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TWO RESULTS ON COFIBERS

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Recently M. Mather has generalized results of T. Ganea concerning homotopy fibers and cofibers. In this paper we present two results on cofibers, one of which substantially extends and clarifies Mather's generalization. Our other result (which is used to prove the first) in part examines the mapping cone of the fiberwise join of two maps. Applications of the results are made to reprove a result of I. M. James on the fiberwise suspension and to give a characterization of coreducible Thom spaces.

1. Introduction. One of the principal problems of homotopy theory is to describe the homotopy fiber F_α and the (homotopy) cofiber C_α of a given map $\alpha: X \to Y$. A very useful general result (indeed, one of the few known) is that F_α is a double mapping cylinder if X is. As an application of this general result one has Ganea's classic gem [1]: if $p: E \to B$ is a Hurewicz fibration with fiber F then the homotopy fiber of the map $E \cup CF \to B$ is the topological join $\Omega B * F$,

Interesting enough Ganea's result can be recovered as well from a cofiber theorem. Namely

THEOREM 1.1. *In the diagram*

$$\begin{array}{ccccc} & & E_B & \longrightarrow & E \\ & {}^{e}\nearrow & \downarrow & & \downarrow p \\ C & \xrightarrow{g} & B & \xrightarrow{\beta} & M \end{array}$$

let p be a Hurewicz fibration with E_B obtained from p by pullback along β. Suppose also that g coclassifies β and that e is the canonically obtained lifting. Then the cofiber of the induced map $C_e \to E$ is $C * F$ where F is the fiber of p.

Under restrictions imposed by his method of proof, Ganea [2] proved this theorem when $E \simeq *$ (in which case the result has the form $\Sigma C_e \simeq C * \Omega M$). The connection between [1] and [2] was observed by Mather in [7]. (Actually, Mather states the result only when $p: E \to M$ is the principal fibration induced by a map $M \to Z$.) One recovers Ganea's first result from (1.1) by letting $M = E \cup CF$ and $C = F = E_B$ (so that $e = 1$, $C_e \simeq *$).

We prove the following generalization of Theorem 1.1 (with notation explained later).

THEOREM 1.2. *In the homotopy commutative diagram*

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \mathfrak{S}} \\ P & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & M \end{array}$$

suppose that the outside square is a homotopy pushout and that the lower square is a homotopy pullback. If the map $\mathfrak{s}_A: A \to E_A$ *is a cofibration then the cofiber of the induced map* $\mathscr{M}(e, f) \to E$ *is the quotient space* $E(f * p_A)/A$.

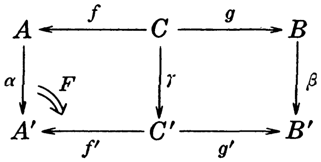
A main ingredient in the proof of (1.2) is Lemma 3.3 below. The other cofiber result of the title (stated as (3.4)) is an immediate consequence of this lemma. The lemma itself is quite useful and seems to have been overlooked heretofore in the literature. In §4 we use it to reprove a result of James on the fiberwise suspension as well as to give a characterization of coreducible Thom spaces.

2. Notation. We work in either the category of based or unbased topological spaces. In each of these categories one has available homotopy pushouts and homotopy pullbacks. We assume the reader is familiar with these concepts, but, primarily to fix the notation we use, we recall some notions concerning homotopy pushouts. For this purpose we employ the double mapping cylinder functor. It is denoted $\mathscr{M}(f, g)$ in the unbased category and $\widetilde{\mathscr{M}}(f, g)$ in the based category. (The reader is referred to [4] for elementary properties of the double mapping cylinder functor, albeit there in an abstract setting.) For convenience the definitions are stated only in the unbased category.

