

ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AS MULTIPLIERS FOR WEIGHT FUNCTIONS

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If $W(x) \geq 1$ is defined on the real line and satisfies (1), a discussion is given of the regularity assumptions which must be imposed on W in order to guarantee the existence of nonzero entire functions φ of arbitrarily small exponential type making $W(x)\varphi(x)$ bounded on the real axis. It is known that such φ exist provided that $\log W(x)$ is uniformly Lip 1. An example is given which shows, among other things, that this is no longer the case if we merely suppose that $\log \log W(x)$ is uniformly Lip 1.

1. Introduction. Suppose we are given a function $W(x) \geq 1$ (henceforth called a *weight*), defined for $-\infty < x < \infty$ and satisfying

$$(1) \quad \int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx < \infty.$$

In connection with various problems, it is important to know when there exist nonzero entire functions $\varphi(z)$ of exponential type for which $W(x)\varphi(x)$ is bounded, or belongs at least to some L_p class, on the real axis. Some of these problems involve the completeness of sets of exponentials on finite intervals ([12], p. 50ff, [1]) or other aspects of approximation theory ([4], pp. 254-288). Let us also mention the question of nonuniqueness for the best H_∞ -approximations to certain functions of unit modulus related to W ([5], § 4), and a closely related one concerning the existence of nonzero positive functions $\omega(x)$ such that

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^2 \omega(x) dx \leq \int_{-\infty}^{\infty} \frac{|f(x)|^2}{W(x)} dx$$

for the Hilbert transform \tilde{f} of each bounded function f having its spectrum in $(-\infty, -a] \cup [a, \infty)$, $a > 0$ ([5], [6]).

If, besides (1), $W(x)$ satisfies certain regularity conditions, there are nonzero entire functions φ of arbitrarily small exponential type making $W(x)\varphi(x)$ bounded for real x . It has long been known, for instance, that this is the case if $W(x)$ is *even and increasing* for $x > 0$ (see, inter alia, [10], p. 81, [7], § 5); the *theorem on the multiplier*, due to Beurling and Malliavin, furnishes the same result

whenever W , satisfying (1), is *itself* an entire function of exponential type, or has a *logarithm uniformly continuous* on the real axis ([2], [8], [11]). The mere relation (1) is certainly not sufficient by itself for the existence of such φ , and it seems very difficult to get a simple description of minimal regularity conditions whose imposition on $W(x)$ would ensure that existence.

For this reason the following question seems natural:

If there is some nonzero entire function Φ of exponential type, with (for instance)

$$(2) \quad \int_{-\infty}^{\infty} |\Phi(x)| W(x) dx < \infty,$$

are there nonzero entire functions φ of arbitrarily small exponential type such that

$$(3) \quad \int_{-\infty}^{\infty} |\varphi(x)| W(x) dx < \infty?$$

The simplest examples show that even here $W(x)$ must have some regularity in order for there to be a positive answer. Take, for instance,

$$W(x) = \left(\frac{\pi x}{\sin \pi x} \right)^2,$$

then (2) holds with

$$\Phi(x) = \left(\frac{\sin \pi x \sin \eta x}{x^2} \right)^2,$$

an entire function of exponential type $2\pi + 2\eta$. Any entire function φ satisfying (3) must, however, have a double zero at least at each of the points $\pm 1, \pm 2, \pm 3, \dots$, and this, by Carlson's theorem ([3], p. 153), cannot happen for a nonzero φ of exponential type $< 2\pi$. This example, which works because $W(n) = \infty$ at the nonzero integers n , seems to have an artificial character; it would be excluded from our discussion by requiring W to satisfy an inequality like

$$W(x') \geq (W(x))^c$$

whenever $|x' - x| \leq l$, with two positive constants c and l . This is a kind of uniform semi-continuity, much weaker than the uniform continuity of $\log W(x)$ which, as noted above, is already enough to ensure existence for nonzero φ of arbitrarily small exponential type satisfying (3).

Let us agree to *henceforth only consider* weights $W(x)$ having at least the following regularity:

There are two positive constants c and l such that, for each x_0 , we have

$$(4) \quad W(x) \geq (W(x_0))^c$$

throughout some interval of length l containing x_0 .

This condition, milder than the one mentioned previously, holds whenever, starting from any function $\Omega(x) \geq 1$ satisfying

$$\int_{-\infty}^{\infty} \frac{\log \Omega(x)}{1+x^2} dx < \infty,$$

we form the geometrically smoothed weight

$$(5) \quad W(x) = \exp \left\{ \frac{1}{2l} \int_{-l}^l \log \Omega(x+t) dt \right\}.$$

Indeed, for any x , at least one of the two relations

$$\begin{aligned} \frac{1}{2l} \int_{-l}^0 \log \Omega(x+t) dt &\geq \frac{1}{2} \log W(x), \\ \frac{1}{2l} \int_0^l \log \Omega(x+t) dt &\geq \frac{1}{2} \log W(x) \end{aligned}$$

must hold. So, since $\log \Omega \geq 0$, we have $W(x') \geq \sqrt{W(x)}$ on at least one of the two intervals $x-l \leq x' \leq x$, $x \leq x' \leq x+l$, giving us (4) with $c = 1/2$.

