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ON ISOMETRIES OF HARDY SPACES ON COMPACT ABELIAN GROUPS

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Let $H^p(m)$, $0 < p \leqq \infty$, be the Hardy spaces on a quotient K of the Bohr group. In this paper we completely determine the isometries of $H^p(m)$, $p \neq 2$, onto itself. Our result is a generalization of a recent work of Muhly who determined the isometries of $H^p(m)$ onto itself under the assumption that the dual group of K is countable, and it may be regarded as a partial answer to a question posed by Muhly.

1. Introduction. Many results have been obtained concerning isometries of Hardy spaces in the theory of uniform algebras. The most fundamental result in this direction is due to de Leeuw, Rudin, and Wermer [2], which states that an automorphism of the classical Hardy space $H^\infty(\boldsymbol{T})$ is induced via composition with the unit circle T of a fractional linear transformation of the unit disc onto itself. Their work was carried on independent of Nagasawa [13], who also described the isometries of $H^\infty(\boldsymbol{T})$ onto itself. On the other hand, Arens [1] completely determined the automorphisms of the uniform algebra of analytic functions on a compact abelian group K whose dual group Γ is archimedean ordered (cf. [11]). This result was extended by Muhly [11] to the uniform algebra of analytic functions induced by a flow which has no periodic orbits. Moreover Muhly [12] has recently given, among other things, the following interesting generalization of this result of Arens to the case of isometries of Hardy spaces $H^p(m)$, $p \neq 2$, on K: Under the assumption that Γ is countable, every isometry of $H^p(m)$, $p \neq 2$, is induced via composition with an affine map of K such that the adjoint of the additive factor of this map preserves the order of Γ. The purpose of this paper is to remove the assumption on Γ. This result provide a partial positive answer to the following question posed by Muhly in [12; §5]:

Is it possible to describe the isometries of ergodic Hardy spaces onto itself without the separability assumptions on phase spaces?

The difficulty is that, in the absence of separability assumptions automorphisms of measure algebras may not have point realizations. On the other hand, although our proof rests on some techniques which were first investigated by Muhly [11], [12], and is given in the context of almost periodic setting, one will find some improvements of the proof given in [12; §3].

In the next section we present some preliminary material which we shall need, and state our main result. In §3, we show that under the assumption that K is metrizable, the automorphisms of $H^\infty(m)$ onto itself are induced via composition with certain Borel isomorphisms. This will be used in §4 for the proof of our theorem stated in §2. In §5, we close with some remarks.

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2. Notations and the main theorem. Let K be a compact abelian group, not a circle, dual to a subgroup Γ of the discrete real line R_d. For $0 < p \leqq \infty$, $L^p(m)$ is the Lebesgue space based on the normalized Haar measure m on K, and $C(K)$ is the space of all complex-valued continuous functions on K. Let $\mathfrak{A}$ be the uniform algebra of all analytic functions in $C(K)$, i.e., the family of all functions f in $C(K)$ whose Fourier coefficient

$$a_\lambda(f) = \int_K \bar{\chi}_\lambda(x) f(x) dm(x)$$

vanishes for all negative λ in Γ, where $\chi_\lambda(x)$ denotes the continuous character on K defined by setting $\chi_\lambda(x) = x(\lambda)$ for any x in K. The Hardy space, $H^p(m)$, $0 < p < \infty$, is the closure of $\mathfrak{A}$ in $L^p(m)$, while $H^\infty(m)$ is defined to be the weak-* closure of $\mathfrak{A}$ in $L^\infty(m)$. Let $\{T_t\}_{t\in R}$ be the transformation group on K such that, for any x in K,

$$T_t(x) = x + e_t$$

where e_t is the element of K defined by $e_t(\lambda) = e^{it\lambda}$ for all λ in Γ. When it is convenient, we will often write $T_t(x)$ for $x + t$. Recall that the map $t \to e_t$ embeds the real line R continuously onto a dense subgroup K_0 of K. A straightforward Fourier series argument shows that the flow $(K, \{T_t\}_{t\in R})$ is strictly ergodic, i.e., the normalized Haar measure m is the unique probability measure which is invariant under the action of $\{T_t\}_{t\in R}$. We refer the reader to Helson's monograph [7] for an up-to-date account of the theory of analyticity on compact abelian groups.

In order to state our main result, we require a little more terminology. For $i = 1, 2$, let K_i, Γ_i, A_i and m_i be as above, and let $\mathfrak{B}_i$ be the Borel field on K_i. A set E in $\mathfrak{B}_i$ is called *conull* if $m_i(E^c) = 0$. We say a map σ from K_1 onto K_2 is an *affine* map if σ may be factored as $\sigma = \sigma_1 \circ \sigma_2$ where σ_2 is a continuous group isomorphism from K_1 onto K_2 and σ_1 is the translation by an element of K_2. Let Γ_i^+ be the subsemigroup of nonnegative elements in Γ_i. Then we say also that the affine σ is *order preserving* if the adjoint

σ_2^* of σ_2 carries Γ_2^+ onto Γ_1^+. We denote by $(\mathfrak{B}_i, m_i)$ the measure algebra of Borel field $\mathfrak{B}_i$ associated with m_i, i.e., $(\mathfrak{B}_i, m_i)$ is the Boolean sigma-algebra of $\mathfrak{B}_i$ mod m_i-null sets. For E_i in $\mathfrak{B}_i$, a map τ is called a *Borel isomorphism* from E_1 to E_2 if τ is one to one, onto, and both τ and τ^{-1} are Borel maps. It is well-known in ergodic theory that, under the assumption that both K_1 and K_2 are compact metric, any sigma-isomorphism σ from $(\mathfrak{B}_1, m_1)$ onto $(\mathfrak{B}_2, m_2)$ has a point realization, i.e., there exist conull sets K_1' and K_2' in $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively, such that σ may be considered as a Borel isomorphism from K_1' onto K_2' (see [17]). Let T be a map from K_1 to K_2. For any function f on K_2, we define $(Tf)(x) = f(Tx)$ for x in K_1.

