

## QUASIREGULAR, PSEUDOCOMPLETE, AND BAIRE SPACES

AARON R. TODD

**J. C. Oxtoby obtains the essence of the classical Baire category theorems in his pseudocomplete spaces. In doing so, he reduces the role of points in his auxiliary definitions of quasiregular spaces and pseudobases. With a natural modification of his definition of pseudobase that continues this reduction, a topological space is quasiregular if it has a pseudobase of closed sets. With this change, strong nesting is explicitly required in the definition of pseudocomplete spaces. The change also leads to an equivalence relation on the topologies of a set  $X$ : Topologies  $\sigma$  and  $\tau$  are  $S$ -related if  $\tau^* = \tau \setminus \{\phi\}$  is a pseudobase for  $\sigma$ . It also leads to conditions for which a topology finer or coarser than a pseudocomplete topology is itself pseudocomplete. Several examples illustrate the utility of quasiregularity, and there is a discussion of extensions of topological spaces. In particular, it is noted that a  $T_1$ -space is quasiregular iff it has a quasiregular compactification, and a topological space has a quasiregular one-point compactification iff the space has pseudobase of closed compact sets.**

1. Introduction. In the concept of pseudocompleteness, Oxtoby [16] effectively distilled the major features of the proofs of the classical Baire category theorems. Often authors who find conditions implying that a space is a Baire space could as easily have shown that the conditions imply the stronger property of pseudocompleteness. In particular, the proofs of the Baire category theorems of Hajek [6] and McCoy [15] could as well serve to give the following results respectively: (1) A relatively Hausdorff compactification  $\langle Y, f \rangle$  of a regular space  $X$  is pseudocomplete, and (2) A quasiregular lightly compact space is pseudocomplete. Such results and necessary definitions are discussed later.

Since a pseudocomplete linear metrizable space is a completely metrizable space (Todd [19]), and an infinite dimensional Banach space has proper dense linear subspaces that are Baire (Kelley, Namioka, *et al.* [11] p. 95, prob. 10.B) pseudocompleteness strictly implies the Baire property. Another way of observing the difference: A product of pseudocomplete spaces is pseudocomplete (Oxtoby [16]) and there exist (nonempty) Baire spaces with squares that are not Baire. The existence of such Baire spaces has an interesting literature (see [16], [1] and especially [4] and [17] for citations).

Other permanence results for pseudocompleteness often resemble those for Baire, yet there are simple results for the Baire property

that are only partially known for pseudocompleteness. Aarts and Lutzer [1] raise questions on two such results: (1) Is the continuous open image of a pseudocomplete space, pseudocomplete, and (2) Is a dense  $G_\delta$ -subspace of a pseudocomplete space, pseudocomplete? A partial result for the latter appeared in Oxtoby ([16], 5.2): *If  $Y$  is a quasiregular space with a pseudobase  $\mathcal{B}$  consisting of sets whose closure is countably compact, then any dense  $G_\delta$ -subspace  $X$  of  $Y$  is pseudocomplete.* Neither question is fully answered here, however, Theorem 4.1 gives an improvement on Oxtoby's result and Corollary 3.8 is cast as a start on the first question.

The fastest definition of a Baire space is the following: A topological space is a *Baire space* if the intersection of each countable family of dense open sets is dense. If such intersections are nonempty the space is *nonmeager*. If the space is meager, then by complementation the space is the union of a countable family of rare sets (a set is *rare* if its closure has empty interior). Fleissner and Kunen [4] give a brief indication of the usefulness of Baire space throughout mathematics.

Quasiregularity and pseudobases, defined in Oxtoby [16], can be seen as usefully reducing the emphasis on points in topology and directing attention to collections of sets with close relationships to the family of open sets: A topological space  $(X, \tau)$  is *quasiregular* if each nonempty open set  $U \in \tau^* = \tau \setminus \{\emptyset\}$  contains the closure of a nonempty open set  $V \in \tau^*$ . That this may be a useful variation on regularity is suggested by the following: A regular  $H$ -closed space is necessarily compact (Willard [20], p. 127, prob. 17 K.3). However, a quasiregular  $H$ -closed space need not be compact as shown by Example 2.5. Nonetheless, it is pseudocompact as shown in § 2.

Oxtoby [16] requires that a pseudobase be a collection  $\mathcal{B}$  of nonempty open sets such that each nonempty open set contains a member of  $\mathcal{B}$ . We further deemphasize points by the following definition of *pseudobase*.

**DEFINITION 1.1.** A collection  $\mathcal{B}$  of subsets of a topological space  $(X, \tau)$  is a pseudobase of the topology  $\tau$  if (i) for each  $B$  in  $\mathcal{B}$ ,  $\text{int}_\tau B \neq \emptyset$  and (ii) for each  $U$  in  $\tau^*$ , there is  $B$  in  $\mathcal{B}$  with  $B$  contained in  $U$ .

This definition leads to an equivalence relation on the family of topologies on a given set  $X$ : Topologies  $\sigma$  and  $\tau$  on  $X$  are in this relation if and only if  $\tau^*$  is a pseudobase for  $\sigma$ . It turns out that if one member of an equivalence class is Baire, then all members are Baire. The definition also simplifies descriptions of properties of spaces in terms of properties of members of a pseudobase. For

example, the quasiregular property is immediately seen to be equivalent to the existence of a pseudobase of closed sets. Also see the result of Oxtoby quoted above and, of course, results given throughout this paper.

The definition of pseudobase entails a change in the definition of *pseudocomplete spaces*:

DEFINITION 1.2. A topological space  $(X, \tau)$  is pseudocomplete if it is quasiregular and there is a sequence  $(\mathcal{B}_n)$  of pseudobases for  $\tau$  such that  $\bigcap_n B_n \neq \phi$ , whenever  $B_n$  is in  $\mathcal{B}_n$  and  $\text{int}_\tau B_n$  contains  $\text{cl}_\tau B_{n+1}$  for  $n = 1, 2, 3, \dots$ .

If a pseudobase is required to consist of open sets, as in Oxtoby's definition of pseudobase, then the strong nesting property  $\text{int}_\tau B_n \supset \text{cl}_\tau B_{n+1}$  in Definition 1.5 becomes, simply,  $B_n \supset \text{cl}_\tau B_{n+1}$  giving Oxtoby's original definition of pseudocompleteness. Therefore Oxtoby's original property is formally stronger than this notion of pseudocompleteness. Whether or not it is strictly stronger is an open question. The results of this paper suggest that the present notion of pseudocompleteness is an appropriate and interesting answer to the unification problem discussed in Aarts and Lutzer [2]. *Is there a natural class of spaces that contains all completely metrizable spaces and all locally compact Hausdorff spaces and for which the conclusion of the Baire category theorem holds?*

The second of the following is a formal result, the others follow from standard arguments found in Oxtoby [16] (or Todd [18] for the third).

THEOREM (Oxtoby). *If a topological space is pseudocomplete, then it is a Baire space.*

THEOREM 1.4. (Oxtoby). *Any completely metrizable topological space or locally compact Hausdorff space is pseudocomplete.*

THEOREM 1.5. (Oxtoby). *The Cartesian product of pseudocomplete spaces is pseudocomplete.*

Other presently known permanence properties of pseudocompleteness are undisturbed, yet the change does ease the establishment of additional interesting properties.