A square

$$(2.1)\qquad \begin{array}{ccc} C & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & F \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & X \end{array}$$

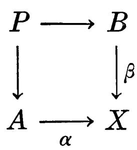
with a homotopy $F: \alpha f \simeq \beta g$ is called a *homotopy pushout* if the map $\mu_F: \mathscr{M}(f, g) \to X$ induced by the homotopy F is a homotopy equivalence. The homotopy class of μ_F actually depends only on the track class of the homotopy F. There is a bijective correspondence between maps $\mathscr{M}(f, g) \to X$ and triples (α, F, β) as in (2.1) above. In particular a diagram



having $F\colon \alpha f \simeq f'\gamma$ and $\beta g = g'\gamma$ induces a map $\mu(\alpha, \gamma, \beta; F)$: $\mathscr{M}(f, g) \to \mathscr{M}(f', g')$ whose defining homotopy is the track sum $i_0F + K\gamma$ where K denotes the defining homotopy of $\mathscr{M}(f', g')$ and $i_0\colon A' \to \mathscr{M}(f', g')$ is the inclusion. Observe that if F is the static homotopy (so that now $\alpha f = f'\gamma$) then $i_0F + K\gamma$ is track equivalent to $K\gamma$; hence $\mu(\alpha, \gamma, \beta; F)$ is homotopic to $\mu_{K\gamma}$. In this case $\mu_{K\gamma}$, denoted $\mu(\alpha, \gamma, \beta)$, is said to be induced functorially by the triple of maps (α, γ, β). If α, γ and β are homotopy equivalences then by [4, Theorem 4.9] (or [8, Corollary 9]) $\mu(\alpha, \gamma, \beta; F)$ is a homotopy equivalence, a fact we often use without citation.

We shall also need the (fiberwise) join construction for maps. This is defined as follows.

If $\alpha\colon A \to X$ and $\beta\colon B \to X$ are arbitrary maps then the (fiberwise) *join* $\alpha * \beta\colon E(\alpha * \beta) \to X$ is the map constructed by considering the (topological) pullback square



and letting $E(\alpha * \beta) = \mathscr{M}(A \leftarrow P \to B)$, with $E(\alpha * \beta) \to X$ being the projection induced by the commutative square. Note that the fiber of $\alpha * \beta$ over $x \in X$ is $\alpha^{-1}(x) * \beta^{-1}(x)$ (the usual join of spaces with the identification topology). Hence if $X = *$ then $E(\alpha * \beta) = A * B$.

We denote by $\alpha * \varepsilon^0$ the fiberwise join of $\alpha\colon A \to X$ and the projection $X \times S^0 \to X$. One sees readily that $E(\alpha * \varepsilon^0)$ is homeomorphic to $\mathscr{M}(\alpha, \alpha)$.

Note. The fiberwise join construction lives most conveniently in the unbased category (for one reason, to maintain its fibration properties). In this paper whenever $E(\alpha * \beta)$ (or any unreduced double mapping cylinder) occurs in a situation requiring base points, we always take the variable base point as explained in [5] and assume that all spaces are well-pointed. The use of this convention to modify our results whenever necessary is left to the reader.

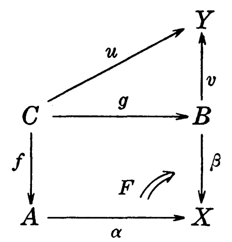
This must be done for example in the based version of Theorem 1.2.

We also point out that since we use the track calculus (as opposed to identification map techniques) absolutely no restrictions on the spaces involved (except possibly well-pointedness) are needed.

3. A lemma.

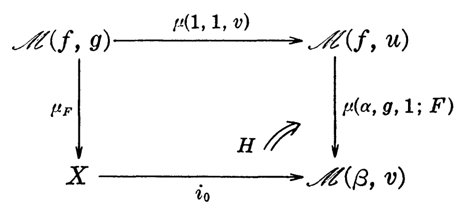
In this section we consider a fixed diagram

(3.1)



with $F\colon \alpha f \simeq \beta g$ and $u = vg$. (Here the square containing F need be neither a homotopy pushout nor a homotopy pullback.) From (3.1) we obtain a square