We may now give the statement of our main theorem which is an analogue of [12; Theorem IV]. It will be proved in §4.

THEOREM 2.1. *For $i = 1, 2$, let Γ_i be an arbitrary dense subgroup of the real line R, but endowed with the discrete topology, and let K_i, m_i, and $H^p(m_i)$, $0 < p \leqq \infty$, be as before. If Ψ is an isometry mapping $H^p(m_1)$ onto $H^p(m_2)$, $p \neq 2$, then there exists a constant c of modulus one and an order preserving affine map σ from K_1 onto K_2 such that*

$$\Psi f = c(f \circ \sigma^{-1}) \tag{2.1}$$

for all f in $H^p(m_1)$. Conversely, such a constant c and an affine map σ determine an isometry via this equation.

By virtue of Lowdenslager's theorem [7; Ch. 2. §2], this theorem may be regarded as an extention to Besicovitch almost periodic functions of a theorem of Arens about isomorphisms of algebras of ordinary analytic almost periodic functions.

In our discussions in the forthcoming sections, we frequently use the following lemma, which is a weak version of the statement in [12; §3. Step. 2].

LEMMA 2.2. *For $i = 1, 2$, let Γ_i, K_i, m_i and $(\mathfrak{B}_i, m_i)$ be as before. Suppose that Ψ is an algebra isomorphism from $H^\infty(m_1)$ onto $H^\infty(m_2)$. Then there is a sigma-isomorphism σ from $(\mathfrak{B}_1, m_1)$ onto $(\mathfrak{B}_2, m_2)$ such that*

$$\int_E \Psi(f)dm_2 = \int_{\sigma^{-1}(E)} fdm_1 \tag{2.2}$$

for any f in $H^\infty(m_1)$ and any E in $(\mathfrak{B}_2, m_2)$. In particular, $m_1(\sigma^{-1}(E)) = m_2(E)$ for any E in $(\mathfrak{B}_2, m_2)$. Moreover, if Γ_1 and Γ_2

are countable, then σ has a point realization.

Proof. For $i=1, 2$, let $\mathfrak{M}_i$ be the maximal ideal space of $H^\infty(m_i)$, and let $\hat{H}^\infty(m_i)=\{\hat{f};\ f \text{ in } H^\infty(m_i)\}$ where the hat $\hat{}$ indicates the Gelfand transform. Recall that $\hat{H}^\infty(m_i)$ is a logmodular algebra on the Shilov boundary $\partial\mathfrak{M}_i$ of $\mathfrak{M}_i$, also recall that $\partial\mathfrak{M}_i$ may be identified with the maximal ideal space of $L^\infty(m_i)$. If we set $\hat{\Psi}(\hat{f})=(\Psi(f))\hat{}$ for each f in $H^\infty(m_1)$, then there is a homeomorphism $\tilde{\sigma}$ mapping $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ such that $\hat{\Psi}(\hat{f})=\hat{f}\cdot\tilde{\sigma}^{-1}$ and $\tilde{\sigma}(\partial\mathfrak{M}_1)=\partial\mathfrak{M}_2$ (see [13] and [11; §4] for details). Let $\tilde{m}_i$ denotes the Radonization of m_i. Then we have that $\tilde{m}_i(U)>0$ for all nonempty open sets U of $\partial\mathfrak{M}_i$ ([4; Ch. I, Corollary 9.2]). Since any nonzero E in $(\mathfrak{B}_i, m_i)$ corresponds to a nonempty open and closed subset $\tilde{E}$ of $\partial\mathfrak{M}_i$, it can be seen that $\tilde{\sigma}$ determines a sigma-isomorphism σ from $(\mathfrak{B}_1, m_1)$ onto $(\mathfrak{B}_2, m_2)$ such that

$$\int_E \Psi(f)dm_2=\int_{\sigma^{-1}(E)} f dm_2\circ\sigma$$

for any f in $H^\infty(m_1)$ (cf. [4; Ch. I, §9]). On the other hand, it is easy to see that m_1 and $m_2\circ\sigma$ are mutually absolutely continuous representing measures of the uniform algebra $\mathfrak{A}_1$. This implies that m_1 and $m_2\circ\sigma$ belong to a same Gleason part. So, since $\{m_1\}$ is a one point part by [4; Ch. VII, §4], we have $m_1=m_2\circ\sigma$. Together with the above equation, we obtain the equation (2.2). When Γ_1 and Γ_2 are countable, both K_1 and K_2 are compact metric spaces. Hence, by the remark above, σ may be identified with a Borel isomorphism from a conull set K_1' in $\mathfrak{B}_1$ onto a conull set K_2' in $\mathfrak{B}_2$. This concludes the proof.

3. Isomorphisms of Hardy spaces on metric groups. In this section we study the properties of Borel isomorphisms which determine isomorphisms of Hardy spaces. Throughout this section we assume that, for $i=1, 2$, Γ_i is a countable dense subgroup of R.

The following proposition is a consequence of [12; Theorem I]. However, we provide here an elementary proof.

