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MONOTONICITY OF PERMANENTS OF CERTAIN DOUBLY STOCHASTIC MATRICES

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Let $p_k(A)$, $k = 1, \cdots, n$, denote the sum of the permanents of all $k \times k$ submatrices of the $n \times n$ matrix A.

We prove that

$$(*)\quad p_k(I_n + P_n) = \frac{n}{n-k}\binom{2n-k-1}{k}, \quad k = 1, \cdots, n-1,$$

where I_n and P_n are respectively the $n \times n$ identity matrix and the $n \times n$ permutation matrix with 1's in positions $(1, 2), (2, 3), \cdots, (n-1, n), (n, 1)$. Using (*), we prove that for $n \geqq 3$ and $A = (I_n + P_n)/2$, the functions

$$p_k((1-\theta)J_n + \theta A), \quad k = 2, \cdots, n,$$

are strictly monotonic increasing in the interval $0 \leqq \theta \leqq 1$. Here J_n is the $n \times n$ matrix all whose entries are equal to $1/n$.

Let A be an $n \times n$ matrix, let $p(A)$ be the permanent of A, let $p_k(A)$, $k = 1, \cdots, n$, be the sum of the permanents of all $\binom{n}{k}^2 k \times k$ submatrices of A and define $p_0(A) = 1$. Note that $p_n(A) = p(A)$.

Denote by Ω_n the set of all $n \times n$ doubly stochastic matrices, by J_n the $n \times n$ matrix all whose entries are equal to $1/n$, by I_n the $n \times n$ identity matrix and by P_n the $n \times n$ permutation matrix with 1's in positions $(1, 2), (2, 3), \cdots, (n-1, n), (n, 1)$.

The van der Waerden conjecture asserts that if $A \in \Omega_n$, then

$$p(A) \geqq p(J_n) = \frac{n!}{n^n},$$

with equality if and only if $A = J_n$.

A stronger version of this conjecture states that the function

$$p((1-\theta)J_n + \theta A),$$

where A is any fixed matrix on the boundary of Ω_n, is strictly increasing in the interval $0 \leqq \theta \leqq 1$. In [2] the above assertion was proved for $A = I_n$ and for $A = (nJ_n - I_n)/(n-1)$. In [5, p. 158, Problem 8] the problem of finding other matrices A, for which the above assertion holds, was posed.

In the present paper we prove this assertion for $A = (I_n + P_n)/2$. We actually prove a stronger result: for $n \geqq 3$ and $A = (I_n + P_n)/2$ the functions

$$h_{A,k}(\theta) = p_k((1-\theta)J_n + \theta A)\ , \quad k = 2, \cdots, n\ ,$$

are strictly increasing in the interval $0 \leqq \theta \leqq 1$.

We start with the following lemma.

LEMMA 1. *Let* $n \geqq 3$ *and let* $A \in \Omega_n$. *If*

$$\frac{p_i(A)}{p_i(J_n)} \leqq \frac{p_{i+1}(A)}{p_{i+1}(J_n)}\ , \quad i = 1, \cdots, n-1\ , \tag{1}$$

with strict inequality for $1 \leqq i < n-1$, *then the functions*

$$h_{A,k}(\theta) = p_k((1-\theta)J_n + \theta A)\ , \quad k = 2, \cdots, n\ ,$$

are strictly increasing in the interval $0 \leqq \theta \leqq 1$.

Proof. By [4, Lemma 2],

$$h_{A,k}(\theta) = p_k(J_n)\sum_{i=0}^{k}\binom{k}{i}(1-\theta)^{k-i}\theta^i\frac{p_i(A)}{p_i(J_n)}\ .$$

Differentiating, we obtain

$$h'_{A,k}(\theta) = kp_k(J_n)\sum_{i=1}^{k-1}\binom{k-1}{i}(1-\theta)^{k-i-1}\theta^i\left(\frac{p_{i+1}(A)}{p_{i+1}(J_n)} - \frac{p_i(A)}{p_i(J_n)}\right)\ . \tag{2}$$

From (1) and (2) follows that

$$h'_{A,k}(\theta) > 0\ , \quad k = 2, \cdots, n\ ,$$

in $0 < \theta < 1$, and so the functions $h_{A,k}(\theta)$ are strictly increasing in the interval $0 \leqq \theta \leqq 1$.