2. **Quasiregular spaces and pseudocompleteness.** Here we discuss the incidence of the quasiregular property and some properties which when associated with it imply pseudocompleteness.

A regular space is clearly quasiregular. The usefulness of the

quasiregular property is that it gives just the local property needed for the standard Baire category argument but without the emphasis on points that regularity has. The following gives an example of a space that is quasiregular yet has no point at which there is a neighborhood base consisting of closed sets.

EXAMPLE 2.1. Let  $X$  be the set of real numbers and  $\sigma$  be the topology on  $X$  generated by basic open sets of the form  $[a, b) \setminus D$  where  $D$  is a countable subset of the half-open interval  $[a, b) = \{x \in X : a \leq x < b\}$  and  $D \cap [a, a + 1/n)$  is finite for all  $n = 1, 2, 3, \dots$ . Now suppose, for some  $\varepsilon > 0$ ,  $[a, a + \varepsilon) \setminus E$  is contained in  $[a, b) \setminus D$  with  $D$  infinite and  $a \notin D$ . But  $\text{cl}_\sigma([a, a + \varepsilon) \setminus E) = [a, a + \varepsilon)$  and this meets  $D$ . Thus  $\text{cl}_\sigma([a, a + \varepsilon) \setminus E) = [a, a + \varepsilon)$  is not contained in  $[a, b) \setminus D$ , so that  $X$  is not regular at  $a$ . On the other hand, let  $a' = (\sup D + b)/2$ , then  $[a', b)$  is a nonempty open subset of  $X$  that is closed and contained in  $[a, b) \setminus D$ . Therefore  $(X, \sigma)$  is a quasiregular space that is regular at no point.  $\square$

The next item is part of the folklore.

PROPOSITION 2.2. *A dense subspace of a quasiregular space  $(Y, \tau)$  is quasiregular.*

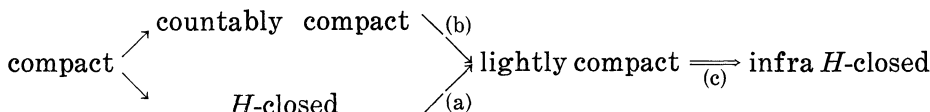
*Proof.* Let  $X$  be a dense subset of  $(Y, \tau)$  and let  $U$  be in  $(\tau|X)^*$ . There is  $W$  in  $\tau^*$  so that  $W \cap X = U$ . As  $(Y, \tau)$  is quasiregular, there is  $V$  in  $\tau^*$  with  $\text{cl}_\tau V$  contained in  $W$ , and so  $V \cap X$  is in  $(\tau|X)^*$  with  $\text{cl}_{\tau|X}(V \cap X) \subset (\text{cl}_\tau V) \cap X \subset W \cap X = U$ . Therefore  $(X, \tau|X)$  is quasiregular.  $\square$

An extension of a quasiregular space need not be quasiregular as the next example shows. By an *extension* of a topological space  $X$ , we mean a pair  $\langle Y, f \rangle$  where  $f$  is a dense embedding of  $X$  in the topological space  $Y$ . A *compactification* of  $X$  is an extension  $\langle Y, f \rangle$  of  $X$  with  $Y$  compact. However, a *one-point compactification* of  $X$  is obtained by appending to  $X$  an object  $\infty$  that is, not in  $X$  and taking as the topology of  $Y = X \cup \{\infty\}$  all open subsets of  $X$  and all complements in  $Y$  of closed compact subsets of  $X$  (see Kelley [10] p. 150).

EXAMPLE 2.3. Let  $Y = Q \cup \{\infty\}$  be a one-point compactification of the rational numbers  $Q$  with the usual topology. Now  $Q$  is a nonempty open subset of  $Y$ , and we suppose that  $U$  is a nonempty open subset of  $Y$  that is contained in  $Q$ .  $U$  is open in  $Q$ . If  $K$  is a subset of  $Q$  that is closed and compact in  $Q$ , then  $K$  is closed and

compact in the reals  $R$ . As  $Q$  is dense in  $R$ ,  $K$  is rare in  $Q$  and so  $U \not\subset K$ . Therefore  $U$  meets  $Y \setminus K$ , and so  $\infty$  is in  $\text{cl}_Y U$ . Thus no nonempty open subset of  $Y$  has its closure contained in  $Q$ . The space  $Y$  is not quasiregular, yet it is an extension of a regular (in fact, completely normal) Hausdorff space  $Q$ .  $\square$

A topological space  $X$  is  $H$ -closed if any continuous image in a Hausdorff space is closed. It is well-known (Liu [13]) that this is characterized by the following: *Each filterbase  $\mathcal{U}$  in the lattice of open sets of  $X$  has an accumulation point*, i.e.,  $\bigcap \{\text{cl} U : U \text{ in } \mathcal{U}\} \neq \emptyset$ . Hajek and Todd [7] have shown that a similar property characterizes the lightly compact spaces (those spaces for which each locally finite family of open sets is finite (Bagley, Connell, and McKnight [3]): *A space  $X$  is lightly compact if each countable filterbase in the lattice of open sets has an accumulation point*. Also in [7], an *infra  $H$ -closed* space was defined to be a space for which any continuous image of it in a first countable Hausdorff space is closed. We have the following implications:



Implications (a) and (c) are discussed in [7]. Bagley, Connell and McKnight [3] cite an example of a lightly compact topological space that is locally compact and completely regular yet not countably compact. This also shows the strictness of (a) as a regular  $H$ -closed space is compact. Implication (b) is also shown to be strict from Example 2.5 which gives a quasiregular  $H$ -closed Hausdorff space that has a sequence  $((n, 0))_n$  with no accumulation point. We now connect this with pseudocompleteness.

**PROPOSITION 2.4.** *If a quasiregular space  $X$  is lightly compact, then it is pseudocomplete.*

*Proof.* Let each  $\mathcal{B}_n$  equal the set of all nonempty open sets of  $X$ . If  $B_n$  is in  $\mathcal{B}_n$  and  $\text{int } B_n$  contains  $\text{cl } B_{n+1}$ , then  $(\text{int } B_n)_n$  is a countable filterbase in the lattice of open sets of the lightly compact space  $X$ , so  $\bigcap_n B_n$  contains the nonempty set  $\bigcap_n \text{cl } B_{n+1}$ . Since  $X$  is also quasiregular, it is pseudocomplete from Definition 1.5.  $\square$

Example 2.3 shows that a lightly compact space need not be a pseudocomplete or even a Baire space.

Bagley, Connell and McKnight [3] show the following: *A topological space is lightly compact if and only if each proper subset  $S$*

with  $S = \text{cl int } S$  is *lightly compact*. Thus, from Proposition 2.4, the closure of any open subset of a lightly compact space is pseudocomplete. Of course this is a property of any pseudocomplete space, in fact, for an open subset  $U$  of a pseudocomplete space, if  $U \subset X \subset \text{cl } U$ , then  $X$  is a pseudocomplete subspace.