PROPOSITION 3.1. *For $i=1, 2$, let Γ_i be a countable dense subgroup of R and let K_i, m_i, $\mathfrak{B}_i$, $\{T_t^{(i)}\}_{t\in R}$, and $H^\infty(m_i)$ be as in §2. If Ψ is an isomorphism from $H^\infty(m_1)$ onto $H^\infty(m_2)$, then we may find a constant $\beta>0$, a conull set K_i' in $\mathfrak{B}_i$, and a Borel isomorphism σ mapping K_1' onto K_2' such that*

(3.1) $\Psi f=f\circ\sigma^{-1}$, *for each f in $H^\infty(m_1)$;*

(3.2) $m_1(E)=m_2(\sigma(E\cap K_1'))$, *for each E in $\mathfrak{B}_1$; and*

$$(\sigma^{-1}T_t^{(2)}\sigma)f(x) = T_{Bt}^{(1)}f(x)\ , \quad m_1\text{-a.e. } x \tag{3.3}$$

for each t in R and each f in $H^\infty(m_1)$. Conversely, such a σ determines an isomorphism from $H^\infty(m_1)$ onto $H^\infty(m_2)$ via the equation (3.1).

In order to prove Proposition 3.1, we need some lemmas. By Lemma 2.2, there exists a conull set K_i' in $\mathfrak{B}_i$, $i = 1, 2$, and a Borel isomorphism σ mapping K_1' onto K_2' which satisfies the equations (3.1) and (3.2). So it suffices to show that this Borel isomorphism σ satisfies the equation (3.3).

We recall that $\mathfrak{A}_i$ is the uniform algebra of all continuous analytic functions on K_i for $i = 1, 2$, and note that, since Γ_i is countable, $\mathfrak{A}_i$ is separable. For x in K_i and $s > 0$, we denote by $m(x, s)$ the regular Borel measure on K_i defined by the equation:

$$\int_{K_i} \phi dm(x, s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x + t)\frac{s}{s^2 + t^2}dt$$

for any ϕ in $C(K_i)$. Since the domain K_1' of σ is conull, it follows from Fubini's theorem that there is a null set N such that, for each x in $K_1\backslash N$, $m(x, s)$ is supported on K_1'. Hence, for x in $K_1\backslash N$, we can define the measure $m(x, s)\circ\sigma^{-1}$ on K_2 by the equation:

$$m(x, s)\circ\sigma^{-1}(E) = m(x, s)(\sigma^{-1}(E \cap K_2'))$$

for each E in $\mathfrak{B}_2$. Let $H^\infty(R)$ denote the Hardy space of boundary values of bounded analytic functions in the upper half-plane.

LEMMA 3.2. *There exists an invariant conull set S_0 in $\mathfrak{B}_1$ which has the following properties: For any fixed x in S_0,*

(i) *$m(x, s)$ is concentrated on the domain K_1' of σ,*

(ii) *the family $\{\phi\circ\sigma(x + t);\ \phi$ is in $\mathfrak{A}_2\}$ of functions of t is weak-* dense in $H^\infty(R)$, and*

(iii) *there is a sequence $\{s_n\}$ with $s_n \to \infty$ such that*

$$\int_{K_1} \phi\circ\sigma dm_1 = \lim_{n\to\infty}\int_{K_1} \phi\circ\sigma dm(x, s_n) \tag{3.4}$$

for each ϕ in $\mathfrak{A}_2$.

Proof. Let $\{\phi_n;\ n = 1, 2, \cdots\}$ be a countable dense subset of $\mathfrak{A}_2$. Then, together with above remark, we may choose an invariant null set N_1 such that, for each x in $K_1\backslash N_1$, $m(x, s)$ is concentrated on K_1' and the function of t, $\phi_n(x + t)$, belongs to $H^\infty(R)$ for $n = 1, 2, \cdots$. It is easy to see that, since Γ_1 is dense in R, $H^\infty(R)$ is generated by $\{e^{i\lambda t};\ \lambda$ is in $\Gamma_1^+\}$ where Γ_1^+ denotes the subsemigroup of nonnegative elements in Γ_1 (Ch. [7; Ch. 3, §1]). Let μ be the probability measure on R defined by the equation $d\mu(t) = dt/\pi(1+t^2)$. Since $H^\infty(m_2)$ is contained in $H^2(m_2)$ and $\chi_\lambda^{(1)} \circ \sigma^{-1}$ belongs to $H^\infty(m_2)$ for each λ in Γ_1^+, there is a subsequence $\{\phi_{n'}\}$ of $\{\phi_n\}$ such that

$$\|\chi_\lambda^{(1)} \circ \sigma^{-1} - \phi_{n'}\|_{L^2(m_2)} \longrightarrow 0 \quad (n' \longrightarrow \infty)\,.$$

On the other hand, it follows from Lemma 2.2 and Fubini's theorem that

$$\begin{aligned}\|\chi_\lambda^{(1)} \circ \sigma^{-1} - \phi_{n'}\|_{L^2(m_2)}^2 &= \int_{K_2} |\chi_\lambda^{(1)} \circ \sigma^{-1}(y) - \phi_{n'}(y)|^2 dm_2(y)\\ &= \int_{K_1} |\chi_\lambda^{(1)}(x) - \phi_{n'}\circ\sigma(x)|^2 dm_1(x)\\ &= \int_{K_1}\left[\int_{-\infty}^{\infty} |\chi_\lambda^{(1)}(x+t) - \phi_{n'}\circ\sigma(x+t)|^2 d\mu(t)\right] dm_1(x)\,.\end{aligned}$$

We set

$$F_{n'}(x) = \int_{-\infty}^{\infty} |\chi_\lambda^{(1)}(x+t) - \phi_{n'}\circ\sigma(x+t)|^2 d\mu(t)\,.$$

Then, since $F_{n'} \to 0$ in $L^1(m_1)$, we may find a subsequence $\{F_j\}$ of $\{F_{n'}\}$ with $F_j(x) \to 0$, m_1-a.e. x. Since Γ_1^+ is countable and $\chi_\lambda^{(1)}(x+t) = \chi_\lambda^{(1)}(x)e^{i\lambda t}$, this implies that there is an invariant null set N_2 such that, for any x in $K_1\backslash N_2$, the family $\{e^{i\lambda t};\ \lambda$ is in $\Gamma_1^+\}$ is contained in the closure of $\{\phi\circ\sigma(x+t);\ \phi$ is in $\mathfrak{A}_2\}$ in $L^2(\mu)$. We recall that $H^\infty(R) = H^2(\mu) \cap L^\infty(R)$ where $H^2(\mu)$ denotes the closure $H^\infty(R)$ in $L^2(\mu)$. Hence the conull set $S_1 = K_1\backslash(N_1 \cup N_2)$ satisfies the properties (i) and (ii).