(3.2)



with homotopy H given by

$$H_s[c, t] = \begin{cases} F\left(c, \dfrac{2t}{2-s}\right), & 0 \leqq t \leqq \dfrac{2-s}{2} \\ [g(c), s+2t-2]\,, & \dfrac{2-s}{2} \leqq t \leqq 1 \end{cases}$$

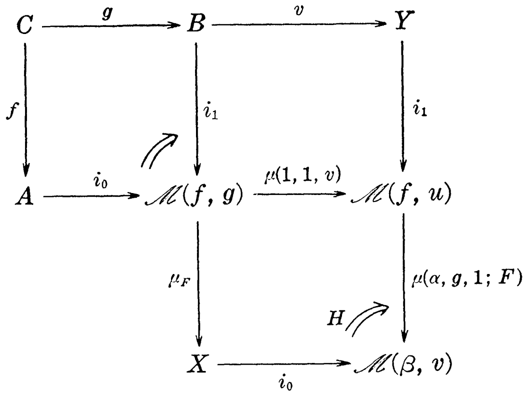
$$H_s(a) = \alpha(a)$$

$$H_s(b) = [bb, s]$$

for $a \in A$, $b \in B$, $c \in C$, and $s, t \in I$.

LEMMA 3.3. *Square* (3.2) *is a homotopy pushout. Moreover if F is the static homotopy then the homotopy H may be replaced by another homotopy so that the square* (3.2) *with* $\mu(\alpha, g, 1; F)$ *replaced by* $\mu(\alpha, g, 1)$ *is a homotopy pushout.*

Proof. Construct the diagram



Now applying in an evident way [8, Lemma 15] twice to this diagram yields (3.2) as a homotopy pushout. The last statement of the theorem clearly follows from the first.

COROLLARY 3.4. *The maps μ_F and $\mu(\alpha, g, 1; F)$ have homotopy equivalent mapping cones.*

4. The fiberwise suspension. Let $\Sigma F \xrightarrow{j} E(p * \varepsilon^0) \to B$ be the fiberwise suspension of a given Hurewicz fibration $F \xrightarrow{i} E \xrightarrow{p} B$. Since $E(p * \varepsilon^0) \cong \mathscr{M}(p, p)$, $p * \varepsilon^0$ has two canonical cross-sections, denoted $\mathfrak{s}_0, \mathfrak{s}_1 \colon B \to E(p * \varepsilon^0)$. James has studied the composite

$$\pi_r B \xrightarrow{\partial} \pi_{r-1} F \xrightarrow{E} \pi_r \Sigma F \xrightarrow{j_\sharp} \pi_r E(p * \varepsilon^0)$$

where ∂ is the boundary homomorphism in the long exact homotopy sequence for p and E is the suspension homomorphism. He proved [3]:

THEOREM 4.1. $j_\sharp \circ E \circ \partial = \mathfrak{s}_{0\sharp} - \mathfrak{s}_{1\sharp} \colon \pi_r B \to \pi_r E(p * \varepsilon^0)$.

As an application of Lemma 3.3, we wish to give an elementary proof of this result. Let $\partial \colon \Omega B \to F$ be defined in the usual way by choosing a lifting function for p. Let $\varepsilon \colon \Sigma\Omega B \to B$ denote the canonical map. Then (4.1) follows immediately from:

THEOREM 4.2. $j \circ \Sigma\partial \simeq \mathfrak{s}_0 \circ \varepsilon - \mathfrak{s}_1 \circ \varepsilon \colon \Sigma\Omega B \to E(p * \varepsilon^0)$.

LEMMA 4.3. *Let $w \colon E \cup CF \to \Sigma F \vee B$ be given by $w(e) = (*, pe)$ for $e \in E$ and $w[y, t] = ([y, t], *)$ for $y \in F$, $0 \leqq t \leqq 1$. Then the homotopy commutative diagram*

$$\begin{array}{ccc} E \cup CF & \xrightarrow{w} & \Sigma F \vee B \\ \downarrow & & \downarrow{\scriptstyle j\nabla\mathfrak{S}_1} \\ B & \xrightarrow[\mathfrak{S}_0]{} & E(p * \varepsilon^0) \end{array}$$

is a homotopy pushout.