Doković [1] (see also [3]) conjectured that (1) holds for all $A \in \Omega_n$. Lemma 1 shows that if the Doković conjecture holds for a certain matrix $A \in \Omega_n$, then the functions $h_{A,k}(\theta)$, $k = 2, \cdots, n$, are increasing in the interval $0 \leqq \theta \leqq 1$.

To apply Lemma 1 for a given A, $p_k(A)$, $k = 2, \cdots, n$, have to be evaluated. Although the evaluation of $p_k(A)$ is in general rather difficult, explicit formulas for $p_k(A)$ are obvious for $A = I_n$ and can be developed for $A = (I_n + P_n)/2$.

For $A = I_n$, we get

$$p_k(I_n) = \binom{n}{k}\ , \quad k = 0, \cdots, n\ .$$

Noting that

$$p_k(J_n) = \binom{n}{k}^2\frac{k!}{n^k}\ , \quad k = 0, \cdots, n\ , \tag{3}$$

(1) follows with strict inequality for $1 \leqq i \leqq n-1$. Hence, for $n \geqq 2$, $p_k((1-\theta)J_n + \theta I_n)$, $k = 2, \cdots, n$, are strictly increasing in $0 \leqq \theta \leqq 1$. For $k = n$, we get the result of Friedland and Minc [2].

To find formulas for $p_k(I_n + P_n)$, it is convenient first to bring some combinatorial results.

LEMMA 2. *Let l and m be positive integers, $m \leqq l$. The number l can be represented as a sum of m positive integers in $\binom{l-1}{m-1}$ different ways. (Two representations differing in the order of the summands are regarded different.)*

Proof. The lemma can be proved easily by induction. We prefer to use power series technique.

Consider

$$\frac{x}{1-x} = \sum_{r=1}^{\infty} x^r , \quad |x| < 1 .$$

It is obvious that the requested number of representations is equal to the coefficient of x^l in the power series of $[x/(1-x)]^m$, which is easily found to be equal to $\binom{l-1}{m-1}$.

LEMMA 3. *Let k, l and n be positive integers, $k < n$. Then*

$$\sum_{m=1}^{\min(l,n-k)} \binom{l}{m} \binom{n-k-1}{n-k-m} = \binom{n-k+l-1}{n-k} , \tag{4}$$

$$\sum_{m=0}^{k} \binom{n-m-1}{n-k-1} \binom{n-k+m-1}{n-k-1} = \binom{2n-k-1}{k} . \tag{5}$$

Proof. We use again power series.

To prove (4), we consider

$$(1+x)^l = \sum_{r=0}^{l} \binom{l}{r} x^r ,$$

$$(1+x)^{n-k-1} = \sum_{r=0}^{l} \binom{n-k-1}{r} x^r .$$

The sum in the lefthand side of (4) is equal to the coefficient of x^{n-k} in the power series of $(1+x)^{n-k+l-1}$, which is $\binom{n-k+l-1}{n-k}$.

To prove (5), we consider

$$\frac{x^{n-k-1}}{(1-x)^{n-k}} = \sum_{r=n-k-1}^{\infty} \binom{r}{n-k-1} x^r , \quad |x| < 1 .$$

The sum in the lefthand side of (5) is equal to the coefficient of x^{2n-k-2} in the power series of $[x^{n-k-1}/(1-x)^{n-k}]^2$, which is $\binom{2n-k-1}{k}$. The proof of the lemma is completed.

Let n and l be positive integers, $l \leqq n$. Let $(n_1, \cdots, n_l)$, $1 \leqq n_1 < n_2 < \cdots < n_l \leqq n$, be a l-combination of $1, \cdots, n$. Let m be the number of r's, $r = 1, \cdots, l$, for which $n_{r+1} \neq n_r + 1$, where n_{l+1} is taken as n_1 and $n+1$ as 1. We say that *the l-combination $(n_1, \cdots, n_l)$ has m gaps*. Obviously, $m \leqq l$ and $m + l \leqq n$; i.e., $0 \leqq m \leqq \min(l, n-l)$.

Take $l < n$ and arrange $1, \cdots, n$ in increasing order (clockwise) in a circle. Then the set $(n_1, \cdots, n_l)$ and its complement have the same number of (connected) components. This number is the number m defined above as the number of gaps of $(n_1, \cdots, n_l)$.