Let $\{t_n\}$ be a arbitrary sequence of positive numbers with $t_n \to \infty$. It is well-known that if g belongs to $C(K_1)$, then (3.4) holds uniformly for this sequence $\{t_n\}$ ([4; Ch. VII, §4]). Let j be any positive integer. Then we may find g in $C(K_1)$ and s_j^1 in $\{t_n\}$ such that

$$\|\phi_1\circ\sigma - g\|_{L^1(m_1)} < (2^{-j})^2\,,$$

and

$$\left|\int_{K_1} g\,dm_1 - \int_{K_1} g\,dm(x, s_j^1)\right| < 2^{-j}$$

for any x in K_1. It follows from Fubini's theorem that

$$\|\phi_1\circ\sigma - g\|_{L^1(m_1)} = \int_{K_1}\left[\int_{K_1} |\phi_1\circ\sigma - g|\,dm(x, s_j^1)\right] dm_1(x)\,.$$

Therefore, if we set $E_j^1 = \left\{x; \int |\phi_1 \circ \sigma - g| \, dm(x, s_j^1) \geqq 2^{-j}\right\}$, then $m_1(E_j^1) < 2^{-j}$, and so

$$\left| \int_{K_1} \phi_1 \circ \sigma dm_1 - \int_{K_1} \phi_1 \circ \sigma dm(x, s_j^1) \right| < 2^{-j}(2 + 2^{-j})$$

for each x in $K^1 \backslash E_j^1$. Since $\sum_{j=1}^{\infty} m_1(E_j^1) < \infty$, we see that $m_1(\liminf_{j\to\infty} K_1 \backslash E_j^1) = 1$ by Borel-Cantelli lemma. So we may choose a null set $N(\phi_1 \circ \sigma)$ and an increasing subsequence $\{s_j^1\}$ of $\{t_n\}$ such that $\phi_1 \circ \sigma$ satisfies (3.4) for each x in $K_1 \backslash N(\phi_1 \circ \sigma)$. Since the right side limit of (3.4) is invariant, $N(\phi_1 \cdot \sigma)$ may be assumed to be invariant. By induction, it can be easily seen that if k is any positive integer, then there exists a subsequence $\{s_j^{k+1}\}$ of $\{s_j^k\}$ and an invariant null set $N(\phi_{k+1} \circ \sigma)$ for which $\phi_{k+1} \circ \sigma$ satisfies (3.4). Let $s_n = s_n^n$, and let $S_0 = S_1 \cap (K_1 \backslash \bigcup_{n=1}^{\infty} N(\phi_n \circ \sigma))$. Then, since $\{\phi_n\}$ is uniformly dense in $\mathfrak{A}_2$, S_0 and $\{s_n\}$ have the desired properties.

It is useful to note that the equation (3.4) can be extended to an ergodic flow. This is an application of Wiener's Tauberian theorem (see [12; Lemma 2.6]).

Next, let S_0 be as in Lemma 3.2, and take an x in S_0. If we set

$$h(\phi) = \int_{K_1} \phi \circ \sigma dm(x, 1)$$

for each ϕ in $\mathfrak{A}_2$, then $h(\phi)$ is a complex homomorphism of $\mathfrak{A}_2$ which lies in a nontrivial Gleason part. Since the maximal ideal space of $\mathfrak{A}_2$ is completely determined ([4; Ch. VII, Theorem 4.1]), we may find an $\hat{x}$ in K_2 and a positive number $A(x)$ such that

$$h(\phi) = \int_{K_2} \phi dm(\hat{x}, A(x)) \tag{3.5}$$

for each ϕ in $\mathfrak{A}_2$. Since $\mathfrak{A}_2$ is a Dirichlet algebra, we have

$$\int_{K_2} f dm(x, 1) \cdot \sigma^{-1} = \int_{K_2} f dm(\hat{x}, A(x))$$

for all f in $C(K_2)$. This shows that $m(x, 1) \cdot \sigma^{-1} = m(\hat{x}, A(x))$. Moreover, since $m(\hat{x}, A(x))$ and $m(\hat{x}, 1)$ are mutually absolutely continuous, it follows easily from Lemma 3.2 that

$$\begin{cases} \Psi L^{\infty}(m(x, 1)) = L^{\infty}(m(\hat{x}, 1)) \,, \quad \text{and} \\ \Psi H^{\infty}(m(x, 1)) = H^{\infty}(m(\hat{x}, 1)) \,, \end{cases} \tag{3.6}$$

where $\Psi(\psi) = \psi \circ \sigma^{-1}$ for each ψ in $L^{\infty}(m(x, 1))$. In order to show the equation (3.3), we have to determine the Borel isomorphism σ on each orbit. This will be accomplished by applying the result of de Leeuw, Rudin, and Wermer [2].

LEMMA 3.3. *Let S_0, x, $\hat{x}$, and $A(x)$ be as above. Then we have:*

$$\psi \circ \sigma(x + t) = \psi(\hat{x} + A(x)t) \qquad dt\text{-a.e.} \tag{3.7}$$

for each ψ in $H^\infty(m(\hat{x}, 1))$.