As an  $H$ -closed space is compact if and only if it is regular, it is interesting to consider an example of Willard ([20] p.127, Prop. 17K. 4):

EXAMPLE 2.5. Let  $N$  be the natural numbers,  $N^* = \{1/n : n \text{ in } N\} \cup \{0\}$  with their usual subspace topologies, and  $X = N \times N^* \cup \{\infty\}$  where  $U$  is an open subset of  $X$  if and only if  $U$  is an open subset of  $N \times N^*$  with its product topology or  $U \setminus \{\infty\}$  is open in  $N \times N^*$  and  $U$  contains  $U_k = \{(n, 1/m) : m, n \text{ in } N \text{ and } n \geq k\} \cup \{\infty\}$  for some natural number  $k$ . Now  $X$  is Hausdorff and, in fact,  $H$ -closed. The latter may be verified by using the following characterization of  $H$ -closed spaces (see Liu [13]): *A space is  $H$ -closed if and only if each open cover has a finite subcollection whose union is dense.* Each  $\text{cl } U_k$  contains the point  $(k, 0)$  so that  $X$  is not regular at the point  $\infty$ , and thus is not compact. The set of isolated points of  $X$  is  $N \times \{1/m : m \text{ in } N\}$  which is dense in  $X$ , so that, as  $X$  is Hausdorff, each nonempty open set  $U$  contains a clopen set  $\{x\}$ . Thus  $X$  is quasiregular, and since it is lightly compact it is pseudocomplete by Proposition 2.4. Therefore the Hausdorff space  $X$  is a noncompact  $H$ -closed space that is pseudocomplete.  $\square$

An infra  $H$ -closed space, even if it is regular, need not be pseudocomplete or even Baire as is now shown.

EXAMPLE 2.6. The space  $X$  discussed in Hajek and Todd [7] is a slight modification of an example of Herrlich [8]. This modification gives a regular Hausdorff space with the property that any continuous image of  $X$  in any first countable Hausdorff space is a singleton. Therefore  $X$  is an infra  $H$ -closed space. From the original construction  $X = \bigcup_n X_n$  where a subset  $U$  of  $X$  is defined to be open if and only if  $U \cap X_n$  is open in  $X_n$  for each  $n = 1, 2, 3, \dots$ . Moreover, each  $X_n$  is a closed subset of  $X_{n+1}$  with empty interior in  $X_{n+1}$ . It is now simple to show that each  $X_n$  is rare in  $X$ , so that  $X = \bigcup_n X_n$  is meager ( $(X \setminus X_n)_n$  is a sequence of open dense sets with empty intersection). Therefore  $X$  is a Hausdorff infra  $H$ -closed space that is regular and not a Baire space.  $\square$

That an  $H$ -closed space may be very close to being pseudocomplete but fail to be so because it is not quasiregular will be seen

shortly. First here is a simple device for enlarging a topology  $\tau$  on a set  $X$ : For any subset  $S$  of  $X$  let  $\tau(S) = \{U \cup V \cap S : U, V \text{ in } \tau\}$ . It is easily shown that  $\tau(S)$  is a topology on  $X$  that is strictly larger than  $\tau$  if  $S$  is not in  $\tau$ . Such enlargements have been studied by Levin [12].

EXAMPLE 2.7. Let  $S$  be a dense subset of a topological space  $(X, \tau)$  and  $\sigma = \tau(S)$ . A little work shows that  $\text{cl}_\tau W = \text{cl}_\sigma W$  for any  $W$  in  $\sigma$ . Using the characterization of  $H$ -closed spaces given in Example 2.5, we see that  $(X, \sigma)$  is  $H$ -closed if  $(X, \tau)$  is. Also if  $X \setminus S$  is  $\tau$ -dense,  $(X, \sigma)$  is not quasiregular in a rather catastrophic way. For suppose  $W_1$  is in  $\sigma^*$ . Then  $W_2 = W_1 \cap S$  is in  $\sigma^*$ . For  $W = U \cup V \cap S$  in  $\sigma^*$ ,  $\text{cl}_\sigma W = \text{cl}_\tau W = \text{cl}_\tau(U \cup V \cap S) \supset \text{cl}_\tau((U \cup V) \cap S) = \text{cl}_\tau(U \cup V)$  contains elements of the  $\tau$ -dense set  $X \setminus S$ . Therefore  $W_2$  does not contain  $\text{cl}_\sigma W$ . Thus each  $W_1$  in  $\sigma^*$  contains some  $W_2$  in  $\sigma^*$  that does not contain the  $\sigma$ -closure of any nonempty  $\sigma$ -open set.

In particular, let  $X = [0, 1]$ ,  $\tau$  be the usual topology and  $S$  be the set of irrational numbers in  $X$ . Both  $\tau$  and  $\sigma = \tau(S)$  coincide on  $S$  which has a complete metric (Kelley [10] p. 207 Prop. 6. K(b)) and so is pseudocomplete.  $(X, \tau)$  is compact, so  $(X, \sigma)$  is  $H$ -closed. Now  $X \setminus S$  is  $\tau$ -dense, and so  $(X, \sigma)$  is not quasiregular, although it is an extension of a pseudocomplete space and so is a Baire space. Therefore  $(X, \sigma)$  is a Hausdorff  $H$ -closed space that is not quasiregular yet is a Baire space since it is an extension of a pseudocomplete space. □

3. A relation on the topologies for a set. Our definition of pseudobase allows a simple description of a close relationship between topologies on a set: Topologies  $\tau$  and  $\sigma$  on a set  $X$  are  $S$ -related if  $\tau^*$  is a pseudobase for  $\sigma$ . Later this will aid in extending a theorem of Oxtoby. Here we discuss implications based on this relation.

PROPOSITION 3.1. *The  $S$ -relation on the topologies for a set  $X$  is an equivalence relation.*

*Proof.* From the definition of pseudobase we see that  $\tau S \sigma$  means: For all  $U$  and  $V$  in  $\tau^*$  and  $\sigma^*$  respectively,  $\text{int}_\sigma U \neq \phi$  and  $\text{int}_\tau V \neq \phi$ . This gives an equivalence relation on the topologies for set  $X$ . □

The usual and the Sorgenfrey topologies on the real line are  $S$ -related, yet are very different topologies. The topologies in the next example, though homeomorphic, give quite different  $S$ -related

topologies.

EXAMPLE 3.2. Let  $\sigma$  be the Sorgenfrey topology on the real line, that is, the basic open sets for  $\sigma$  are of the form  $[a, b)$ . Let  $\tau$  be the topology on the line with basic open sets of the form  $(a, b]$ . Each of these topologies are  $S$ -related to the usual topology of the line and so are  $S$ -related to each other. Moreover neither topology is finer than the other,  $\sigma \cap \tau$  is the usual topology on the line and  $\sigma \vee \tau$ , the topology generated by their union, is the discrete topology.  $\square$

The next proposition emphasizes the closeness of  $S$ -related topologies.