If condition (4) on W is fulfilled and there exists any nonzero Φ of exponential type satisfying (2), we can find another entire Ψ of exponential type (perhaps larger than that of Φ), having only real zeros, and such that

$$(6) \quad \int_{-\infty}^{\infty} |\Psi(x+i)| W(x) dx < \infty.$$

To see this, start with the Hadamard product representation

$$(7) \quad \Phi(z) = Az^p e^{iaz} \prod_{\lambda} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda};$$

here, λ runs through the zeros Φ has away from the origin, and p is a nonnegative integer. Since $W \geq 1$, (2) implies $\int_{-\infty}^{\infty} |\Phi(x)| dx < \infty$, so $|\Phi(x)|$ is bounded for real x ([3], p. 98). This makes the constant α in (7) real. In the right-hand side of (7), let us now throw away all factors corresponding to purely imaginary zeros λ , and replace each remaining λ by a real one, $\tilde{\lambda}$, related to it through the formula

$$1/\tilde{\lambda} = \Re(1/\lambda).$$

This procedure gives us a new entire function $\Phi_0(z)$ of exponential type which is easily seen to satisfy the inequality $|\Phi_0(x)| \leq |\Phi(x)|$ for real x , and hence also

$$\int_{-\infty}^{\infty} |\Phi_0(x)| W(x) dx < \infty ,$$

(According to [10], p. 30, the type of Φ_0 is less than or equal to that of Φ ; here, the boundedness of Φ on the real axis makes the two types equal. See [12], pp. 55-56 and [3], p. 115.)

Calling A the type of Φ_0 in the upper half plane, we have, by Poisson's formula,

$$(8) \quad \log |\Phi_0(x+i)| = A + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\Phi_0(t)| dt}{(x-t)^2 + 1} ,$$

because Φ_0 has only real zeros ([3], p. 92). Jensen's inequality, combined with (8), now yields

$$\begin{aligned} \log |\Phi_0(x+i)| + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log W(x+t)}{t^2 + 1} dt \\ \leq A + \log \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\Phi_0(t)| W(t) dt}{(x-t)^2 + 1} \right\} \\ \leq A + \log \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} |\Phi_0(t)| W(t) dt \right\} , \end{aligned}$$

so

$$|\Phi_0(x+i)| \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log W(x+t)}{t^2 + 1} dt \right\}$$

is bounded for $-\infty < x < \infty$.

Condition (4), however, yields

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log W(x+t)}{t^2 + 1} dt \geq \frac{cl \log W(x)}{\pi(l^2 + 1)} ,$$

so $|\Phi_0(x+i)|(W(x))^k \leq K$ on the real axis with two positive constants k and K . Since $W(x) \geq 1$, we see that $|\Phi_0(x+i)|^N W(x)$ is bounded for real x if N is any integer $> 1/k$, and the formula

$$\Psi(z) = \frac{\sin^2 z}{z^2} (\Phi_0(z))^N$$

now provides us with a nonzero entire function Ψ of exponential type having only real zeros and satisfying (6).

Suppose that we merely have *some* function $\Omega(x) \geq 1$ and a nonzero Φ of exponential type such that

$$\int_{-\infty}^{\infty} |\Phi(x)| \Omega(x) dx < \infty .$$

An argument like the preceding one can then be carried through with Ω instead of W , up to the point where an inequality

$$|\Phi_0(x + i)| \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log \Omega(x + t)}{t^2 + 1} dt \right\} < K, \quad -\infty < x < \infty,$$

is obtained. However, if W is the smoothed weight obtained from Ω by (5), we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log \Omega(x + t)}{t^2 + 1} dt \geq \frac{2l \log W(x)}{\pi(l^2 + 1)},$$

and we see that (6) *still* holds with a suitable entire Ψ having only real zeros.

On account of (6), the existence of entire φ of arbitrarily small exponential type satisfying (3) is seen to boil down to existence of such φ for which

$$(9) \quad \frac{|\varphi(x)|}{|\Psi(x + i)|} \leq \text{const.}, \quad -\infty < x < \infty.$$

Taken generally, this statement has a *converse*:

If, for every weight W fulfilling (4), the existence of one entire Φ of exponential type satisfying (2) implies that of entire φ with arbitrarily small type satisfying (3), then, for every entire Ψ bounded on the real axis and having all its zeros there, we do have entire φ of arbitrarily small type satisfying (9).

To see this, let $\Psi(z)$, having only real zeros, be of exponential type A in $\Im z > 0$ and satisfy $|\Psi(x)| \leq 1, -\infty < x < \infty$. A Phragmén-Lindelöf theorem ([3], p. 82) makes $|\Psi(x + i/2)| \leq e^{A/2}$, and we can take the weight

$$W(x) = \frac{e^{A/2}}{|\Psi(x + i/2)|}.$$

A relation similar to (8),

$$\log |\Psi(x + i/2)| = \frac{1}{2} A + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\log |\Psi(t)| dt}{4(x - t)^2 + 1},$$

yields, on differentiation,

$$\left| \frac{d \log |\Psi(x + i/2)|}{dx} \right| \leq 2 \log \left| \frac{e^{A/2}}{\Psi(x + i/2)} \right|,$$

because $\log |\Psi(t)| \leq 0$. Our W thus satisfies (4) with, say, $l = 1$ and $c = 1/e^2$.