Proof. For any function f on K_i and y is K_i, we define

$$\Phi_y(f)(t) = f(y + t)\ , \quad t \text{ in } R\ .$$

Since each function in $H^\infty(m(y, 1))$ is the almost every limit of a sequence in $\mathfrak{A}_i$, it is easy to see that Φ_y is an isomorphism from $H^\infty(m(y, 1))$ onto $H^\infty(R)$. We consider the following diagram:

$$\begin{array}{ccc} H^\infty(m(\hat{x}, 1)) & \xrightarrow{\ \Psi^{-1}\ } & H^\infty(m(x, 1)) \\ \Phi_{\hat{x}} \downarrow & & \downarrow (\Phi_x) \\ H^\infty(R) & \xrightarrow[\ \Phi_x \Psi^{-1} \Phi_{\hat{x}}^{-1}\ ]{} & H^\infty(R) \end{array}$$

Then, according to a theorem of de Leeuw, Rudin, and Wermer [2], there is a fractional linear transformation $\alpha_x(t)$ of the upper half-plane onto itself such that

$$(\Phi_x \Psi^{-1} \Phi_{\hat{x}}^{-1}) f(t) = f(\alpha_x(t))$$

for each f in $H^\infty(R)$. Let ϕ be a function in $\mathfrak{A}_2$. Then we have

$$\begin{aligned} \phi \circ \sigma(x + t) &= (\Phi_x \Psi^{-1} \Phi_{\hat{x}}^{-1})(\Phi_{\hat{x}} \phi)(t) \\ &= (\Phi_{\hat{x}} \phi)(\alpha_x(t)) \\ &= \phi(\hat{x} + \alpha_x(t)) \qquad dt\text{-a.e.} \end{aligned}$$

We claim that there exist real numbers p and q with $p > 0$ such that $\alpha_x(t) = pt + q$. Suppose not. Then we may choose some real numbers a, b, and c such that $\alpha_x^{-1}(u) = (au + b)(u + c)^{-1}$ and $ac - b > 0$. Let $\{s_n\}$ be sequence as in Lemma 3.2. Then, for each ϕ in $\mathfrak{A}_2$, we see from (3.2) and Lemma 3.2 that

$$\begin{aligned} \int_{K_2} \phi dm_2 &= \int_{K_1} \phi \circ \sigma dm_1 \\ &= \lim_{n \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \phi \circ \sigma(x + t) \frac{s_n}{s_n^2 + t^2} dt \\ &= \lim_{n \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\hat{x} + \alpha_x(t)) \frac{s_n}{s_n^2 + t^2} dt\ . \end{aligned} \tag{3.8}$$

On the other hand, a quick calculation show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(\hat{x}+\alpha_x(t))\frac{s_n}{s_n^2+t^2}dt$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(\hat{x}+u)\frac{s_n(ac-b)(s_n^2+a^2)^{-1}}{(u+(s_n^2c+ab)(s_n^2+a^2)^{-1})^2+(s_n(ac-b)(s_n^2+a^2)^{-1})^2}du\,.$$

Since $\phi(\hat{x}+u)$ is continuous as a function of u, this implies that (3.8) equals to $\phi(\hat{x}-c)$. Hence we see that m_2 is the point mass at $\hat{x}-c$ since $\mathfrak{A}_2$ is a Dirichlet algebra. Thus we have a contradiction. We may now assert that $p=A(x)$ and $q=0$. By setting $v=pt+q$, it follows from (3.5) that

$$\int_{-\infty}^{\infty}\phi(\hat{x}+v)\frac{p}{p^2+(v-q)^2}dv=\int_{-\infty}^{\infty}\phi(\hat{x}+\alpha_x(t))\frac{1}{1+t^2}dt$$
$$=\int_{-\infty}^{\infty}\phi(\hat{x}+t)\frac{A(x)}{A(x)^2+t^2}dt$$

for each ϕ in $\mathfrak{A}_2$. Therefore, since $\mathfrak{A}_2$ is a Dirichlet algebra, we obtain

$$\frac{p}{p^2+(t-q)^2}=\frac{A(x)}{A(x)^2+t^2}$$

for any t in R. From this equation, it is easy to see that $p=A(x)$ and $q=0$. Thus the equation (3.7) holds for each ϕ in $\mathfrak{A}_2$. However, since any ψ in $H^\infty(m(\hat{x},1))$ is the almost every limit of a sequence in $\mathfrak{A}_2$, it follows easily from (3.6) that ψ satisfies also the equation (3.7). So the proof is complete.

We remark here that $A(x)$ is invariant as a function of x. In fact, for x in S_0 and u in R, we have

$$f\circ\sigma(x+u+t)=f(\hat{x}+A(x)u+A(x)t)\qquad dt\text{-a.e.}\,.$$

This shows that $(x+u)^\wedge=\hat{x}+A(x)u$ and $A(x+u)=A(x)$.

Proof of Proposition 3.1. Let S_0 be an invariant conull set as in Lemma 3.2. For any x in S_0, let $\hat{x}$ and $A(x)$ be as in Lemma 3.3. Then, for each positive λ in Γ_1, since the function of t, $\chi_\lambda^{(1)}(x+t)$, belongs to $H^\infty(m(x,1))$, we see that

$$\sigma^{-1}\chi_\lambda^{(1)}(\hat{x}+s)=\chi_\lambda^{(1)}(x+A(x)^{-1}s)\qquad ds\text{-a.e.}\,,$$

where $\sigma^{-1}\chi_\lambda^{(1)}$ is defined by the equation $\sigma^{-1}\chi_\lambda^{(1)}(y)=\chi_\lambda^{(1)}(\sigma^{-1}(y))$. Hence we have, for any t in R,

$$(3.9)\qquad \begin{cases} T_t^{(2)}(\sigma^{-1}\chi_\lambda^{(1)})(\hat{x}+s)=\sigma^{-1}\chi_\lambda^{(1)}(\hat{x}+s+t) \\ \qquad\qquad =\chi_\lambda^{(1)}(x+A(x)^{-1}(s+t))\qquad ds\text{-a.e.}\,. \end{cases}$$