Proof. Apply Lemma 3.3 to the diagram

$$\begin{array}{ccc} & \overset{0}{\nearrow} & B \\ & & \uparrow{\scriptstyle p} \\ F & \xrightarrow{i} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & B\,. \end{array}$$

Proof of (4.2). Since the composite $\Omega B \xrightarrow{\partial} F \xrightarrow{i} E$ is null homotopic, we obtain (by fixing a particular null homotopy) a map $\varphi\colon \Sigma\Omega B \to E \cup CF$. Moreover, the diagram

$$\begin{array}{ccc} \Sigma\Omega B & \xrightarrow{\omega} & \Sigma\Omega B \vee \Sigma\Omega B \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \Sigma\partial \vee \varepsilon} \\ E \cup CF & \xrightarrow{w} & \Sigma F \vee B \end{array}$$

is commutative where ω is suspension comultiplication. Also the composite $\Sigma\Omega B \xrightarrow{\varphi} E \cup CF \to B$ is easily seen to be homotopic to $\varepsilon\colon \Sigma\Omega B \to B$. Hence (4.2) follows from (4.3).

Let $\mu\colon \Sigma F \to C_p$ be the canonical inclusion. (When C_p is referred to as the Thom space then the map μ is called the homotopy Thom class.) Recall that C_p is said to be *coreducible* if there is a map $r\colon C_p \to \Sigma F$ with $r\mu \simeq 1$.

By applying Lemma 3.3 to the diagram

$$\begin{array}{ccc} & \nearrow & * \\ & & \uparrow \\ F & \xrightarrow{i} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & B \end{array}$$

we obtain:

PROPOSITION 4.4. *If* $q\colon E \cup CF \to \Sigma F$ *is the canonical quotient* *map then the following homotopy commutative square is a homotopy pushout.*

$$\begin{array}{ccc} E \cup CF \xrightarrow{q} \Sigma F \xrightarrow{-1} \Sigma F \\ \downarrow & & \downarrow \mu \\ B & \xrightarrow[i_1]{} & C_p \end{array}$$

(*Here the map* $-1\colon \Sigma F \to \Sigma F$ *intervenes because* C_p *is defined with vertex at* $t = 0$.)

We note that this square actually occurs in Proposition 1.6 of [1], although of course it is not there asserted to be a homotopy pushout.

COROLLARY 4.5. *C_p is coreducible if and only if $q\colon E \cup CF \to \Sigma F$ factors through $E \cup CF \to B$.*

COROLLARY 4.6. *Suppose C_p is coreducible with retraction $r\colon C_p \to \Sigma F$. If $\alpha \in \pi_n B$ then $E\partial(\alpha) = -ri_1\alpha \in \pi_n \Sigma F$ where $i_1\colon B \to C_p$.*

Proof. For $q \circ \varphi \simeq \Sigma\partial\colon \Sigma\Omega B \to \Sigma F$, with φ as in the proof of (4.2).

Let $\rho\colon \Omega B \times F \to F$ be the usual map with $\rho|_{\Omega B \times \{*\}} = \partial\colon \Omega B \to F$. Combining Theorem 1.4 of [1] with Corollary 4.5 above we get the next result.

PROPOSITION 4.7. *If C_p is coreducible then the Hopf construction on ρ is trivial:*

$$h(\rho) \simeq 0\colon \Omega B * F \longrightarrow \Sigma F .$$

We remark that using (4.3) one may prove a result slightly more general than (4.7). Namely, if $p * \varepsilon^0$ is retractible then $h(\rho) \simeq 0$: $\Omega B * F \to \Sigma F$. ($p * \varepsilon^0$ is retractible if ΣF is a homotopy retract of $E(p * \varepsilon^0)$.) Since this involves changing the topology on $E(p * \varepsilon^0)$ to insure (without restrictive assumptions) that $p * \varepsilon^0$ has the homotopy lifting property, we omit the proof.

