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JOINT BROWDER SPECTRUM

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There exist in the literature many notions of joint spectra which have been then generalized to joint essential spectra and in some instances to joint Browder spectra. The purpose of this note is to develop a notion of joint Browder spectrum which uses ideas of Arveson and Waelbrook. It is shown that this notion of joint Browder spectrum has many of the useful properties of the Browder spectrum.

Polynomial joint Browder spectrum. For an operator (bounded operator) A on a Banach space $X, \lambda \in \rho_b(A)$ if $\lambda - A$ is Fredholm and there exists a deleted open neighborhood N of λ, such that $\mu \in N$ implies that $\mu - A$ is invertible. The Browder essential spectrum of $A, \sigma_b(A)$, is the complement of $\rho_b(A)$. Thus $\sigma_b(A) = \sigma_e(A) \cup \{\text{accumulation of points of } \sigma(A)\}$, where $\sigma_e(A)$ and $\sigma(A)$ denote the Fredholm spectrum and the spectrum of A [4], respectively. Also $\sigma_b(A) = \sigma(\pi(A))$, where $\pi: \mathfrak{a} \to \mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$, $\mathfrak{a}$ is a maximal commutative subalgebra of $\mathscr{B}(X)$ ($\mathscr{B}(X)$ is the algebra of all bounded operators on X) containing the double commutant of A, and $\mathscr{K}$ is the ideal of compact operators on X. This characterization of $\sigma_b(A)$ is due to Gramsch and Lay [5]. Let A_1 and A_2 be two commuting operators on X. Let $\mathscr{B}$ be any commutative Banach algebra containing A_1 and A_2. Then the joint spectrum of the pair $A = (A_1, A_2)$ with respect to the algebra $\mathscr{B}$ is usually defined as $\{(\phi(A_1), \phi(A_2)): \phi$ is in the maximal ideal space of $\mathscr{B}\}$ and is denoted by $\sigma_{\mathscr{B}}(A_1, A_2) = \sigma_{\mathscr{B}}(A)$. Arveson [1] defines $\sigma(A_1, A_2) = \sigma(A) = \{\lambda = (\lambda_1, \lambda_2): p(\lambda) \in \sigma(p(A))$ for all polynomials $p: \boldsymbol{C}^2 \to \boldsymbol{C}\}$ and shows that $\sigma(A) = \sigma_{\mathscr{B}}(A)$, where $\mathscr{B}$ is the norm closure of the rational algebra generated by A_1 and A_2 [1], $\boldsymbol{C}$ is the set of all complex numbers and $\boldsymbol{C}^2$ is the two-fold Cartesian product of $\boldsymbol{C}$. Following Arveson we define the joint Browder spectrum of $A = (A_1, A_2)$, denoted by $\sigma_b^p(A)$, as and defined by

$$\sigma_b^p(A) = \{\lambda = (\lambda_1, \lambda_2): p(\lambda) \in \sigma_b)(p(A)) \text{ for all polynomials } p: \boldsymbol{C}^2 \longrightarrow \boldsymbol{C}\}.$$

Also the joint Fredholm spectrum

$$\sigma_e^p(A) = \{\lambda = (\lambda_1, \lambda_2): p(\lambda) \in \sigma_e(p(A)) \text{ for all polynomials } p: \boldsymbol{C}^2 \longrightarrow \boldsymbol{C}\}.$$

When there is no confusion, we shall write the polynomial joint Browder spectrum and the polynomial joint Fredholm spectrum as $\sigma_b(A)$ and $\sigma_e(A)$, respectively.

In [7] Snow has defined the joint Browder spectrum of A_1 and A_2 in terms of the joint Fredholm spectrum of Schechter and Snow [6]. Our treatment of the subject is different from theirs. Although all our results hold for n-tuples of operators, for brevity we shall discuss them for pairs of operators only. Throughout the text we shall write a pair of operators as $A = (A_1, A_2)$ and a pair of complex numbers as $\lambda = (\lambda_1, \lambda_2)$ unless otherwise mentioned. In the first part we shall obtain an analogue of Arveson's theorem and a generalization of Gramsch and Lay's result. In the later part we shall obtain a relationship between $\sigma_b(A)$ and $\sigma_e(A)$, where $A=(A_1, A_2)$.