Let $\beta(x) = A(x)^{-1}$. Then $\beta(x)$ is invariant as a function of x by above remark. It follows from (3.9) that, for any x in S_0 and any t in R,

$$\begin{aligned}(\sigma^{-1}T_t^{(2)}\sigma)\chi_\lambda^{(1)}(x+s) &= \sigma(T_t^{(2)}(\sigma^{-1}\chi_\lambda^{(1)}))(x+s)\\ &= T_t^{(2)}(\sigma^{-1}\chi_\lambda^{(1)})(\hat{x}+A(x)s)\\ &= \chi_\lambda^{(1)}(x+A(x)^{-1}(A(x)s+t))\\ &= \chi_\lambda^{(1)}(x+s+\beta(x)t)\\ &= T_{\beta(x)t}^{(1)}\chi_\lambda^{(1)}(x+s) \qquad ds\text{-a.e. .}\end{aligned}$$

Recall that $\chi_\lambda^{(1)}(x+s) = e^{i\lambda s}\chi_\lambda^{(1)}(x)$. So we obtain

$$(3.10)\qquad \begin{cases}(\sigma^{-1}T_t^{(2)}\sigma)\chi_\lambda^{(1)}(x) = T_{\beta(x)t}^{(1)}\chi_\lambda^{(1)}(x)\\ \qquad\qquad = e^{i\beta(x)\lambda t}\chi_\lambda^{(1)}(x) \qquad m_1\text{-a.e. } x\,,\end{cases}$$

for each t in R and each λ in Γ_1. We have to show $\beta(x)$ is a constant β as a function of x. Since the system $(K_1, m_1, \{T_t^{(1)}\}_{t\in R})$ is ergodic, it suffices to show that $\beta(x)$ is measurable as a function of x. For this, we note that $(\sigma^{-1}T_t^{(2)}\sigma)\chi_\lambda^{(1)}(x)$ is measurable with respect to (t, x). Hence it follows from (3.10) that $e^{i\beta(x)\lambda t}$ is measurable as a function of (t, x). From this fact, we see easily that $\beta(x)$ is measurable. Recall that the space of all analytic polynomials on K_1 is weak-* dense in $H^\infty(m_1)$. So since $\sigma^{-1}T_t^{(2)}\sigma$ is a measure preserving transformation on (K_1, m_1), (3.10) implies that the equation (3.3) holds for each f in $H^\infty(m_1)$. This completes the proof of Proposition 3.1.

Since $H^\infty(m_1) + \bar{H}^\infty(m_1)$ is weak-* dense in $L^\infty(m_1)$, the equation (3.3) assert that $\sigma^{-1}T_t^{(2)}\sigma$ is equal to $T_{\beta t}^{(1)}$ as a sigma-isomorphism from the measure algebra $(\mathfrak{A}_1, m_1)$ onto itself. However, since K_1 is metric, we may strengthen it as follows:

$$(3.3')\qquad \sigma^{-1}T_t^{(2)}\sigma(x) = T_{\beta t}^{(1)}(x) \qquad m_1\text{-a.e. } x\,.$$

4. The proof of main result. In this section we present a proof of Theorem 2.1. For $i = 1, 2$, let Γ_i be an arbitrary dense subgroup of R but endowed with the discrete topology (cf. §5, Remark (c)). For any countable subgroup $\tilde{\Gamma}_i$ of Γ_i, we set $H^\infty(m_i, \Gamma_i)$ is the space of all functions f in $H^\infty(m_i)$ whose frequencies lie in $\tilde{\Gamma}_i$, i.e.,

$$H^\infty(m_i, \tilde{\Gamma}_i) = \{f \in H^\infty(m_i);\ f \sim \sum_{\lambda\in\Gamma_i} a(f)\chi_\lambda^{(i)}\}$$

where $\sum_\lambda a_\lambda(f)\chi_\lambda^{(i)}$ denotes the Fourier series of f.

LEMMA 4.1. *Under the assumption of Theorem* 2.1, *let* Ψ *be an isomorphism from* $H^\infty(m_1)$ *onto* $H^\infty(m_2)$. *If* $S^{(1)}$ *is a countable subset of* Γ_1, *then there exist countable subgroups* $\tilde{\Gamma}_1$ *and* $\tilde{\Gamma}_2$ *of* Γ_1 *and* Γ_2, *respectively, which have the following properties*:

(4.1) $$\tilde{\Gamma}_1 \supset S^{(1)} ;$$

(4.2) *both* $\tilde{\Gamma}_1$ *and* $\tilde{\Gamma}_2$ *are dense in* R; *and*

(4.3) $$\Psi(H^\infty(m_1, \tilde{\Gamma}_1)) = H^\infty(m_2, \tilde{\Gamma}_2) .$$