For example, if $n = 6$ and $l = 3$, then the number of gaps of (1, 2, 3) and (1, 2, 6) is 1, of (1, 3, 4) is 2 and of (1, 3, 5) is 3. If $n = l$, then $m = 0$.

We denote by $\binom{n}{l,\, m}$ the number of l-combinations of $1, \cdots, n$ having m gaps. $\binom{n}{l,\, m}$ is thus defined for all nonnegative integers l, m, n satisfying $0 < l \leqq n$, $0 \leqq m \leqq \min(l, n-l)$. We also define $\binom{n}{0,\, 0} = 1$. From the definition of $\binom{n}{l,\, m}$ follows that

$$\sum_{m=0}^{\min(l,n-l)} \binom{n}{l,\, m} = \binom{n}{l} .$$

In the following lemma we obtain a formula for $\binom{n}{l,\, m}$.

LEMMA 4. *Let l, m, n be positive integers satisfying $0 < l \leqq n-1$, $0 < m \leqq \min(l, n-l)$. Then*

$$\binom{n}{l,\, m} = \frac{n}{m}\binom{l-1}{m-1}\binom{n-l-1}{m-1} . \tag{6}$$

Proof. $\binom{n}{l,\, m}$ is equal to the number of l-combinations of $1, \cdots, n$ with m gaps. We first find the number of l-combinations of the form $(1, n_2, \cdots, n_l)$ with m gaps.

Arrange the numbers $1, \cdots, n$ in a circle and take a l-combination $(1, n_2, \cdots, n_l)$ with m gaps. As $l < n$, the set $(1, n_2, \cdots, n_l)$ and its complement have each m components. Let m_i and m_i', $i = 1, \cdots, m$, be the number of elements in the ith component of $(1, n_2, \cdots, n_l)$ and its complement respectively. We have

$$(7) \qquad \begin{cases} \sum_{i=1}^{m} m_i = l \,, \\ \sum_{i=1}^{m} m_i' = n - l \,. \end{cases}$$

It is obvious that there is a $1-1$ correspondence between the l-combinations of the form $(1, n_2, \cdots, n_l)$ with m gaps and the $2m$-tuples $(m_1, m_1', \cdots, m_m, m_m')$ of positive integers satisfying (7). By Lemma 2, the number of these $2m$-triples is $\binom{l-1}{m-1}\binom{n-l-1}{m-1}$. Hence, the number of l-combinations of the form $(1, n_2, \cdots, n_l)$ with m gaps is $\binom{l-1}{m-1}\binom{n-l-1}{m-1}$.

For each of the numbers $1, \cdots, n$ we get $\binom{l-1}{m-1}\binom{n-l-1}{m-1}$ l-combinations with m gaps. Assembling all these combinations, each combination with l gaps is repeated m times. Hence, to get the number of these combinations, $\binom{n-1}{m-1}\binom{n-l-1}{m-1}$ has to multiplied by n and divided by m. Formula (6) is thus proved.

In the following lemma we obtain formulas for $p_k(I_n + P_n)$, $k = 0, \cdots, n$.

LEMMA 5. *Let $n \geqq 2$. Then*

$$(8) \qquad p_k(I_n + P_n) = \begin{cases} \dfrac{n}{n-k}\dbinom{2n-k-1}{k} \,, & k = 0, \cdots, n-1 \,, \\ 2 \,, & k = n \,. \end{cases}$$

Proof. Formula (8) is easily verified for $k = 0$ and $k = n$.

Let $1 \leqq k \leqq n - 1$. $p_k(I_n + P_n)$ is equal to the number of different diagonals of 1's of length k in $I_n + P_n$. (Where diagonals of length k in the $n \times n$ matrix $I_n + P_n$ are defined in the obvious way.) Each such diagonal is composed of l elements of I_n and $k - l$ elements of P_n.