It is interesting to note one further consequence of the results of this section. Suppose that the base space B of the given fibration p is a suspension space ΣX. Then as in (5.2) below there is a clutching function $\gamma\colon X \times F \to F$ and as in [6, §5] one has the twisted Whitehead product map $W_p\colon X * F \to \Sigma X \vee \Sigma F$.

THEOREM 4.8. *Let* $p: E \to \Sigma X$ *be a Hurewicz fibration with fiber* F *and clutching function* $\gamma: X \times F \to F$. *Then the following conditions are equivalent*:

(i) C_p *is coreducible.*

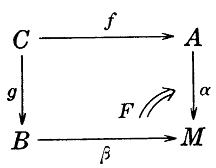
(ii) $h(\gamma) \simeq 0: X * F \to \Sigma F$.

(iii) $W_p \simeq (r \vee r) \circ W: X * F \to \Sigma X \vee \Sigma F$ *where* r *denotes parameter reversal and* W *denotes the (untwisted) generalized Whitehead product map for the spaces* X *and* F.

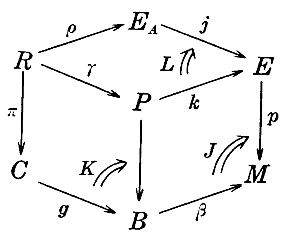
Proof. By (4.7), (i) $\Rightarrow$ (ii). Also (as is classically known or see [5]), because the base is a suspension space, $\mu: \Sigma F \to C_p$ is coclassified by $h(\gamma): X * F \to \Sigma F$. Hence (ii) $\Rightarrow$ (i). Finally (ii) $\Leftrightarrow$ (iii) follows from Corollary 4.4 of [6].

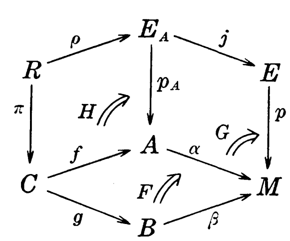
5. Proof of Theorem 1.2.

In the diagram of Theorem 1.2, let $\alpha = p\mathfrak{S}$ and let g be the composite $C \xrightarrow{e} P \to B$. By hypothesis, the square



is a homotopy pushout (for some given homotopy F). We "pullback" the map $p: E \to M$ over this homotopy pushout (cf. [8, Lemma 31]) to obtain diagrams





in which H, G, K, J are homotopy pullbacks, and homotopies $F\pi + \alpha H + G\rho$ and $\beta K + J\gamma + pL$ are track equivalent. Since F is a homotopy pushout, [8, Theorem 25] implies that L is a homotopy pushout; i.e., $\mu_L: \mathscr{M}(\gamma, \rho) \to E$ is a homotopy equivalence.

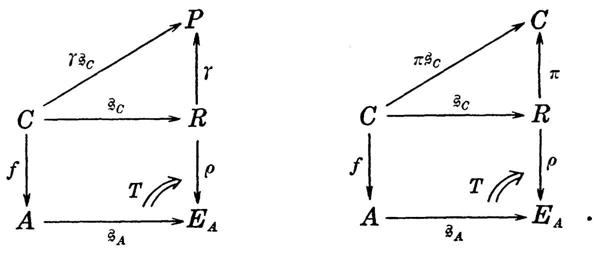
Let $\mathfrak{s}_A: A \to E_A$ and $\mathfrak{s}_C: C \to R$ be the maps induced by $\mathfrak{S}: A \to E$. In the diagram of Theorem 1.2 let k denote the map $P \to E$ and let $U: ke \simeq \mathfrak{s}f$ be the given homotopy. Then $\mathfrak{s}_A$ and $\mathfrak{s}_C$ satisfy:

$$V: e \simeq \gamma\mathfrak{s}_C\,, \quad T: \mathfrak{s}_A f \simeq \rho\mathfrak{s}_C\,, \quad W: j\mathfrak{s}_A \simeq \mathfrak{s}\,, \quad \pi\mathfrak{s}_C \simeq 1_C\,.$$

Because J is a homotopy pullback we have:

(5.1) The homotopies $kV + L\mathfrak{s}_C - jT - Wf$ and U are track equivalent.