PROPOSITION 3.3. *If  $\sigma$  and  $\tau$  are topologies on a set  $X$  and  $\tau^*$  is a pseudobase for  $\sigma$ , then for any subset  $A$  of  $X$ ,*

$$\text{int}_\sigma A \subset \text{cl}_\sigma \text{int}_\tau A \quad \text{and} \quad \text{cl}_\sigma A \supset \text{int}_\sigma \text{cl}_\tau A .$$

*Proof.* Let  $x$  be in  $\text{int}_\sigma A$  and  $W$  be a  $\sigma$ -open neighborhood of  $x$ . Now  $U = \text{int}_\tau (W \cap \text{int}_\sigma A) \neq \emptyset$ , and the  $\tau$ -open set  $U$  is contained in  $A$ , so  $W$  meets  $\text{int}_\tau A$ . Thus  $x$  is in  $\text{cl}_\sigma \text{int}_\tau A$  and so  $\text{int}_\sigma A$  is contained in  $\text{cl}_\sigma \text{int}_\tau A$ . Using  $\mathcal{C}A = X \setminus A$ , for each subset  $A$  of  $X$ , we have  $\mathcal{C} \text{int}_\sigma \text{cl}_\tau A = \mathcal{C} \mathcal{C} \text{cl}_\sigma \mathcal{C} \mathcal{C} \text{int}_\tau \mathcal{C} A = \text{cl}_\sigma \text{int}_\tau \mathcal{C} A$  which contains  $\text{int}_\sigma \mathcal{C} A$  by the first part of the proposition. Taking complements, we obtain  $\text{int}_\sigma \text{cl}_\tau A \subset \mathcal{C} \text{int}_\sigma \mathcal{C} A = \text{cl}_\sigma A$ , which is the second inclusion of the proposition.  $\square$

If one member of an  $S$ -equivalence class is a Baire space then all members are by the following:

PROPOSITION 3.4. *Suppose  $\sigma$  and  $\tau$  are topologies on a set  $X$ . If  $\tau^*$  is a pseudobase for  $\sigma$  and  $(X, \tau)$  is a Baire space, then  $(X, \sigma)$  is a Baire space.*

*Proof.* Let  $V$  be a dense open set of  $(X, \sigma)$  and  $U = \text{int}_\tau V$ . Using both inclusions of Proposition 3.3, we obtain  $X = \text{cl}_\sigma V = \text{cl}_\sigma \text{int}_\tau V \subset \text{cl}_\sigma \text{int}_\tau V = \text{cl}_\sigma U$ , giving  $\text{cl}_\tau U \supset \text{int}_\tau \text{cl}_\sigma U \supset \text{int}_\tau X = X$ . Therefore  $U$  is a dense open set of  $(X, \tau)$ . Now suppose  $(V_n)$  is a sequence of dense open sets of  $(X, \sigma)$  and  $V$  is in  $\sigma^*$ . Let  $U_n = \text{int}_\tau V_n$  and  $U = \text{int}_\tau V$ . Since each  $U_n$  is a dense open set of the Baire space  $(X, \tau)$ , and  $U$  is a nonempty  $\tau$ -open set,  $U$  meets  $\bigcap_n U_n$ . Therefore  $V (\supset U)$  meets  $\bigcap_n V_n (\supset \bigcap_n U_n)$ , and so  $\bigcap_n V_n$  is dense in  $(X, \sigma)$ . Hence  $(X, \sigma)$  is a Baire space.  $\square$



From this proposition, we see immediately that the Sorgenfrey line is a Baire space as its topology is  $S$ -related to the usual topology, which is completely metrizable.

The next several propositions use a stronger connection between topologies than that of being  $S$ -related.

**PROPOSITION 3.5.** *Suppose a topological space  $(Y, \tau)$  is the image under a continuous function  $f$  of a topological space  $(X, \sigma)$  and each member of  $\sigma^*$  contains a member of  $f^{-1}\tau \setminus \{\phi\}$ . If  $(Y, \tau)$  is quasiregular then  $(X, \sigma)$  is quasiregular.*

*Proof.* Let  $V$  be in  $\sigma^*$ . There is  $U$  in  $\tau^*$  with  $f^{-1}U \subset V$  and  $W$  in  $\tau^*$  with  $\text{cl}_\tau W \subset U$ , so that  $\phi \neq f^{-1}W = \text{int}_\sigma f^{-1}W \subset f^{-1}\text{cl}_\tau W \subset f^{-1}U \subset V$ . Since  $f^{-1}\text{cl}_\tau W$  is  $\sigma$ -closed,  $f^{-1}W$  is a nonempty  $\sigma$ -open set whose  $\sigma$ -closure is contained in  $V$ . Therefore  $(X, \sigma)$  is quasiregular. □

**COROLLARY 3.6.** *Suppose  $\sigma$  and  $\tau$  are topologies on a set  $X$ ,  $\tau$  is coarser than  $\sigma$  and each member of  $\sigma^*$  contains a member of  $\tau^*$ . If  $(X, \tau)$  is quasiregular, then  $(X, \sigma)$  is quasiregular.*

*Proof.* Apply the previous proposition for  $f$  the inclusion map of  $(X, \sigma)$  in  $(X, \tau)$ . □

**PROPOSITION 3.7.** *Hypotheses are as in Corollary 3.6. If  $(X, \tau)$  is quasiregular and  $(X, \sigma)$  is pseudocomplete, then  $(X, \tau)$  is pseudocomplete.*

*Proof.* Suppose  $(\mathcal{B}_n)$  is a sequence of pseudobases of  $\sigma$  verifying the pseudocomplete property for  $(X, \sigma)$ . Since each member of  $\tau^*$  is a member of  $\sigma^*$  and each member of  $\sigma^*$  contains a member of  $\tau^*$  the topologies  $\sigma$  and  $\tau$  are  $S$ -related. Thus  $\sigma^*$  is a pseudobase for  $\tau$ , and it is simple to verify that each  $\mathcal{B}_n$  is a pseudobase for  $(X, \tau)$ . Now suppose  $B_n \in \mathcal{B}_n$  with

$$\text{int}_\tau B_n \supset \text{cl}_\tau B_{n+1} .$$

Then as  $\tau \subset \sigma$ , we have

$$\text{int}_\sigma B_n \supset \text{int}_\tau B_n \text{ and } \text{cl}_\tau B_{n+1} \supset \text{cl}_\sigma B_{n+1} ,$$

so that

$$\text{int}_\sigma B_n \supset \text{cl}_\sigma B_{n+1} ,$$

and thus  $\bigcap_n B_n \neq \phi$ . Therefore, as  $(X, \tau)$  is quasiregular, it is

pseudocomplete. □

A slightly different way of looking at this result is the following.

**COROLLARY 3.8.** *If a quasiregular topological space  $(Y, \tau)$  is the image under a continuous one-to-one function  $f$  of a pseudocomplete topological space  $(X, \sigma)$  and  $f\sigma \setminus \{\phi\}$  is a pseudobase for  $\tau$ , then  $(Y, \tau)$  is pseudocomplete.*

*Proof.* The topology  $\eta = f^{-1}\tau$  on  $X$  is contained in  $\sigma$ . Also  $(X, \eta)$  is quasiregular. Now for  $V$  in  $\sigma^*$ ,  $U = \text{int}_\tau fV \neq \phi$ , and, since  $f$  is one-to-one,  $f^{-1}U$  is in  $\eta^*$  with  $f^{-1}U \subset V$ . Thus  $\eta \subset \sigma$  and each member of  $\sigma^*$  contains a member of  $\eta^*$  so Proposition 3.7 applies, and  $(X, \eta)$  is pseudocomplete. Now  $(Y, \tau)$  is homeomorphic to  $(X, \eta)$ , so  $(Y, \tau)$  is pseudocomplete. □

Note that the requirement that  $f\sigma \setminus \{\phi\}$  be a pseudobase is weaker than requiring  $f$  to be both continuous and open.