There is certainly a nonzero entire Φ of exponential type satisfying (2), for instance, $\Phi(z) = (\sin^2 z/z^2)\Psi(z + i/2)$. Suppose now that

φ is any entire function, of exponential type equal, say, to a . Then ([3], p. 93),

$$\log |\varphi(x + i/2)| \leq \frac{a}{2} + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\log |\varphi(t)| dt}{4(x-t)^2 + 1},$$

so, since

$$\log |\Psi(x + i)| = \frac{A}{2} + \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\log |\Psi(t + i/2)|}{4(x-t)^2 + 1} dt,$$

we have, by Jensen's inequality,

$$\log \left| \frac{\varphi(x + i/2)}{\Psi(x + i)} \right| \leq \frac{a}{2} - A + \log \frac{2}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{|\varphi(t)| W(t)}{4(x-t)^2 + 1} dt \right\},$$

in terms of our weight W . If, now, φ satisfies (3), we see that (9) holds with $\varphi(x + i/2)$ standing in place of $\varphi(x)$.

Truth of the

AFFIRMATION. For every nonzero Ψ of exponential type bounded on the real axis and having all its zeros there, there exist nonzero entire φ of arbitrarily small exponential type such that $\varphi(x)/\Psi(x + i)$ is bounded for real x

is thus both necessary and sufficient for the validity of the following

PROPOSITION. For weights W having the regularity property (4), existence, for any particular one of them, of a nonzero entire Φ of exponential type satisfying (2) implies that of nonzero entire φ with arbitrarily small type satisfying (3).

This makes proof or disproof of the affirmation rather important, in spite of the special nature of the factor $1/\Psi(x + i)$ appearing in its formulation. In the following section an example is constructed which shows the affirmation to be *false*. Therefore the proposition is also false.

2. **The example.** Starting with some large value of p , say $p = 8$, let us put

$$(10) \quad x_p = \exp p^{1/3}.$$

Take $\Delta_8 = x_8$, and, for $p > 8$, write

$$(11) \quad \Delta_p = x_p - x_{p-1}.$$

We form the even function

$$F(z) = \prod_{p \geq 8} \left(1 - \frac{z^2}{x_p^2}\right)^{[\Delta_p]}$$

($[\Delta_p]$ means the greatest integer $\leq \Delta_p$); convergence of the infinite product on the right is easily verified, and $F(z)$ is an entire function having a zero of multiplicity $[\Delta_p]$ at each point $\pm x_p$, $p \geq 8$.

For $t > 0$, denote by $n(t)$ the number of zeros of $F(z)$ (counting multiplicities) lying between 0 and t . If $p > \infty$ and t lies between x_p and x_{p+1} ,

$$n(t) = [\Delta_8] + \dots + [\Delta_p]$$

lies between $x_p - p$ and x_p , where, by (10), $p = (\log x_p)^3$. Since $x_{p+1}/x_p \rightarrow 1$, we clearly have

$$\frac{n(t)}{t} \longrightarrow 1, \quad t \longrightarrow \infty,$$

and this implies, as is well known ([3], p. 136), that $F(z)$ is of exponential type π .

Given any $\eta > 0$ (which we now choose and fix once and for all), let us show that there is a nonzero even entire function $g(z)$ of exponential type $\leq \eta$ having only real zeros with $F(z)g(z)$ bounded on the real axis. This we can do by appealing to the old result (about even weights $W(x)$ increasing on the positive real axis) mentioned near the beginning of the introduction. For $x_p \leq t < x_{p+1}$, we have $t \geq x_p \geq n(t) \geq x_p - p \geq t - p - \Delta_{p+1}$. By (10) and (11), for large p ,

$$(12) \quad \Delta_p \sim \Delta_{p+1} \sim \frac{1}{3} p^{-2/3} x_p = \frac{1}{3} \frac{x_p}{(\log x_p)^2},$$

so, if $x_p \leq t < x_{p+1}$,

$$t - \left(\frac{1}{3} + o(1)\right) \frac{x_p}{(\log x_p)^2} - (\log x_p)^3 \leq n(t) \leq t,$$

where the term $o(1)$ tends to zero as $p \rightarrow \infty$. We can thus surely write

$$(13) \quad t - \frac{t}{(\log t)^2} \leq n(t) \leq t$$

for all sufficiently large values of t ($n(t) \leq t$ for all $t > 0$).

The above definition of the function F can be rewritten as $\log |F(x)| = \int_0^\infty \log |1 - x^2/t^2| dn(t)$; after integrating by parts and chang-

ing variables ([7], pp. 127-128), this becomes, for $x > 0$,

$$\log |F(x)| = 2 \int_0^1 \frac{\frac{n(x\tau)}{\tau} - \tau n\left(\frac{x}{\tau}\right)}{1 - \tau^2} d\tau .$$

Since $n(t)$ is increasing, the integral on the right is

$$\leq 2n(x) \log \frac{1}{\lambda} + 2 \int_0^\lambda \frac{\frac{n(x\tau)}{\tau} - \tau n\left(\frac{x}{\tau}\right)}{1 - \tau^2} d\tau$$

for any positive number λ less than 1. Substituting (13) into this expression, we find

$$\log |F(x)| \leq 2x \log \frac{1}{\lambda} + 2x \int_0^\lambda \frac{d\tau}{[\log(x/\tau)]^2 (1 - \tau^2)}$$

and, taking $\lambda = 1 - 1/(\log x)^2$, we get

$$(14) \quad \log |F(x)| \leq \frac{Cx \log \log x}{(\log x)^2}$$

for large x , C being a constant. (The same result may also be obtained directly by just plugging $H(x) = x/(\log x)^2$ into Theorem 51, p. 35 of [12].)