Let $(n_1, \cdots, n_l)$ be a l-combination of $1, \cdots, n$ with m gaps. The number of 1's in P_n belonging either to the rows $n_1, \cdots, n_l$ or to the columns $n_1, \cdots, n_l$ is $l + m$. Hence, the diagonal of length l consisting of 1's in positions $(n_1, n_1), (n_2, n_2), \cdots, (n_l, n_l)$ can be augmented, using elements of P_n, to $\binom{n-l-m}{k-l}$ different diagonals of 1's of length k. As there are $\binom{n}{l, m}$ l-combinations with m gaps, the number of diagonal of length k which originate in a l-combination with m gaps is $\binom{n}{l, m}\binom{n-l-m}{k-l}$. Summing up over all possible m and l, we obtain

$$p_k(A) = \sum_{l=0}^{k} \sum_{m=0}^{\min(l,n-k)} \binom{n}{l,\, m} \binom{n-l-m}{k-l}.$$

Noting that $\binom{n}{0,\, 0} = 1$ and, as $k < n$, $\binom{n}{l,\, 0} = 0$ for $l = 1, \cdots, k$, it follows that

$$p_k(A) = \binom{n}{k} + \sum_{l=1}^{k} \sum_{m=1}^{\min(l,n-k)} \binom{n}{l,\, m} \binom{n-l-m}{k-l}.$$

Using now Lemma 4, we obtain

$$p_k(A) = \binom{n}{k} + \sum_{l=1}^{k} \sum_{m=1}^{\min(l,n-k)} \frac{n}{m} \binom{l-1}{m-1} \binom{n-l-1}{m-1} \binom{n-l-m}{k-l}.$$

As

$$\frac{n}{m} \binom{l-1}{m-1} \binom{n-l-1}{m-1} \binom{n-l-m}{k-l} = \frac{n}{l} \binom{n-l-1}{n-k-1} \binom{l}{m} \binom{n-k-1}{n-k-m},$$

it follows that

$$p_k(A) = \binom{n}{k} + n \sum_{l=1}^{k} \frac{1}{l} \binom{n-l-1}{n-k-1} \sum_{m=1}^{\min(l,n-k)} \binom{l}{m} \binom{n-k-1}{n-k-m},$$

and using (4), we obtain

$$p_k(A) = \binom{n}{k} + n \sum_{l=1}^{k} \frac{1}{l} \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k}.$$

But

$$\frac{1}{l} \binom{n-k+l-1}{n-k} = \frac{1}{n-k} \binom{n-k+l-1}{n-k-1}.$$

So

$$p_k(A) = \binom{n}{k} + \frac{n}{n-k} \sum_{l=1}^{k} \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k-1} \tag{9}$$
$$= \frac{n}{n-k} \sum_{l=0}^{k} \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k-1}.$$

Formula (8) follows from (5) and (9).

We bring now our main result.

THEOREM. *Let $n \geqq 3$ and let $A = (I_n + P_n)/2$. Then the functions*

$$h_{A,k}(\theta) = p_k((1-\theta)J_n + \theta A)\ , \quad k = 2, \cdots, n\ ,$$

are strictly increasing in the interval $0 \leqq \theta \leqq 1$.

Proof. By Lemma 1, it is sufficient to show that

$$\frac{2p_i(I_n + P_n)}{p_i(J_n)} \leqq \frac{p_{i+1}(I_n + P_n)}{p_{i+1}(J_n)}\ , \quad i = 1, \cdots, n-1\ , \tag{10}$$

with strict inequality for $1 \leqq i < n-1$.

For $i = n-1$, (10) holds with equality sign.

For $i = 1, \cdots, n-2$, (3) and (8) imply

$$\frac{p_{i+1}(I_n + P_n)}{p_{i+1}(J_n)} = \frac{(i+1)!\,[(n-i-1)!]^2 n^{i+2}}{(n-i-1)(n!)^2}\binom{2n-i-2}{i+1}\ . \tag{11}$$

From (11) follows

$$\frac{p_{i+1}(I_n + P_n)}{p_{i+1}(J_n)} - \frac{2p_i(I_n + P_n)}{p_i(J_n)} = \frac{2in^{i+1}(2n-i-2)!\,[(n-i-1)!]^2(n-i-1)}{(n!)^2(2n-2i-1)!}\ .$$

Hence (10) holds with strict inequality for $1 \leqq i < n-1$, and the proof of our theorem is completed.

We note that the theorem holds also for all $n \times n$ matrices A which can be obtained from $(I_n + P_n)/2$ by permutations of rows and columns.

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