We offer two ways of obtaining topologies  $\tau$  and  $\sigma$  on a set  $X$  with  $\tau \subset \sigma$  and  $\tau^*$  a pseudobase of  $\sigma$ .

**EXAMPLE 3.9.** Let  $\tau$  be a topology on  $X$  and  $S$  a subset of  $X$  such that  $S \subset \text{cl}_\tau \text{int}_\tau S$ . Let  $\sigma = \tau(S) = \{U \cup V \cap S : U, V \text{ in } \tau\}$ . As in Example 2.7,  $\tau \subset \sigma$ , strictly if  $S$  is not in  $\tau$ . Now for  $U \cup V \cap S \neq \phi$  either  $\phi \neq U \subset U \cup V \cap S$  or  $\phi \neq V \cap \text{int}_\tau S \subset U \cup V \cap S$ , so that  $\tau^*$  is a pseudobase for  $\sigma$ . □

The next example is related to Example 2.1.

**EXAMPLE 3.10.** Suppose  $\tau$  is a topology on a set  $X$  with each  $U$  in  $\tau^*$  uncountable. Let

$$\mathcal{V} = \{U \setminus D : U \in \tau, D \subset U, D \text{ countable, } D^d \text{ finite}\},$$

where  $A^d$  is the derived set of a subset  $A$  of  $X$ , that is, the set of all points  $x$  in  $X$  such that  $x$  is in  $\text{cl}_\tau(A \setminus \{x\})$ . Now as  $(U_1 \setminus D_1) \cap (U_2 \setminus D_2) = U_1 \cap U_2 \setminus (D_1 \cup D_2)$ , it is easily seen that  $\mathcal{V}$  is in fact, a base for a topology  $\sigma$  on  $X$ . Also  $U \setminus (D \cup D^d) = U \setminus \text{cl}_\tau D \subset U \setminus D$ , and clearly  $\tau \subset \sigma$ , so  $\tau^*$  is a pseudobase for  $\sigma$ . (Here it is useful to have each  $U \in \tau^*$  uncountable,  $D$  countable, and  $D^d$  finite.) By Corollary 3.6,  $(X, \sigma)$  is quasiregular if  $(X, \tau)$  is quasiregular. Note that if  $a$  is a nonisolated point of  $(X, \tau)$  at which  $\tau$  is first countable, then  $(X, \sigma)$  is not regular at  $a$ . □

4. **Lightly compact pseudobases and pseudocompleteness.** In this section we shall use the relation discussed in the previous section and light compactness to extend a theorem of Oxtoby [16] on topological spaces with pseudobases of closed countably compact sets.

**THEOREM 4.1.** *Suppose  $X$  is a dense  $G_\delta$ -subset of a quasiregular topological space  $(Y, \tau)$  that has a pseudobase  $\mathcal{C}$  of lightly compact sets. If  $\sigma$  is a topology on  $Y$  finer than  $\tau$  with  $\tau^*$  as a pseudobase, then the subspace  $(X, \sigma|X)$  is a pseudocomplete space.*

*Proof.* Let  $C$  be in  $\mathcal{C}$ . As  $(Y, \tau)$  is quasiregular, there is  $U$  in  $\tau^*$  with  $\text{cl}_\tau U \subset \text{int}_\tau C$ . The set  $\text{cl}_\tau U$  is lightly compact since  $C$  is lightly compact,  $\text{cl}_\tau U = (\text{cl}_\tau U) \cap C = \text{cl}_{\tau|C} U$ , and  $U \in \tau|C$ . We may therefore assume that  $\mathcal{C}$  consists of closed lightly compact sets of  $(Y, \tau)$ . Since  $\tau \subset \sigma$  and  $\tau^*$  is a quasibase for  $\sigma$ , the topological space  $(Y, \sigma)$  is quasiregular by Corollary 3.6. Also  $\text{cl}_\sigma X \supset \text{int}_\sigma \text{cl}_\tau X = \text{int}_\sigma Y = Y$ , so that the dense subspace  $(X, \sigma|X)$  of  $(Y, \sigma)$  is quasiregular by Proposition 2.2.

Now let  $(G_n)$  be a sequence of  $\tau$ -open sets with  $X = \bigcap_n G_n$  and let  $\mathcal{B}_n = \{C \cap X : C \in \mathcal{C} \text{ and } C \subset G_n\}$ . For  $C \cap X$  in  $\mathcal{B}_n$ , we have  $\phi \neq \text{int}_\tau C \subset \text{int}_\sigma C$ , so  $\phi \neq (\text{int}_\sigma C) \cap X \subset \text{int}_{\sigma|X}(C \cap X)$ . For  $V$  in  $\sigma^*$ , there is  $U$  in  $\tau^*$  with  $U \subset V$  and  $C$  in  $\mathcal{C}$  with  $C \subset U \cap G_n$  so that  $C \cap X \in \mathcal{B}$  and  $C \cap X \subset U \cap X \subset V \cap X$ . We have shown that each  $\mathcal{B}_n$  is a pseudobase for  $(X, \sigma|X)$ .

Suppose  $B_n \in \mathcal{B}_n$  and  $\text{int}_{\sigma|X} B_n \supset \text{cl}_{\sigma|X} B_{n+1}$ . There are  $C_k$  in  $\mathcal{C}$  with  $C_k \subset G_k$  and  $C_k \cap X = B_k$ . Let  $V_k \in \sigma$  be such that  $V_k \cap X = \text{int}_{\sigma|X} C_k \cap X$ . Now  $\bigcap_{k=1}^n V_k \supset \bigcap_{k=1}^n V_k \cap X = V_n \cap X \neq \phi$  and  $C_k = \text{cl}_\sigma C_k \supset \text{cl}_\sigma (V_k \cap X) = \text{cl}_\sigma V_k \supset V_k$ , so that  $\bigcap_{k=1}^n \text{int}_\tau C_k \supset \text{int}_\tau \bigcap_{k=1}^n V_k \neq \phi$ . Therefore  $\{\bigcap_{k=1}^n \text{int}_\tau C_k\}_{n \geq 1}$  is a countable filterbase of open sets of the lightly compact  $\tau$ -closed subspace  $(C_1, \tau|C_1)$ , and so  $\bigcap_n C_n \supset \bigcap_n \text{cl}_\tau \text{int}_\tau C \neq \phi$ . But each  $C_n \subset G_n$  so that  $\bigcap_n C_n \subset \bigcap_n G_n = X$ , and  $\bigcap_n B_n = \bigcap_n C_n \cap X = (\bigcap_n C_n) \cap X = \bigcap_n C_n \neq \phi$ . Therefore  $(X, \sigma|X)$  is a pseudocomplete space. □

The following is an immediate application of the theorem and will be used later.