Next we consider the following diagrams:



We apply Lemma 3.3 to each of these diagrams to get (after modifications) the following homotopy pushouts:

$$\begin{array}{ccc} \mathscr{M}(\mathfrak{s}_C, f) & \longrightarrow & \mathscr{M}(e, f) \\ {\scriptstyle \mu_{-T}}\downarrow & & \downarrow{\scriptstyle \mu_U} \\ E_A & \xrightarrow[j]{} & E \end{array} \qquad \begin{array}{ccc} \mathscr{M}(\mathfrak{s}_C, f) & \longrightarrow & A \\ {\scriptstyle \mu_{-T}}\downarrow & & \downarrow \\ E_A & \longrightarrow & E(f * p_A) \end{array} .$$

In the first square, apart from parameter reversal, we have replaced $\mathscr{M}(\rho, \gamma)$ by E and used (5.1) to identify the corresponding map $\mathscr{M}(e, f) \to E$ as μ_U. In the second square, parameter reversal is also taken into account and $\mathscr{M}(f, \pi\mathfrak{s}_C)$ has been replaced by A since $\pi\mathfrak{S}_C \simeq 1_C$. The corresponding map $A \to E(f * p_A)$ may be taken to be the composite

$$A \xrightarrow{\mathfrak{S}_A} E_A \xrightarrow{i_1} E(f * p_A) .$$

Now applying Corollary 3.4 twice it follows that $\mu_U \colon \mathscr{M}(e, f) \to E$ and $A \to E(f * p_A)$ have homotopy equivalent mapping cones. Finally, if $A \to E(f * p_A)$ is a cofibration (which is the case if $\mathfrak{s}_A \colon A \to E_A$ is a cofibration), then its mapping cone is homotopy equivalent to the quotient space $E(f * p_A)/A$. This completes the proof of Theorem 1.2.

REMARK 5.2. A topological pullback of a Hurewicz fibration is a homotopy pullback. Hence if the map $p \colon E \to M$ in Theorem 1.2 is a Hurewicz fibration then in the proof the squares containing G, H and J can be taken to be topological pullbacks with G, H and J being static homotopies. In this case $R = C \times_A E_A$ with π and ρ the corresponding projections. The map $\gamma \colon C \times_A E_A \to E_B$ $(P = E_B)$ is called a *clutching function* for p over M. If furthermore A is a one point space then $M \simeq C_g$ and $e \colon C \to E_B$ is called the *characteristic* *function* of p. Of course when B is also a one point space, so that the base space M is (homotopy equivalent to) the suspension SC, this use of characteristic function coincides with the usual one.

Suppose now that $p: E \to C_g$ is a Hurewicz fibration with characteristic function $e: C \to E_B$, as in (5.2). By Theorem 1.2 the map $C_e \to E$ has cofiber the topological join $C*F$ where $F = p^{-1}(*)$, $*$ = the vertex of C_g. Actually this result can be improved if F itself is a suspension.

THEOREM 5.3. *Let* $p: E \to C_g$ *be a Hurewicz fibration over a mapping cone with characteristic function* $e: C \to E_B$. *If* p *has fiber a suspension space* SD *then the map* $C_e \to E$ *is coclassified by a map* $D*C \to C_e$. *Hence* $E \simeq E_B \bigcup_e C(C) \cup C(D*C)$.

The result is classically known when $B = *$ (so that $C_g = SC$ is a suspension). We omit the proof of (5.3) since a more general result is proven in [5, (6.3)].

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