The right hand member of (14) is *increasing* for large x , and satisfies

$$\int_e^\infty \frac{Cx \log \log x}{(\log x)^2} \frac{dx}{x^2} < \infty .$$

An old theorem (see [10], p. 81, [7], § 5, or [12], pp. 26-27—it is not necessary to resort to the Beurling-Malliavin multiplier theorem here) now furnishes us with a nonzero even entire function $g(z)$ having only real zeros and of exponential type $\leq \eta$, such that $F(x)g(x)$ is bounded for real x ; putting

$$\Psi(z) = F(z)g(z) ,$$

we have a nonzero even entire function Ψ of exponential type $\leq \pi + \eta$, *bounded on the real axis and having only real zeros, among which are the $[A_p]$ -fold ones at the points $\pm x_p$.*

I say now that there can be no nonzero entire function φ of exponential type $a < \pi$ for which $|\varphi(x)/\Psi(x+i)|$ is bounded on the real axis. Suppose there were. The argument used to pass from Φ to Φ_0 in § 1 shows that, if such a φ existed, there would be a nonzero entire f , also of exponential type a , but having only real

zeros, such that

$$(15) \quad \left| \frac{f(x)}{\Psi(x+i)} \right| \leq 1,$$

say, for $-\infty < x < \infty$.

This, however, is impossible. We may, by a Phragmén-Lindelöf theorem ([3], p. 82), assume without loss of generality that $|\Psi(x+2i)| \leq 1$ for real x ; then, by that same theorem,

$$(16) \quad |\Psi(z+2i)| \leq \exp\{(\pi + \eta)|\Im z|\}.$$

Denote by $\mathfrak{N}(\rho, z_0)$ the number of zeros of Ψ in the circle $|z - z_0| \leq \rho$; Jensen's formula says that

$$\log |\Psi(x_p + 2i)| = \frac{1}{2\pi} \int_0^{2\pi} \log |\Psi(x_p + 2i + re^{i\theta})| d\theta - \int_0^r \frac{\mathfrak{N}(\rho, x_p + 2i)}{\rho} d\rho.$$

Because $\Psi(z)$ has a zero of order $[A_p]$ at x_p , substitution of (16) into the last relation yields

$$(17) \quad \log |\Psi(x_p + 2i)| \leq \left(2 + \frac{2\eta}{\pi}\right)r - [A_p] \log \frac{r}{2}, \quad r \geq 2.$$

If f , of exponential type $a < \pi$, satisfies (15), then, since $\Psi(z)$ has only real zeros, the ratio $f(z)/\Psi(z+i)$ is of negative exponential type in the half plane $\Im z > 0$. A Phragmén-Lindelöf theorem ([3], p. 82) now implies that

$$(18) \quad |f(x_p + i)| \leq |\Psi(x_p + 2i)|.$$

Calling $N(\rho, z_0)$ the number of zeros of $f(z)$ in the circle $|z - z_0| \leq \rho$, Jensen's formula reads

$$(19) \quad \log |f(x_p + i)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(x_p + i + re^{i\theta})| d\theta - \int_0^r \frac{N(\rho, x_p + i)}{\rho} d\rho.$$

Because the zeros of f are real,

$$\int_0^r \frac{N(\rho, x_p + i)}{\rho} d\rho \leq N(r, x_p) \log r, \quad r \geq 1.$$

We can represent $f(z)$ as a Hadamard product like the one on the right in (7), putting, without loss of generality, $\alpha = 0$. From such a representation, it is easy to see that

$$|f(z)| \geq |f(\Re z)|.$$

Using this relation and the previous one in (19), one finds

$$\log |f(x_p + i)| \geq \frac{1}{\pi} \int_{-r}^r \frac{\log |f(x_p + s)|}{\sqrt{r^2 - s^2}} ds - N(r, x_p) \log r, \quad r \geq 1,$$

which, combined with (18) and (17), yields

$$(20) \quad \frac{1}{\pi} \int_{-r}^r \frac{\log |f(x_p + s)|}{\sqrt{r^2 - s^2}} ds \leq \left(2 + \frac{2\eta}{\pi}\right)r + [\Delta_p] \log 2 \\ + N(r, x_p) \log r - [\Delta_p] \log r, \quad r \geq 2.$$