**COROLLARY 4.2.** *A topological space with a pseudobase of closed lightly compact sets is pseudocomplete.*

As a quasiregular lightly compact space has a pseudobase of closed lightly compact sets Proposition 2.4 is a corollary of the above.

**COROLLARY 4.3.** *Suppose  $(Y, \tau)$  has a pseudobase  $\mathcal{C}$  of closed lightly compact sets. If  $\sigma$  is a topology on  $Y$  finer than  $\tau$  with  $\tau^*$  as a pseudobase, then any dense  $G_\delta$ -subspace of  $(Y, \sigma)$  is pseudo-complete.*

*Proof.* First note a few facts: By Corollary 4.2  $(Y, \tau)$  is pseudo-complete, and so, by Theorem 1.3 it is a Baire space. By Proposition 3.4,  $(Y, \sigma)$  is also a Baire space. From the proof of that proposition, the  $\tau$ -interior of a dense open subset of  $(Y, \sigma)$  is dense in  $(Y, \tau)$  and *vice versa*.

Let  $Z$  be a dense  $G_\delta$ -subspace of  $(Y, \sigma)$ . There is a sequence  $(H_n)$  of dense open sets of  $(Y, \sigma)$  with  $Z = \bigcap_n H_n$ . Using the above noted facts, we see that  $X = \bigcap_n \text{int}_\tau H_n$  is a dense  $G_\delta$ -subset of  $(Y, \tau)$ . As it contains  $\bigcap_n \text{int}_\sigma \text{int}_\tau H_n$ , which is dense in  $(Y, \sigma)$ ,  $X$  is also dense in  $(Y, \sigma)$ .

From Theorem 4.1,  $(X, \sigma|X)$  is a pseudocomplete space. By Corollary 3.6,  $(Y, \sigma)$  is quasiregular, and so, by Proposition 2.2,  $(Z, \sigma|Z)$  is quasiregular. Thus  $(Z, \sigma|Z)$  is a quasiregular extension of the pseudocomplete space  $(X, \sigma|X)$  and is, itself, pseudocomplete by Proposition 5.3.  $\square$

It turns out that the Sorgenfrey line is pseudocomplete in Oxtoby's sense (for each  $\mathcal{B}_n$  use the family of nonempty basic open sets  $[a, b)$  with rational endpoints for even  $n$  and irrational for odd  $n$ ). We get the following from the previous corollary.

**COROLLARY 4.4.** *Each dense  $G_\delta$ -subspace of the Sorgenfrey line is pseudocomplete.*

*Proof.* Let  $Y$  be the set of real numbers,  $\tau$  be the usual topology,  $\sigma$  be the Sorgenfrey topology, and  $\mathcal{C}$  be the family of  $\tau$ -compact intervals  $[a, b]$  with  $a < b$ . Now  $\mathcal{C}$  is a pseudobase for  $\tau$  consisting of  $\tau$ -lightly compact  $\tau$ -closed intervals,  $\sigma$  is finer than  $\tau$  and  $\tau^*$  is a pseudobase for  $\sigma$ . Corollary 4.3 applies, and so any dense  $G_\delta$ -subspace of  $(Y, \sigma)$  is pseudocomplete.  $\square$

The following leads to interesting Sorgenfrey type topologies to which the theorem may be applied.

**EXAMPLE 4.5.** Let  $X = R \times R$  be the Cartesian plane with  $\tau$  as the usual product topology. The family of sets of the form

$$(a, b) \times (c, d) \cup \{(a, c)\}$$

is a base for a topology  $\sigma$  on  $X$  that is stronger than  $\tau$  for which

$\tau^*$  is a pseudobase. Moreover,  $\mathcal{C} = \{[a, b] \times [c, d] : a, b, c, d \in R, a < b, c < d\}$  is a pseudobase for  $\tau$  of  $\tau$ -closed and  $\tau$ -compact sets. By Theorem 4.1,  $(X, \sigma)$  is pseudocomplete. (The same conclusion may be obtained for topologies based on  $\sigma$  as in Example 3.10). It may be noted that  $(X, \sigma)$  is strictly stronger than the product topology on  $X$  when each factor is given the Sorgenfrey topology. This product topology is pseudocomplete by Corollary 4.4 and Theorem 1.5. □

In another direction we shall improve on the Baire category theorem of Hajek [6]. An extension  $\langle Y, f \rangle$  of a topological space  $X$  is a *relatively Hausdorff extension* if  $fx$  and  $fy$  have disjoint neighborhoods whenever  $x$  is in  $X$ ,  $y$  is in  $Y$  and  $fx \neq fy$ .

**PROPOSITION 4.6.** *If  $Y$  is a relatively Hausdorff extension of a topological space  $X$  and each point of a dense subset  $S$  of  $Y$  has a neighborhood whose closure is compact, then  $Y$  has a pseudobase of closed compact sets. Moreover  $Y$  is pseudocomplete.*

*Proof.* For convenience, we consider  $X$  as a dense subspace of  $(Y, \tau)$ , so that the relatively Hausdorff property means that elements,  $x$  and  $y$  of  $X$  and  $Y$  respectively, have disjoint neighborhoods whenever  $x \neq y$ . Let  $\mathcal{B} = \{\text{cl } U : U \in \tau^* \text{ and } \text{cl } U \text{ is compact}\}$ . We shall show that  $\mathcal{B}$  is a pseudobase for  $\tau$ . Let  $W$  be in  $\tau^*$ . As  $S$  is dense in  $(Y, \tau)$  there is an  $s$  in  $W \cap S$ . There is an open neighborhood  $U$  of  $s$  with  $\text{cl } U$  compact. As  $X$  is dense in  $Y$ , there is an  $x$  in  $U \cap W \cap X$ . Let  $F = (\text{cl } U) \setminus W$ .

Since  $Y$  is a relatively Hausdorff extension of  $X$ , for each  $y$  in  $F$ , there are disjoint open neighborhoods  $U_y$  and  $V_y$  of  $x$  and  $y$  respectively. As the open sets  $\{V_y : y \in F\}$  cover the compact set  $F$ , there is a finite subcollection  $\{V_{y(k)}\}_{k=1}^n$  that covers  $F$ . Let  $U_0 = \bigcap_{k=1}^n U_{y(k)} \cap U \cap W$ , so that  $U_0$  is an open neighborhood of  $x$  that is contained in  $U$  and disjoint from the open neighborhood  $V = \bigcap_{k=1}^n V_{y(k)}$  of  $F = (\text{cl } U) \setminus W$ . Thus  $U_0 \subset (\text{cl } U) \setminus V \subset W$ . Since  $(\text{cl } U) \setminus V$  is closed,  $W$  contains  $\text{cl } U_0$ . Moreover, as  $\text{cl } U$  is compact, so is  $\text{cl } U_0$ . Finally  $x$  is in  $U_0$ , so  $U_0$  is in  $\tau^*$ , and therefore  $\text{cl } U_0$  is a member of  $\mathcal{B}$  that is contained in  $W$ . Also each member of  $\mathcal{B}$  has nonempty interior, so  $\mathcal{B}$  is a pseudobase for  $\tau$  consisting of closed compact sets, and from Corollary 4.2  $(Y, \tau)$  is pseudocomplete. □

Hajek [5] shows that a topological space is regular and Hausdorff if and only if it has a relatively Hausdorff compactification. Then in [6], Hajek shows that such an extension of a nonempty topological space is nonmeager. This is now strengthened in the

following immediate result.

