol{b}_0), L(\boldsymbol{b}_1), \cdots, L(b_k)\}$$

which is our second result.

Our final result comes from the observation that

$$\int_{\mathscr{B}_k \cap \mathscr{H}} x_1^{p_1} \cdots x_n^{p_n} dv_k = \frac{1}{p!} \int_{\mathscr{B}_k} \frac{\partial^p}{\partial a_1^{p_1} \cdots \partial a_n^{p_n}} [L(\boldsymbol{x})]_-^p dv_k \, .$$

COROLLARY. *Let $\mathscr{H}$ be the half-space defined by $a_0 + a_1x_1 + \cdots + a_nx_n \leqq 0$. Then*

$$\begin{aligned} v_n(\mathscr{A}_n \cap \mathscr{H})/v_n(\mathscr{A}_n) &= D\{(x)_-^n : a_0, a_0 + a_1, \cdots, a_0 + a_n\} \\ &= D\{(x + a_0)_-^n : 0, a_1, \cdots, a_n\} \end{aligned}$$

where $v_n(\mathscr{A}_n) = 1/n!$, *and*

$$v_{n-1}(\mathscr{A}_{n-1} \cap \mathscr{H})/v_{n-1}(\mathscr{A}_{n-1}) = D\{(x + a_0)_-^{n-1} : a_1, \cdots, a_n\}$$

where $\mathscr{A}_{n-1}$ is the $(n-1)$-face with vertices $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$, and $v_{n-1}(\mathscr{A}_{n-1}) = \sqrt{n}/(n-1)!$

Varsi [5] discovered an algorithm equivalent to the first result of the corollary by dissecting $\mathscr{A}_n$ into $O(2^n)$ simplexes; his algorithm requires $O(n^2)$ computations and is hence quite efficient. Ali [1] observed that Varsi's algorithm was equivalent to divided differences and proved this result using the Fourier-Stieltjes transform; he also proved the second result of the corollary. The technique of integrating a "plus function" was used in [2] to find the volume cut off a hypercube by a half-space, a result which was obtained previously in [3, pp. 48–50] using a more elementary technique which, curiously enough, involved a density function defined on 2^n overlapping regions of space.

Added in proof. The third part of the theorem can be put in a more useful form. For example, using L_i for $L(\boldsymbol{b}_i)$, the centroid $\mathscr{B}_k \cap \mathscr{H}$ is given by

$$\frac{D\{(x)_-^{k+1} : L_0, L_0, L_1, L_2, \cdots, L_k\}\boldsymbol{b}_0 + 0\{(x)^{k+1} : L_0, L_1, L_1, L_2, \cdots, L_k\}\boldsymbol{b}_1 + \cdots}{(k+1)D\{(x)_-^k : L_0, L_1, L_2, \cdots, L_k\}} \, .$$

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