Let $\mathfrak{a}$ be a maximal commutative subalgebra of $\mathscr{B}(X)$ containing the double commutant of A_1, A_2. We need the following lemmas to prove our main result. For an operator A_1 we shall denote the image of A_1 in the algebra $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$ by $\hat{A}_1$. We shall write $\hat{A} = (\hat{A}_1, \hat{A}_2)$.

LEMMA 1. *$\{\omega(\hat{A}) = (\omega(\hat{A}_1), \omega(\hat{A}_2)) \colon \omega$ is in maximal ideal space of $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}\} \subseteqq \sigma_b(A)$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2) \in \omega(A) = (\omega(A_1), \omega(A_2))$. Then for any polynomial p of two variables, $\omega(p(\lambda) - p(\hat{A})) = 0$ and hence $p(\lambda) - p(\hat{A})$ is not invertible in $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$. Thus by the above mentioned result of Gramsch and Lay [5], $p(\lambda)$ is in $\sigma_b(p(A))$ for all polynomials p.

LEMMA 2. *$\sigma_b(A)$ is a nonempty compact subset of $\sigma_b(A_1) \times \sigma_b(A_2)$.*

Proof. $\sigma_b(A)$ being the intersection of closed sets is closed. Let $\lambda = (\lambda_1, \lambda_2) \in \sigma_b(A)$, by taking $p(z_1, z_2) = z_i$, it follows that $\lambda_i \in \sigma_b(A_i)$ for $i = 1, 2$. Lemma 1 assures that $\sigma_b(A)$ is nonempty.

LEMMA 3. *Let $p(\lambda_1, \lambda_2)$ be a polynomial with no zeros on $\sigma_b(A)$. Then $p(\hat{A})$ is invertible in $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$.*

Proof. Suppose $p(\hat{A}) = p(\hat{A}_1, \hat{A}_2)$ is not invertible in $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$, then there exists an ω in the maximal ideal space of $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$ such that $\omega(p(\hat{A})) = 0$. Let $\lambda = (\lambda_1, \lambda_2) = \omega(\hat{A})$. Thus $p(\lambda) = 0$, while $\lambda \in \sigma_b(A)$.

Let Z be any compact subset of C^2. Let Rat (Z) denote the set of all rational functions on Z, that is, all quotients p/q of polynomials p and q (in two variables) for which q has no zeros on Z. Consider in particular $Z = \sigma_b(A) = \sigma_b(A_1, A_2)$. Now for $f \in \text{Rat}\,(Z)$, $f = p/q$, define $f(\hat{A}) = p(\hat{A}) \cdot q(\hat{A})^{-1}$ (Lemma 3 guarantees that $q(\hat{A})$ is invertible.) Thus there is an algebraic homomorphism of Rat (Z)

into $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$. Let $\operatorname{Rat}(\hat{A}) = \operatorname{Rat}(\hat{A}_1, \hat{A}_2)$ denote the norm closure of the image of $\operatorname{Rat}(Z)$ in $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$.

LEMMA 4. *Let* $Z = \sigma_b(A_1, A_2) = \sigma_b(A)$. *Then* $\sigma_b(f(A)) = f(Z)$ *for all* f *in* $\operatorname{Rat}(Z)$.

Proof. Let $f \in \operatorname{Rat}(Z)$ be such that f has no zeros on Z. Thus $f = g/h$, where g and h are polynomials having no zeros on Z. By Lemma 3, both $g(\hat{A})$ and $h(\hat{A})$ are invertible in $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$, and hence $f(\hat{A}) = g(\hat{A})h(\hat{A})^{-1}$ is invertible. Thus $0 \notin f(Z)$ implies that $0 \notin \sigma_b(f(\hat{A}))$. A simple translation argument shows that $\sigma_b(f(\hat{A})) \subseteqq f(Z)$.

Let $\lambda = (\lambda_1, \lambda_2) \in Z$. Then $f - f(\lambda)$ has the form g/h where g and h are polynomials and h has no zeros on Z. Also $g(\lambda) = 0$, by definition of $\sigma_b(A)$, $0 = g(\lambda) \in \sigma_b(g(A))$. Thus $g(\hat{A})$ is not invertible for $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$ and $f(\lambda) \in \sigma_b(f(\hat{A}))$. This proves the lemma.