**COROLLARY 4.7.** *A relatively Hausdorff compactification  $\langle Y, f \rangle$  of a topological space  $X$  has a pseudobase of closed compact sets. Moreover  $Y$  is pseudocomplete.*

5. **Extensions.** In this section are discussed some results on extensions and pseudocompleteness including a characterization of spaces with quasiregular one-point compactifications.

After constructing, for each topological space  $X$ , a Baire extension  $X_F$ , McCoy [14] asks if each regular space has a regular Baire extension. In this context, a natural modification of this question is to replace “regular” by “quasiregular”. It is easy to see that McCoy’s Baire extension  $X_F$  of  $X$  is quasiregular if  $X$  is, so the modified question has an affirmative answer. Aarts and Lutzer [1] give a stronger result by constructing, for each quasiregular space, a pseudocomplete extension. This extension is, in fact, an  $H$ -closed extension, and so we may characterize quasiregular spaces in a manner similar to Hajek’s [5] characterization of regular Hausdorff spaces: *A topological space is quasiregular if and only if it has a quasiregular  $H$ -closed extension.* In case, the space is  $T_1$  (singletons are closed), we have the following.

**PROPOSITION 5.1.** *A  $T_1$ -space  $(X, \tau)$  is quasiregular if and only if it has a quasiregular compactification.*

*Proof.* One implication is clear as a dense subspace of a quasiregular space is quasiregular. We shall be using the Wallman compactification  $\langle W(X), \varphi_x \rangle$  of the  $T_1$ -space  $(X, \tau)$  so we note: (i)  $W(X) = \{\mathcal{F} \text{ an ultrafilter in the lattice of closed sets of } (X, \tau)\}$ , (ii) a basis for the closed sets of  $W(X)$  is the family of all sets of the form  $C(A) = \{\mathcal{F} \in W(x) : A \in \mathcal{F}\}$  where  $A$  is closed in  $(X, \tau)$ , and (iii) the dense embedding  $\varphi_x$  of  $X$  in  $W(X)$  is given by  $\varphi_x(x) = \{A \subset X : X \setminus A \in \tau \text{ and } x \in A\}$  (see Kelley [10] p. 167, prob. 6.R.) Now let  $V$  be a nonempty open subset of  $W(X)$ , so there is a proper closed set  $A$  of  $X$  with  $W(X) \setminus C(A) \subset V$ . As  $(X, \tau)$  is quasiregular, there is  $U$  in  $\tau^*$  with  $\text{cl}_\tau U \subset X \setminus A$ . We need only show that

$$W(X) \setminus C(X \setminus U) \subset C(\text{cl}_\tau U) \subset W(X) \setminus C(A).$$

Suppose  $\mathcal{F}$  is in  $W(X) \setminus C(X \setminus U)$ . Then  $X \setminus U$  is not in  $\mathcal{F}$ . But, as  $(X \setminus U) \cup \text{cl}_\tau U = X$ , we have  $\text{cl}_\tau U$  in  $\mathcal{F}$  so that  $\mathcal{F}$  is in  $C(\text{cl}_\tau U)$ . Now, as  $\text{cl}_\tau U \subset X \setminus A$ ,  $\text{cl}_\tau U$  is disjoint from  $A$ . Therefore  $\text{cl}_\tau U$  is in no closed ultrafilter containing  $A$ , and so  $\mathcal{F}$  in  $C(\text{cl}_\tau U)$ , does not contain  $A$ . Thus  $\mathcal{F}$  is not in  $C(A)$ , and the second inclusion holds.

Hence  $\langle W(X), \varphi_X \rangle$  is a quasiregular compactification of  $X$ . □

Here we shall verify that a quasiregular extension of a pseudocomplete space is pseudocomplete. The argument seems somewhat more delicate than for which pseudobases are collections of open sets ([1] Prop. 2.2(a)).

LEMMA 5.2. *If a quasiregular topological space  $(Y, \sigma)$  has a dense subspace  $X$  with pseudobase  $\mathcal{B}$ , then  $\mathcal{C} = \{C \subset Y: C \cap X \in \mathcal{B}, \text{ and } \text{int}_\sigma C \supset \text{int}_{\sigma|X}(C \cap X)\}$  is a pseudobase for  $(Y, \sigma)$ .*

*Proof.* In the above definition of  $\mathcal{C}$ , each  $C \cap X$  is in  $\mathcal{B}$  so  $\text{int}_{\sigma|X}(C \cap X)$  is nonempty, and as  $\text{int}_\sigma C$  contains  $\text{int}_{\sigma|X}(C \cap X)$ , each member of  $\mathcal{C}$  has a nonempty  $\sigma$ -interior. We need to show that each  $V$  in  $\sigma^*$  contains a member of  $\mathcal{C}$ . As  $(Y, \sigma)$  is quasiregular there is  $V_1$  in  $\sigma^*$  with  $\text{cl}_\sigma V_1 \subset V$ . There is  $B$  in  $\mathcal{B}$  with  $B \subset V_1 \cap X$ . Let  $W$  be in  $\sigma^*$  with  $W \cap X = \text{int}_{\sigma|X} B$ , and set  $C = W \cup B$ . Now  $C \cap X = W \cap X \cup B \cap X = (\text{int}_{\sigma|X} B) \cup B = B \in \mathcal{B}$ , and  $\text{int}_\sigma C \supset W \supset W \cap X = \text{int}_{\sigma|X} B$ , so that  $C$  is a member of  $\mathcal{C}$ . Moreover,  $W \subset \text{cl}_\sigma W = (\text{cl}_\sigma(W \cap X)) \subset \text{cl}_\sigma B \subset \text{cl}_\sigma(V_1 \cap X) = \text{cl}_\sigma V_1 \subset V$ , so that  $C = W \cup B \subset V$ . Therefore  $\mathcal{C}$  is a pseudobase for  $Y$ . □

Now we have the extension property of pseudocompleteness.