Proof. We may easily find a countable subgroup $D_1^{(1)}$ of Γ_1 such that $S^{(1)} \subset D_1^{(1)}$ and $D_1^{(1)}$ is dense in R. Recall that if f belongs to $L^1(m_2)$, then the nonzero Fourier coefficients of f are at most countable. So we may find a countable subgroup $D_1^{(2)}$ of Γ_2 such that $D_1^{(2)}$ is dense in R and $\Psi\chi_\lambda^{(1)}$ belongs to $H^\infty(m_2, D_1^{(2)})$ for each λ in $D_1^{(1)}$. On the other hand, it follows from Lemma 2.2 that Ψ is continuous with respect to weak-* topology. Hence we have $\Psi(H^\infty(m_1, D_1^{(1)})) \subset H^\infty(m_2, D_1^{(2)})$. Similarly, it can be seen that there is a countable subgroup $D_2^{(1)}$ of Γ_1 such that $D_1^{(1)} \subset D_2^{(1)}$ and $H^\infty(m_1, D_2^{(1)}) \supset \Psi^{-1}(H^\infty(m_2, D_1^{(2)}))$. Repeat the procedure to find a countable subgroup $D_2^{(2)}$ of Γ_2. We continue in this way infinitely, if necessary. Then we obtain increasing sequences $\{D_n^{(1)}\}$ and $\{D_n^{(2)}\}$ of countable subgroups of Γ_1 and Γ_2 which satisfy

$$\Psi(H^\infty(m_1, D_n^{(1)})) \subset H^\infty(m_2, D_n^{(2)}) ,$$

and

$$\Psi(H^\infty(m_1, D_{n+1}^{(1)})) \supset H^\infty(m_2, D_n^{(2)}) ,$$

for any positive integer n. Let $\tilde{\Gamma}_1 = \bigcup_{n=1}^\infty D_n^{(1)}$ and let $\tilde{\Gamma}_2 = \bigcup_{n=1}^\infty D_n^{(2)}$. Then we see easily that $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have the desired properties, and the proof is complete.

The following lemma makes essential use of the results in [15] and is proved in [12; §3, Step 1]. However, we give here the sketch of the proof for the shake of completeness.

LEMMA 4.2. *Under the assumption of Theorem* 2.1, *let* Ψ *be an isometry mapping* $H^p(m_1)$ *onto* $H^p(m_2)$, $p \neq 2$. *Then the restriction of* Ψ *to* $H^\infty(m_1)$ *is a constant multiple of an algebra isomorphism from* $H^\infty(m_1)$ *onto* $H^\infty(m_2)$.

Sketch of proof. We set $g = \Psi(1)$, and let $d\nu = |g|^p dm_2$. Then, since g is a nonzero function in $H^p(m_2)$, we have $L^\infty(\nu) = L^\infty(m_2)$. Define $A(f) = g^{-1}\Psi(f)$ for each f in $H^\infty(m_1)$. Then, as Rudin shows in [15; Theorem 2], A is an algebra homomorphism which is isometric in the supremum norm, mapping $H^\infty(m_1)$ into $L^\infty(m_2)$. The properties of weak-* Dirichlet algebras imply that A carries $H^\infty(m_1)$ into $H^\infty(m_2)$. Similarly since Ψ^{-1} has the same properties as Ψ, we find a g' in $H^p(m_1)$ and an algebra homomorphism A' mapping $H^\infty(m_2)$ into $H^\infty(m_1)$ such that $\Psi^{-1}(f) = g'A'(f)$ for all f in $H^\infty(m_2)$. On the other hand, it follows from the definition of A' that $\Psi^{-1}(\phi g) = A'(\phi)\Psi^{-1}(g)$ for each ϕ in $H^\infty(m_2)$ and the above g in $H^p(m_2)$ since A' is a homomorphism and Ψ^{-1} is continuous. Hence, since $\Psi^{-1}(g) = 1$, if we set $f = \Psi^{-1}(\phi g)$, then f belongs to $H^\infty(m_1)$ and $A(f) = g^{-1}\Psi(f) = \phi$. So A maps $H^\infty(m_1)$ onto $H^\infty(m_2)$. By Lemma 2.2, there is a sigma-isomorphism σ from $(\mathfrak{B}_1, m_1)$ onto $(\mathfrak{B}_2, m_2)$ satisfying the equation (2.2) with Ψ replaced by A. This implies that A may be extended to an isometry mapping $H^p(m_1)$ onto $H^p(m_2)$. Since ΨA^{-1} is an isometry mapping $H^p(m_2)$ onto itself, it is shown that g is a unimodular constant. So the restriction of Ψ to $H^\infty(m_1)$ has the desired form.

Proof of Theorem 2.1. We attend only to the direct half since the converse is straightforward. By Lemma 4.2, it suffices to prove under the hypotheses that $p = \infty$ and the isometry Ψ is an algebra isomorphism from $H^\infty(m_1)$ onto $H^\infty(m_2)$. So it follows from Lemma 2.2 that Ψ holds the equation (2.2) for some sigma-isomorphism σ from $(\mathfrak{B}_1, m_1)$ onto $(\mathfrak{B}_2, m_2)$. For any λ in Γ_1, we set $S^{(1)} = \{\lambda\}$ in Lemma 4.1. Then it can be seen that there exist countable subgroups $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ of Γ_1 and Γ_2, respectively, which satisfy the properties (4.1), (4.2), and (4.3). For $i = 1, 2$, let $\tilde{K}_i$ be the dual group of $\tilde{\Gamma}_i$, and let $\tilde{m}_i$ and $\{\tilde{T}_t^{(i)}\}_{t\in R}$ be the objects associated with $\tilde{K}_i$ as in §2. Recall that $\tilde{K}_i$ is isomorphically homeomorphic to the quotient group $K_i/\tilde{\Gamma}_i^\perp$ where $\tilde{\Gamma}_i^\perp$ denotes the annihilator of $\tilde{\Gamma}_i$ (cf. [14; 2.1.2]). We denote by ρ_i the canonical map from K_i onto $K_i/\tilde{\Gamma}_i^\perp$. Since $H^\infty(\tilde{m}_i)$ may be identified with $H^\infty(m_i, \tilde{\Gamma}_i)$, it follows from (4.3) that Ψ defines an isomorphism $\tilde{\Psi}$ from $H^\infty(\tilde{m}_1)$ onto $H^\infty(\tilde{m}_2)$. Since $\tilde{\Gamma}_i$ is a countable dense subgroup of R, we see from Proposition 3.1 that there is a positive constant β such that, for each ν in $\tilde{\Gamma}_1$ and t in R,

$$(4.4) \qquad \tilde{\Psi}^{-1}\tilde{T}_t^{(2)}\tilde{\Psi}(\chi_\nu^{(1)}\rho_1^{-1})(\tilde{x}) = \tilde{T}_{\beta t}^{(1)}(\chi_\nu^{(1)}\rho_1^{-1})(\tilde{x}) \qquad \tilde{m}_1\text{-a.e. } \tilde{x} .$$