LEMMA 5. $\operatorname{Rat}(\hat{A})$ *is inverse closed in* $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$.

Proof. Suppose $\hat{s} \in \operatorname{Rat}(\hat{A})$ and $\hat{s}^{-1} \in \mathfrak{a}/\mathfrak{a} \cap \mathfrak{K}$. Since $\hat{s} \in \operatorname{Rat}(\hat{A})$, there exists a sequence $f_n \in \operatorname{Rat}(\sigma_b(\hat{A}))$ such that $\|\hat{s} - f_n(\hat{A})\| \rightarrow 0$. Since $\hat{s}^{-1}$ exists, for large enough n, $(f_n(\hat{A}))^{-1}$ exists and $(f_n(\hat{A}))^{-1} \rightarrow \hat{s}^{-1}$. By Lemma 4, f_n has no zeros on $\sigma_b(A)$. Let $g_n = 1/f_n$. Then $g_n \in \operatorname{Rat}(\sigma_b(A))$ and $\|\hat{s}^{-1} - g_n(\hat{A})\| \rightarrow 0$ which implies that $\hat{s}^{-1} \in \operatorname{Rat}(\hat{A})$. Hence the result.

THEOREM 6. *Let* $\mathfrak{m}$ *be the maximal ideal space of* $\operatorname{Rat}(\hat{A})$. *Then* $\sigma_b(A) = \{\omega(\hat{A}) : \omega \in \mathfrak{m}\}$.

Proof. For any $\omega \in \mathfrak{m}$, let $\lambda = (\lambda_1, \lambda_2) = (\omega(\hat{A}_1), \omega(\hat{A}_2)) = \omega(\hat{A})$. For each polynomial $p(z_1, z_2)$ we have $\omega(p(\lambda) - p(\hat{A})) = 0$. Thus $p(\lambda) - p(\hat{A})$ is not invertible in $\operatorname{Rat}(\hat{A})$, and by Lemma 5, it is not invertible in $\mathfrak{a}/\mathfrak{a} \cap \mathscr{K}$. Thus by the result of Gramsch and Lay, $p(\lambda) - p(\hat{A})$ is not invertible. Thus $p(\lambda) \in \sigma_b(p(A))$ for every polynomial p and hence $\lambda \in \sigma_b(A)$.

Conversely, let $\lambda \in \sigma_b(A)$. Then for every $f \in \operatorname{Rat}(Z)$, $Z = \sigma_b(A)$, we have $|f(\lambda)| \leqq \sup_Z |f(z_1, z_2)| = \sup |\sigma(f(\hat{A}))| \leqq \|f(\hat{A})\|$. Thus $f(\hat{A}) \rightarrow f(\lambda)$ is a bounded, densely defined homomorphism of $\operatorname{Rat}(\hat{A})$ and so there is an $\omega \in \mathfrak{m}$ such that $\omega(f(\hat{A})) = f(\lambda)$. The conclusion now follows by using $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$, where $f_i(z_1, z_2) = z_i$, $i = 1, 2$.

Theorem 6 is the analogue of a result of Arveson [1, page 276]. The techniques used here are heavily based on the definition of $\sigma_b(A_1, A_2)$ in terms of polynomials of two variables.

We let $\tilde{A}_1, \tilde{A}_2$ denote the image of A_1 and A_2 in the Calkin algebra, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2)$; then an analogue of Lemma 3 will assure that $q(\tilde{A})$ is invertible in the Calkin algebra provided q has no zeros on $\sigma_e(A)$. Thus there is an algebraic homomorphism of $\mathrm{Rat}\,(\sigma_e(A))$ into the Calkin algebra, and we let $\mathrm{Rat}\,(\tilde{A}) = \mathrm{Rat}\,(\tilde{A}_1, \tilde{A}_2)$ denote the norm closure of the image of $\mathrm{Rat}\,(\sigma_e(A))$ in the Calkin algebra. The analogues of Lemmas 4, 5 and Theorem 6 can now be proved in terms of $\sigma_e(A)$ to get the following theorem.

THEOREM 7. *Let* $\mathfrak{m}'$ *be the maximal ideal space of* $\mathrm{Rat}\,(\tilde{A})$. *Then* $\sigma_e(A) = \{(\omega(\tilde{A})\colon \omega \in \mathfrak{m}'\}$.