PROPOSITION 5.3. *If a topological space  $\langle Y, f \rangle$  is a quasiregular extension of a pseudocomplete space  $X$ , then  $Y$  is pseudocomplete.*

*Proof.* For convenience, assume that  $(X, \tau)$  is a dense pseudocomplete subspace of quasiregular space  $(Y, \sigma)$  so that  $\tau = \sigma|X$ . Suppose  $(\mathcal{B}_n)$  is a sequence of pseudobases for  $(X, \tau)$  verifying that  $(X, \tau)$  is a pseudocomplete space. Define the sequence  $(\mathcal{C}_n)$  by  $\mathcal{C}_n = \{C \subset Y: C \cap X \in \mathcal{B}_n \text{ and } \text{int}_\sigma C \supset \text{int}_\tau(C \cap X)\}$ . By Lemma 5.2 each  $\mathcal{C}_n$  is a pseudobase for  $(Y, \sigma)$ . Suppose  $C_n \in \mathcal{C}_n$  with  $\text{int}_\sigma C_n \supset \text{cl}_\sigma C_{n+1}$ , and note that  $B_n = C_n \cap X \in \mathcal{B}_n$ . Now  $B_n = C_n \cap X \supset (\text{int}_\sigma C_n) \cap X \supset (\text{int}_\tau B_n) \cap X = \text{int}_\tau B_n$ , so that  $\text{int}_\tau B_n = (\text{int}_\sigma C_n) \cap X \supset (\text{cl}_\sigma C_{n+1}) \cap X \supset (\text{cl}_\sigma B_{n+1}) \cap X = \text{cl}_\sigma B_{n+1}$ . Therefore, by assumption,  $\bigcap_n B_n \neq \phi$ , and so  $\bigcap_n C_n \neq \phi$ . Thus  $(Y, \sigma)$  is a pseudocomplete space. □

Topological spaces in which compact sets are closed are called KC-spaces. The following definition will be useful to us: An extension  $\langle Y, f \rangle$  of a topological space  $X$  is called a *relatively KC-extension* if  $fK$  is closed in  $Y$  for each compact subset  $K$  of  $X$ .

**PROPOSITION 5.4.** *If a topological space  $(X, \tau)$  has a pseudobase  $\mathcal{B}$  of compact sets and  $\langle Y, f \rangle$  is a relatively KC-extension of  $X$  then there is a dense open subset  $U$  of  $Y$  contained in the embedding  $fX$  of  $X$  in  $Y$ . Moreover,  $f\mathcal{B}$  is a pseudobase for  $Y$  of closed compact subsets.*

*Proof.* We shall treat  $(X, \tau)$  as a dense subspace of  $(Y, \sigma)$  so that  $\tau = \sigma|X$ . Let  $U = \cup \{\text{int}_\sigma B : B \in \mathcal{B}\}$ . Clearly  $U$  is  $\sigma$ -open and contained in  $X$ . Let  $V$  be in  $\sigma^*$ . Since  $X$  is  $\sigma$ -dense, there is a member  $B$  of  $\mathcal{B}$  contained in  $V \cap X$  and  $\text{int}_\sigma \text{cl}_\sigma B$  is nonempty. Since  $B$  is compact and  $Y$  is a relatively KC-extension of  $X$ ,  $\text{cl}_\sigma B = B$ . Now  $\phi \neq \text{int}_\sigma \text{cl}_\sigma B = \text{int}_\sigma B \subset B \subset V \cap X$ , so  $V$  meets  $U$ . Therefore  $U$  is a  $\sigma$ -dense open set of  $Y$  that is contained in  $X$ . It is clear that  $\mathcal{B}$  is a pseudobase for  $(Y, \sigma)$  consisting of  $\sigma$ -closed  $\sigma$ -compact sets.  $\square$

The next gives a simple kind of converse to the above and will not be proved.

**PROPOSITION 5.5.** *If a topological space  $(Y, \sigma)$  has a pseudobase  $\mathcal{C}$  of closed (lightly) compact sets, then for any open set  $U$  and any subset  $X$  of  $Y$  with  $U \subset X \subset \text{cl}_\sigma U$ , the subspace  $(X, \sigma|X)$  has a pseudobase of closed (lightly) compact sets.*

We now consider some results connected with one-point compactifications.

**PROPOSITION 5.6.** *If a topological space  $(X, \tau)$  has a pseudobase  $\mathcal{B}$  of closed compact sets, then the one-point compactification  $X^* = X \cup \{\infty\}$  of  $X$  has  $\mathcal{B}$  as a pseudobase of closed compact sets.*

*Proof.* If  $(X, \tau)$  is compact, the result is simple. Suppose otherwise. Since each member  $B$  of  $\mathcal{B}$  is closed and compact in  $(X, \tau)$ ,  $X^* \setminus B$  is, by definition, open in  $X^*$ . Therefore  $B$  is a closed subset of  $X^*$ . It was at this point in the proof of Proposition 5.4 that the relatively KC-property was used, so this present proposition is essentially a corollary of that proposition.  $\square$

By a *locally compact* space is meant a space such that each point has a compact neighborhood.

**PROPOSITION 5.7.** *If a topological space  $(X, \tau)$  is quasiregular and locally compact, then it has a pseudobase of closed compact sets.*



*Proof.* Let  $\mathcal{B} = \{\text{cl } V : V \in \tau^* \text{ and } \text{cl } V \text{ is compact}\}$ . Suppose  $U$  is in  $\tau^*$  and  $x$  is in  $U$ . There is a compact neighborhood  $N$  of  $x$ . Since  $N \cap U$  is a neighborhood of  $x$ , there is a  $V$  in  $\tau^*$  such that  $\text{cl } V$  is contained in  $(\text{int } N) \cap U$ . Thus the closed set  $\text{cl } V$  is compact and is a member of  $\mathcal{B}$  that is contained in  $U$ . Therefore  $\mathcal{B}$  is a pseudobase for  $(X, \tau)$  of closed compact sets.  $\square$

We may now give a characterization of topological spaces with one-point compactifications that are quasiregular.

**THEOREM 5.8.** *The one-point compactification  $X^* = X \cup \{\infty\}$  of a topological space  $(X, \tau)$  is quasiregular if and only if  $(X, \tau)$  has a pseudobase of closed compact sets.*

*Proof.* Suppose that  $X^*$  is quasiregular. Since it is compact, it has a pseudobase  $\mathcal{C}$  of closed compact sets by Proposition 5.7. Now  $X$  is an open subspace of  $X^*$  so that  $\mathcal{B} = \{C \subset X : C \in \mathcal{C}\}$  is a pseudobase for  $(X, \tau)$  of closed compact sets of  $(X, \tau)$ . Conversely, suppose  $(X, \tau)$  has a pseudobase  $\mathcal{B}$  of closed compact sets. By Proposition 5.6,  $X^*$  has a pseudobase of closed compact sets, and so is quasiregular.  $\square$

**6. Questions.** The notion of pseudocompleteness discussed here forms an answer to the unification problem mentioned in §1. A related question is what permanence properties does such a unifying property share with the Baire property. Aarts and Lutzer [1, 2] cite two specific questions for Oxtoby's notion of pseudocompleteness that also remain open for the notion defined in this paper:

*Question 6.1.* *Is a dense  $G_\delta$ -subspace of a pseudocomplete space pseudocomplete?*

*Question 6.2.* *Is the open continuous image of a pseudocomplete space pseudocomplete?*

Some partial results for these are given in §§3 and 4, and, in a much restricted form, the second question has also been considered in Todd [18].

The following remains open:

*Question 6.3.* *Is the notion of pseudocompleteness discussed here equivalent to Oxtoby's pseudocompleteness?*

Oxtoby [16] discusses countability conditions on pseudobases.

Pseudobases consisting of sets satisfying a compactness condition have been discussed here. What other interesting local properties might be usefully expressed in terms of pseudobases?

For more on Baire extensions, see R. A. McCoy and J. Porter, *General Topology and its Applications*, 7 (1977), 34-58.

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ST. JOHNS UNIVERSITY  
STATEN ISLAND, NY 10301