We notice that $\rho_i T_t^{(i)} = \tilde{T}_t^{(i)}\rho_i$ and that if $\tilde{N}$ is $\tilde{m}_i$-null set, then $\rho_i^{-1}(\tilde{N})$ is also m_i-null set. So it follows from (4.4) that

$$(4.5) \qquad \Psi^{-1}T_t^{(2)}\Psi(\chi_\nu^{(1)})(x) = T_{\beta t}^{(1)}(\chi_\nu^{(1)})(x) \qquad m_1\text{-a.e. } x .$$

We note here that β is independent to $\tilde{\Gamma}_1$. Indeed, since $T_{\beta t}^{(1)}\chi_\nu^{(1)}(x) = e^{i\beta\nu t}\chi_\nu^{(1)}(x)$, it can be seen that β is uniquely determined from each ν

in Γ_1 with $\nu \neq 0$. For any fixed λ' in Γ_1, we may assume that λ' belongs to $\tilde{\Gamma}_1$ by setting $S^{(1)} = \{\lambda, \lambda'\}$ in Lemma 4.1. So β is independent to $\tilde{\Gamma}_1$. Therefore the equation (4.5) holds for each ν in Γ_1. Thus, we have

$$T_t^{(2)}(\Psi \chi_\lambda^{(1)})(y) = e^{i\beta\lambda t}(\Psi \chi_\lambda^{(1)})(y) \qquad m_2\text{-a.e. } y$$

for each λ in Γ_1 and t in R. This implies that $\Psi \chi_\lambda^{(1)}$ is an eigen function for $\{T_t^{(2)}\}_{t\in R}$ with eigenvalue $\beta\lambda$. It follows, therefore, the map $\lambda \to \beta\lambda$ is a group isomorphism mapping Γ_1 into Γ_2. Similarly, we see that $\lambda \to \beta^{-1}\lambda$ is also a group isomorphism mapping Γ_2 into Γ_1 since (4.5) holds with $\chi_\nu^{(1)}$ replaced by $\Psi^{-1}\chi_\lambda^{(2)}$. Hence $\lambda \to \beta\lambda$ maps Γ_1 onto Γ_2. Recall that each eigenvalue of $\{T_t^{(2)}\}_{t\in R}$ is simple, meaning that if f and g are eigenfunction with same eigenvalue, then g is a constant multiple of f (cf. [5]). So we may find a constant $C_{\beta\lambda}$ with $|C_{\beta\lambda}| = 1$ such that

$$\Psi \chi_\lambda^{(1)}(y) = C_{\beta\lambda}\chi_{\beta\lambda}^{(2)}(y) \qquad m_2\text{-a.e. } y\ .$$

Since Ψ is an algebra homomorphism, it is easy to see that $C_{\nu+\nu'} = C_\nu \cdot C_{\nu'}$ for each ν and for each ν' in Γ_2. This shows that $\nu \to C_\nu$ is a character of Γ_2. There is, therefore, a y_0 in K_2 satisfying

$$\Psi \chi_\lambda^{(1)}(y) = \chi_{\beta\lambda}^{(2)}(y + y_0) \qquad m_2\text{-a.e. } y\ .$$

Let σ_1 be the translation by $-y_0$, and let σ_2 be the inverse of the adjoint of the above map $\lambda \to \beta\lambda$. Then, since $\chi_{\beta\lambda}^{(2)}(y) = \chi_\lambda^{(1)}(\sigma_2^{-1}(y))$ for y in K_2, we have that

$$\Psi \chi_\lambda^{(1)}(y) = \chi_\lambda^{(1)}(\sigma_2^{-1}(y + y_0)) \qquad m_2\text{-a.e. } y\ ,$$

for each λ in Γ_1. This shows that the sigma-isomorphism σ may be identified with the affine map $\sigma_1 \cdot \sigma_2$. Hence we see that (2.1) holds for each f in $H^\infty(m_1)$ with $c = 1$. This completes the proof.

5. Remarks. (a) Let X be a compact Hausdorff space upon which $\{S_t\}_{t\in R}$ acts as a locally compact transformation group, and let $\mathfrak{A}$ be the uniform algebra of analytic functions induced by $\{S_t\}_{t\in R}$. We assume that X is not metric and there are no periodic orbits in X. If $\mathfrak{C}$ is a countable subset of $\mathfrak{A}$, then there exists a closed separable subalgebra $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ such that $\mathfrak{C} \subset \tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}$ is invariant, i.e., for any f in $\tilde{\mathfrak{A}}$, $S_t f(x) = f(S_t(x))$ belongs to $\tilde{\mathfrak{A}}$. This implies that $\tilde{\mathfrak{A}}$ may be regarded a uniform algebra on a compact metric space. From this fact, by the similar way as in § 4, we can extend Proposition to 3.1 the ergodic Hardy spaces induced by $\{S_t\}_{t\in R}$.

(b) The author does not know, under the assumption of Theorem 2.1, whether one can characterize the isometries from $H^p(m_1)$ into

$H^p(m_2)$, $p \neq 2$. Forelli [3] answered this question for the classical Hardy spaces.

(c) By [14; 2.5.2], we see that the Bohr group contains an infinite compact metric group. This fact implies that there exists an uncountable subgroup Γ of R_d with $\Gamma \neq R_d$, where R_d denotes the discrete real line.

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