This inequality will force f to be identically zero. Fix a parameter $\lambda < 1$, and, with $R_p = \Delta_p/2$, integrate the r in (20) over the range $[\lambda R_p, R_p]$, noting that, for large p , λR_p is > 2 by (12). After this integration, the left side of (20) becomes

$$\int_{-R_p}^{R_p} \log |f(x_p + s)| J(s/R_p) ds,$$

where

$$J(s/R_p) = \frac{1}{\pi} \int_{\max(|s|, \lambda R_p)}^{R_p} (r^2 - s^2)^{-1/2} dr \\ = \frac{1}{\pi} \arg \cosh(R_p/|s|) - \frac{1}{\pi} \arg \cosh \max(\lambda R_p/|s|, 1)$$

is bounded above by a positive constant C_λ independent of R_p for $0 < |s| \leq R_p$. (We may take $C_\lambda = (1/\pi) \arg \cosh(1/\lambda)$.) According to (15), there is no loss of generality in assuming that $|f(x)| \leq 1$, $-\infty < x < \infty$, so, if p is large, our integration of (20) yields

$$(21) \quad C_\lambda \int_{-R_p}^{R_p} \log |f(x_p + s)| ds \leq (1 - \lambda^2) \left(\frac{1}{4} + \frac{\eta}{4\pi} \right) \Delta_p^2 \\ + \frac{1 - \lambda}{2} \Delta_p [\Delta_p] \log 2 \\ + \frac{1 - \lambda}{2} \left[N\left(\frac{\Delta_p}{2}, x_p\right) \log\left(\frac{\Delta_p}{2}\right) - [\Delta_p] \log\left(\frac{\lambda \Delta_p}{2}\right) \right] \Delta_p.$$

One verifies that $\Delta_{p+1} > \Delta_p$, hence the intervals $[x_p - R_p, x_p + R_p]$ do not overlap. Since $R_p = o(x_p)$, it will clearly be enough to show, with the help of (21), that

$$\sum_p \frac{1}{x_p^2} \int_{-R_p}^{R_p} \log |f(x_p + s)| ds = -\infty,$$

for this will imply that

$$\int_{-\infty}^{\infty} \frac{\log |f(t)|}{t^2 + 1} dt = -\infty,$$

making $f \equiv 0$ by a well known theorem ([3], p. 85).

To this end, observe that the *right side* of (21) is, for large p , bounded above by

$$A\Delta_p^2 + B\left\{N\left(\frac{\Delta_p}{2}, x_p\right) - [\Delta_p]\right\}\Delta_p \log \Delta_p$$

with two positive constants A and B , and that

$$\sum_8^\infty \Delta_p^2/x_p^2 < \infty$$

by (12). We therefore need only establish

$$(22) \quad \sum_p \frac{1}{x_p^2} \left\{N\left(\frac{\Delta_p}{2}, x_p\right) - [\Delta_p]\right\} \Delta_p \log \Delta_p = -\infty .$$

Here, it is convenient to denote the number of zeros of $f(z)$ on the interval $[0, t]$ by $\nu(t)$, thus,

$$N\left(\frac{\Delta_p}{2}, x_p\right) = \nu\left(x_p + \frac{1}{2}\Delta_p\right) - \nu\left(x_p - \frac{1}{2}\Delta_p\right) .$$

Take a fixed parameter $\gamma > 1$ and *close* to 1 (the exact manner of choosing γ will be described later on), and put

$$(23) \quad X_m = \gamma^m, \quad m = 1, 2, 3, \dots .$$

We think of $\{X_m\}$ as a coarse sequence of points moving out towards ∞ along the real axis, and of $\{x_p\}$ as a finely distributed sequence of points interspersed among the X_m . Picking a very large m , let us estimate the sum

$$(24) \quad \sum_{x_m < x_p \leq x_{m+1}} \left\{ \nu\left(x_p + \frac{1}{2}\Delta_p\right) - \nu\left(x_p - \frac{1}{2}\Delta_p\right) - [\Delta_p] \right\} \frac{\Delta_p \log \Delta_p}{x_p^2}$$

from above.

Since $f(z)$ is of exponential type a , is bounded on the real axis, and has only real zeros, Levinson's density theorem ([10], pp. 33-41; [3], pp. 136-138; [9]) implies that

$$\frac{\nu(t)}{t} \longrightarrow \frac{a}{\pi} \quad \text{as } t \longrightarrow \infty ,$$

therefore, if $X' \geq \gamma X$ and X is large enough, we have

$$\nu(X') - \nu(X) \leq \left(\frac{a}{\pi} + \varepsilon\right)(X' - X)$$

with an arbitrary $\varepsilon > 0$. For large enough m , this yields, on account of (23),

$$(25) \quad \sum_{X_m < x_p \leq X_{m+1}} \left\{ \nu \left(x_p + \frac{1}{2} \Delta_p \right) - \nu \left(x_p - \frac{1}{2} \Delta_p \right) \right\} \\ \leq \left(\frac{a}{\pi} + 2\varepsilon \right) (X_{m+1} - X_m),$$

because the intervals $[x_p - \Delta_p/2, x_p + \Delta_p/2]$ don't overlap and $\Delta_p = o(x_p)$ by (12). According to (10) and (23),

$$(m \log \gamma)^3 < p \leq ((m+1) \log \gamma)^3 \quad \text{for } X_m < x_p \leq X_{m+1},$$

so (12) gives, with an arbitrary $\varepsilon > 0$,

$$\frac{\Delta_p \log \Delta_p}{x_p^2} \leq \frac{(1+\varepsilon)p^{-1/3}}{3X_m} \leq \frac{(1+\varepsilon)}{3mX_m \log \gamma}$$

if $X_m < x_p \leq X_{m+1}$ and m is large. Using this and (25), we find, by (23),

$$(26) \quad \sum_{X_m < x_p \leq X_{m+1}} \left\{ \nu \left(x_p + \frac{1}{2} \Delta_p \right) - \nu \left(x_p - \frac{1}{2} \Delta_p \right) \right\} \frac{\Delta_p \log \Delta_p}{x_p^2} \\ \leq \frac{1}{3} \left(\frac{a}{\pi} + o(1) \right) \frac{\gamma - 1}{\log \gamma} \cdot \frac{1}{m},$$

where the $o(1)$ term tends to zero as $m \rightarrow \infty$.