Theorems 6 and 7 and the result of Arveson [1, page 276] now allow us to identity $\sigma_b(A)$, $\sigma_e(A)$ and the $\sigma(A)$ defined above with the respective maximal ideal spaces of the respective finitely generated rational algebras $\mathrm{Rat}\,(\hat{A})$, $\mathrm{Rat}\,(\tilde{A})$ and $\mathrm{Rat}\,(A)$. Now we want to see how these three spectra are related to each other. In the case of a single operator T, it is well known that $\sigma_e(T) \subseteqq \sigma_b(T) \subseteqq \sigma(T)$ and $\sigma_b(T) = \sigma_e(T) \cup \{\text{accumulation points of } \sigma(T)\}$. We shall show that similar results hold in the case of joint spectra.

First let us note that if $(\lambda_1, \lambda_2) \notin \sigma(A_1, A_2)$ then there exists a nonconstant polynomial p such that $p(\lambda_1, \lambda_2) \notin \sigma(p(A_1, A_2))$ and hence $(\lambda_1, \lambda_2) \notin \sigma_b(A_1, A_2)$. Thus $\sigma_b(A_1, A_2) \subseteqq \sigma(A_1, A_2)$, similarly $\sigma_e(A_1, A_2) \subseteqq \sigma_b(A_1, A_2)$.

THEOREM 8. $\sigma_b(A_1, A_2) = \sigma_e(A_1, A_2) \cup \{$*accumulation points of* $\sigma(A_1, A_2)\}$.

Let $\lambda = (\lambda_1, \lambda_2)$ be an accumulation point of $\sigma(A)$. For a nonconstant polynomial p, by using the continuity of p, it follows that $p(\lambda)$ is an accumulation point of $\sigma(p(A))$ and hence is contained in $\sigma_b(p(A))$. We have already noted that $\sigma_e(A) \subseteqq \sigma_b(A)$. Thus we have shown that $\sigma_e(A) \cup \{\text{accumulation points of } \sigma(A)\}$ is contained in $\sigma_b(A)$.

Let $\lambda = (\lambda_1, \lambda_2)$ be an isolated point of $\sigma(A)$ and suppose $\lambda \notin \sigma_e(A)$. Thus $\sigma(A) = \{\lambda\} \cup F_1$ where $F_1 = \sigma(A)\backslash\{\lambda\}$. By the previously mentioned identification the maximal ideal space of $\mathrm{Rat}\,(A)$ is $\{\lambda\} \cap F_1$. So there exists an idempotent E [9, page 96] such that $\mathrm{Rat}\,(A) = E\,\mathrm{Rat}\,(A) \oplus (I - E)\,\mathrm{Rat}\,(A)$ and the maximal ideal space of the two subalgebras can be identified with $\{\lambda\}$ and F_1 respectively. Thus $\{\lambda\} = \{(\lambda_1, \lambda_2)\} = \sigma(EA) = \sigma(EA_1, EA_2)$ and $F_1 = \sigma((I-E)A)$. Let $\tilde{E}$ be the image of E in the Calkin algebra, then $\mathrm{Rat}\,(\tilde{A}) = \tilde{E}\,\mathrm{Rat}\,\tilde{A} \oplus (\tilde{I} - \tilde{E})$ $\mathrm{Rat}\,(\tilde{A})$. This decomposition of the algebra now gives rise to a decomposition of the maximal ideal space of $\mathrm{Rat}\,\tilde{A}$ [9, page 96]; using Theorem 7 it follows that $\sigma_e(A) = \sigma_e(EA) \cup \sigma_e((I-E)A)$. Thus in particular $\sigma_e(EA) \subseteqq \sigma_e(A)$, but $\sigma_e(EA) \subseteqq \sigma(EA)$ and hence $\sigma_e(EA) \subseteqq \sigma(EA) \cap \sigma_e(A) = \varnothing$ (because $\lambda \notin \sigma_e(A)$). Thus EA is compact; i.e., E is a finite-dimensional idempotent, hence $\hat{E} = 0$. By Theorem 6, since E is compact, $\sigma_b(A) = \sigma_b((I - E)A)$. Also $\sigma_b((I-E)A) \subseteqq \sigma((I - E)A)$, hence $\lambda \notin \sigma_b(A)$.

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