When m is large we obtain, on the other hand, from (23) and (12),

$$(27) \quad \sum_{X_m < x_p \leq X_{m+1}} [\Delta_p] \Delta_p \log \Delta_p / x_p^2 \\ \geq (1 - o(1)) \sum_{(m \log \gamma)^3 < p \leq ((m+1) \log \gamma)^3} 1/9p = (1 - o(1)) \cdot \frac{1}{9} \log \left(\frac{m+1}{m} \right)^3 \\ = \frac{1}{3} (1 - o(1)) \cdot \frac{1}{m},$$

where, again, $o(1) \rightarrow 0$ as $m \rightarrow \infty$.

Substituting (26) and (27) into (24) we find that that expression is

$$\leq \frac{1}{3} \left[\frac{a}{\pi} \frac{\gamma - 1}{\log \gamma} + o(1) - 1 \right] \cdot \frac{1}{m}$$

if m is large. The type, a , of f , was, however, supposed $< \pi$. We can therefore fix the parameter γ used to define tht X_m so close to 1 that

$$\frac{a}{\pi} \frac{\gamma - 1}{\log \gamma} < 1$$

and then, for all sufficiently large m , the sum in (24) will be

$$\leq -\frac{c}{m}$$

with a positive constant c .

But this implies (22). Therefore, as was already stated, $f(z) \equiv 0$. Our assertion is proved.

3. Discussion. So, there we are. We have a multiplier theorem for weights $W(x)$ equal to entire functions of exponential type, but *not* for weights equal to the *reciprocals* of such functions, even though the functions' zeros stay away from the real axis.

One's first thoughts are that the methods used to prove Beurling and Malliavin's multiplier theorem, applying, as they do, to weights W equal to functions of exponential type and satisfying (1), should also apply to weights $W(x)$ of the form $1/|\Psi(x+i)|$ with functions Ψ of exponential type bounded on the real axis and having all their zeros there. Our example shows, however, that this cannot be. What is going wrong?

The known proofs of the Beurling-Malliavin theorem ([2], [8], [11]) make use of *two* functionals involving $\log W$. One is the left-hand side of (1), and the other is quadratic in $\log W$. During most of the remainder of this section, *let us assume that $W(x) \geq 1$ is even and that $W(0) = 1$* . Then, if

$$W(x) = |F(x+i)|$$

with an entire F of exponential type having all its zeros on the real axis,

$$(28) \quad \frac{\log W(x)}{x} = -\int_0^\infty \log \left| \frac{x+t}{x-t} \right| d\left(\frac{\nu(t)}{t}\right),$$

where $\nu(t) = (1/\pi)[\arg F(i) - \arg F(t+i)]$ is an increasing function of t . (See, for instance, [8], pp. 286-287.) As Beurling and Malliavin first observed ([11]), this formula represents $(\log W(x))/x$ as a Green potential. The quadratic functional coming into play in proofs of the multiplier theorem is that Green potential's *energy*,

$$(29) \quad \mathcal{E}\left(\frac{\log W(x)}{x}\right) = \int_0^\infty \int_0^\infty \log \left| \frac{x+t}{x-t} \right| d\left(\frac{\nu(t)}{t}\right) d\left(\frac{\nu(x)}{x}\right).$$

This energy is always nonnegative, and the proofs require that *it be finite* as well as fulfillment of (1).

In our present case, where $W(x) = |F(x+i)| \geq 1$, the fact that $\nu(t)$ increases means that

$$\mathcal{E}\left(\frac{\log W(x)}{x}\right) \leq \int_0^\infty \frac{\nu(x)}{x} \frac{\log W(x)}{x^2} dx.$$

But, F being of exponential type, we have $\nu(x)/x \leq \text{const.}$, so here finiteness of $\mathcal{E}(\log W(x)/x)$ follows from (1), because $|F(i)| = W(0) = 1$ and the evenness of $W(x)$ make $\log W(x) = O(x^2)$ near 0.

Suppose, however, that $W(x) = 1/|\Psi(x+i)|$ with an even Ψ of exponential type having only real zeros such that $|\Psi(i)| = 1$ and $|\Psi(x+i)| \leq 1$ for x real. Then,

$$(30) \quad \frac{\log W(x)}{x} = \int_0^\infty \log \left| \frac{x+t}{x-t} \right| d\left(\frac{\nu(t)}{t}\right),$$

where, here, $\nu(t) = (1/\pi)[\arg \Psi(i) - \arg \Psi(t+i)]$ is again an increasing function of t . Relation (28) has been *exactly turned around!* The energy $\mathcal{E}((\log W(x))/x)$ is again given by (29) but here, because of the change of sign, we have

$$(31) \quad \mathcal{E}\left(\frac{\log W(x)}{x}\right) \leq \int_0^\infty \frac{\log W(x)}{x^2} d\nu(x)$$

with, in fact, both sides finite or infinite together, because their difference,

$$\int_0^\infty \frac{\nu(x)}{x} \frac{\log W(x)}{x^2} dx,$$

is always finite in the present circumstance.

The difference between the two choices for $W(x)$, as $|F(x+i)|$ or as $1/|\Psi(x+i)|$, is thus that in the first case the energy is bounded by a multiple of the left side of (1), while in the second it must be estimated by the right side of (31). Regarding the second case, where $W(x) = 1/|\Psi(x+i)|$, it turns out that if $\mathcal{E}((\log W(x))/x)$ is finite, any of the proofs of the multiplier theorem can be pushed through, yielding the existence of nonzero entire φ of arbitrarily small exponential type satisfying (3). Here, the right side of (31) equals

$$\frac{1}{\pi} \int_0^\infty \frac{1}{x^2} \log \left| \frac{\Psi(i)}{\Psi(x+i)} \right| \left(\frac{\partial \log |\Psi(z+i)|}{\partial y} \right)_{z=x} dx,$$

so, if this expression is finite, we have a multiplier theorem.

The example of § 2 shows that the expression just written cannot be replaced by

$$\int_0^\infty \frac{1}{x^2} \log \left| \frac{\Psi(i)}{\Psi(x+i)} \right| dx.$$

Some kind of regularity of $\log |\Psi(x+i)|$ is thus needed in addition

to the property

$$\left| \frac{d \log |\Psi(x+i)|}{dx} \right| \leq \log \left| \frac{\text{const.}}{\Psi(x+i)} \right|$$

which that function always has in the present circumstance (see the end of § 1). It would be interesting to know exactly how much is needed.

That some kind of smoothness is involved seems indicated by another phenomenon. If $W(x) \geq 1$ is any continuous weight tending to ∞ as $x \rightarrow \pm\infty$ and satisfying (1), the weighted exponentials

$$e^{i\lambda x}/W(x) \quad \text{with} \quad -A \leq \lambda \leq A$$

are certainly *not dense* in the space of continuous functions vanishing at $\pm\infty$ for any finite A . The ingenious extreme-point method of de Branges ([4], pp. 270-280) now provides us with an entire function $S(z)$ of exponential type exactly equal to A (in both the upper and lower half planes), having only simple real zeros—call them x_n —such that

$$(32) \quad \sum_n \frac{W(x_n)}{|S'(x_n)|} \leq 1,$$

whilst

$$(33) \quad \int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1+x^2} dx < \infty.$$

For any $a > 0$, the theorem on the multiplier furnishes, thanks to (33), a nonzero entire φ of exponential type $\leq a$, bounded on the real axis and satisfying $|S(x)\varphi(x)| \leq 1$ there. Since the product is of exponential type $\leq A + a$, by Bernstein's theorem ([3], pp. 206 and 210 ff),

$$\left| \frac{d}{dx} (S(x)\varphi(x)) \right| \leq A + a, \quad -\infty < x < \infty.$$

Putting $x = x_n$ and using (32), we see that

$$(34) \quad |W(x_n)\varphi(x_n)| \leq A + a$$

for all n . Here, by Levinson's density theorem which was already applied in § 2, the zeros x_n of $S(z)$ have density A/π on both the positive and negative real axes. That is, assuming the x_n indexed according to their natural order on the real line,

$$\lim_{n \rightarrow \infty} \frac{x_n}{n} = \lim_{n \rightarrow -\infty} \frac{x_n}{n} = \frac{\pi}{A}.$$

This makes (34) seem close to the uniform bound

$$|W(x)\varphi(x)| \leq \text{const.}, \quad -\infty < x < \infty,$$

especially since A can be chosen very large to begin with. (Note, however, that the function $S(z)$, and hence the φ of type \leq the given number a will depend on A !) The idea here is that if $W(x)$ is regular enough, (34) perhaps does imply a uniform bound on $W(x)\varphi(x)$. Again, exactly how much regularity would be needed to draw this conclusion is not clear.

The trouble is that de Brange's method does not give very precise information about the *location* of the points x_n . Essentially all that is known about them is that they have density A/π in both the positive and negative directions.

4. An application. The construction of §2 can be used to obtain an example of a kind of quasi-analyticity.

We shall exhibit a sequence of real numbers λ_k , defined for $k = \pm 1, \pm 2, \pm 3, \dots$, such that

(i) $\lambda_{-k} = -\lambda_k$ (symmetry)

(ii) the exponentials $e^{i\lambda_k t}$ are complete in $L_2(-A, A)$ if $A < \pi$

(iii) the exponentials $e^{i\lambda_k t}$ are not complete in $L_2(-A, A)$ if $A > \pi$

(iv) if $u(t)$, defined on $[0, \infty)$, is such that

$$(35) \quad \int_0^\infty |u(t) - \sum_k A_k(n) e^{i\lambda_k t}|^2 e^{-t} dt \xrightarrow{n} 0$$

for a sequence of finite sums $\sum_k A_k(n) e^{i\lambda_k t}$, and $u(t) \equiv 0$ a.e., on any interval $[0, 2h]$, $h > 0$, then $u(t) \equiv 0$ a.e., on $[0, \infty)$.

What is interesting here is our being able to do the construction with λ_k having the symmetry property (i). If symmetry were not required, we could just take the one-sided sequence $\lambda_k = k$, $k = 1, 2, 3, \dots$, and it would have properties (ii)-(iv).

Start with the sequence $\{x_p\}$, $p = 8, 9, \dots$, used in §2, and take any small positive number l less than x_8 and all the $\Delta_p = x_p - x_{p-1}$ for $p > 8$. On each of the intervals $[x_p - l, x_p]$, choose $[\Delta_p]$ different points; the collection of all these points, enumerated in increasing order, will be the one-sided sequence $\{\lambda_k\}$, $k = 1, 2, \dots$. For $k < 0$ we take λ_k as $-\lambda_{-k}$. It is claimed that properties (ii)-(iv) hold with these λ_k .

Let us here use $n(t)$ to denote the number of points λ_k between 0 and t . As at the beginning of §2, we easily see that

$$\frac{n(t)}{t} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty,$$

and it is well known that this implies property (ii). (See, for instance, [10], p. 3; [3], p. 235; [12], p. 22.)

Just as at the beginning of § 2, we verify that our present distribution function $n(t)$ satisfies (13). As in § 2, we can obtain, for any $\eta > 0$, an entire function of exponential type $\leq \pi + \eta$, bounded on the real axis, whose zeros include all the points λ_k , $k = \pm 1, \pm 2, \pm 3, \dots$. This is easily seen to imply property (iii) ([12], p. 26).

Property (iv) remains to be established. Suppose that $u(t)$ is identically zero on $[0, 2h]$ with $h > 0$, and that (35) holds. It is convenient to write

$$(36) \quad v(t) = e^{-t/2}u(t),$$

$$(37) \quad S_n(t) = e^{-t/2} \sum_k A_k(n) e^{i\lambda_k t};$$

then the Fourier transforms

$$\hat{v}(\lambda) = \int_0^\infty e^{i\lambda t} v(t) dt$$

and

$$\hat{S}_n(\lambda) = \int_0^\infty e^{i\lambda t} S_n(t) dt$$

belong to the space H_2 (for the upper half plane), and (35) implies, by Plancherel's theorem, that

$$(38) \quad \int_{-\infty}^\infty |\hat{v}(\lambda) - \hat{S}_n(\lambda)|^2 d\lambda \xrightarrow{n} 0.$$

Since $v(t) \equiv 0$ a.e., on $[0, 2h]$, we even have

$$(39) \quad \hat{v}(\lambda) \in e^{2ih\lambda} H_2.$$

From (37) we see by direct calculation that each function $\hat{S}_n(\lambda)$ is not only analytic for $\Im \lambda > 0$ but also meromorphic in the whole complex plane, with its poles lying among the points $-(\lambda_k + i/2)$.

Take now any number η with $0 < \eta < h$ and use the argument mentioned above to get a nonzero entire function $\Psi(\lambda)$ of exponential type $\leq \pi + \eta$, bounded on the real axis and having only real zeros, among which figure all the points λ_k , $k = \pm 1, \pm 2, \pm 3, \dots$.

I claim that

$$(40) \quad g(\lambda) = e^{-ih\lambda} \hat{v}(\lambda) \Psi(\lambda + i/2)$$

is an entire function of exponential type $\leq \pi + \eta - h < \pi$. For each n ,

$$(41) \quad g_n(\lambda) = e^{-ih\lambda} \hat{S}_n(\lambda) \Psi(\lambda + i/2)$$

is equal to a finite sum

$$\sum_k iA_k(n)e^{-ih\lambda} \frac{\Psi(\lambda + i/2)}{\lambda + \lambda_k + i/2}$$

which is clearly entire and of exponential type, since $-\lambda_k = \lambda_{-k}$ is, for each k , a zero of $\Psi(\lambda)$. We see that, for each n ,

$$(42) \quad |g_n(\lambda)| \leq C_n e^{h\Im\lambda} e^{(\pi+\eta)|\Im\lambda|};$$

in fact, since $|\Psi(\lambda + i/2)|$ is bounded for real λ , we have

$$\int_{-\infty}^{\infty} |g_n(\lambda)|^2 d\lambda \leq \text{const}.$$

by (38), and the C_n in (42) can be taken independent of n ([3], p. 98). From (38) and (40) we now see that $g(\lambda)$ is also an entire function of exponential type, and, by (42),

$$|g(\lambda)| \leq C e^{(\pi+\eta-h)|\Im\lambda|} \quad \text{in } \Im\lambda < 0.$$

But the same holds for, say, $\Im\lambda \geq 1$ by (39) and (40), since Ψ is of exponential type $\leq \pi + \eta$. So $g(\lambda)$ does have exponential type $< \pi$.

However, by (40),

$$(43) \quad \frac{|g(\lambda + i/2)|}{|\Psi(\lambda + i)|} = e^{h/2} |\hat{v}(\lambda + i/2)|,$$

which is bounded for real λ by (39). The reasoning of §2 now applies with almost no change to the function Ψ introduced here, and, from (43), leads to the conclusion that $g(\lambda) \equiv 0$ because g is of type $< \pi$. This makes $u(t) \equiv 0$ a.e., by (40) and (36), and property (iv) holds.

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