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# SPACES OF REPRESENTATIONS AND ENVELOPING L.M.C. *-ALGEBRAS

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**Given a l.m.c. *-algebra $E$ with a b.a.i., the space of representations $\mathscr{R}(E)$ and the enveloping algebra $\mathscr{E}(E)$ of $E$ are defined. Under a suitable condition for the extreme points of $E$, $\mathscr{R}(E)$, $\mathscr{R}(\mathscr{E}(E))$ coincide topologically, a fact contributing to the openess of the map defining the topology of $\mathscr{R}(E)$. Furthermore, one gets $\mathscr{E}(E)=\varprojlim_{\alpha}\mathscr{E}(E_\alpha)$, within a topological algebraic isomorphism, where $(E_\alpha)$ is the inverse system of Banach algebras corresponding to $E$.**

**1. Introduction.** There is a vast literature concerning representation theory of abstract Banach *-algebras (resp. $C^*$-algebras). On the other hand, due to recent considerations, it would be interesting and useful to have these results extended within the frame of (non-normed) topological *-algebras, a fact arising not only from the part of pure mathematics (e.g., function algebras), but also from that of applications in theoretical physics (:quantum mechanics).

The present paper provides within the context of l.m.c. *-algebras, extensions of various results referred to Banach *-algebras (resp. $C^*$-algebras) representation theory. More specifically, if $E$ is a l.m.c. *-algebra with a b.a.i., $\mathscr{B}(E)$ will denote the non-zero extreme points of $\mathscr{P}(E)$ (:continuous positive linear forms on $E$), and $\mathscr{R}(E)$ the equivalence classes of all continuous topologically irreducible representations of $E$. The set $\mathscr{R}(E)$ endowed with the final topology $\tau_{\delta_E}$ induced on it by the map $\delta_E\colon \mathscr{B}(E)\to\mathscr{R}(E)$ (:an extension of the classical "Gel' fand-Naimark-Segal map"; Th. 3.4) is called the *space of representations* of $E$. Thus, the paper is mainly concerned with the study of $\mathscr{R}(E)$ and the openess of the map $\delta_E$. To this study, the notion of the *enveloping algebra* $\mathscr{E}(E)$ of $E$ having by its definition the crucial $C^*$-property (Def. 4.1), plays an important role. Now, the openess of $\delta_{\mathscr{E}(E)}$, with $E$ a $bQ$ l.m.c. *-algebra with a b.a.i. (Def. 4.2) is obtained, leading thus to the required openess of $\delta_E$ (Th. 4.2), based besides on the fact that the spaces $\mathscr{B}(E)$, $\mathscr{R}(E)$ coincide topologically with the corresponding ones of $\mathscr{E}(E)$, when $\mathscr{B}(\mathscr{E}(E))$ is locally equicontinuous (Th. 4.1).

Furthermore, $\mathscr{E}(E/N(p_\alpha))$, $\mathscr{E}(E_\alpha)$ are isomorphic as topological algebras (Lemma 4.3) where $(E/N(p_\alpha))$, $(E_\alpha)$ are the inverse systems of normed respectively Banach algebras corresponding to $E$ [1], a fact further applied to get an inverse limit decomposition of $\mathscr{E}(E)$ in terms of $(\mathscr{E}(E_\alpha))$ (Th. 4.3).

**2. Preliminaries.** We introduce in this section the notation and terminology applied throughout.

A *representation* $\phi$ (or a *-representation) of a *-algebra $E$ is an involution preserving homomorphism of $E$ into the $C^*$-algebra $\mathscr{L}(H_\varphi)$ of all bounded linear operators on some Hilbert space $H_\varphi$ (:representation space of $E$).

A representation $\phi$ on a Hilbert space $H_\varphi$ is *topologically irreducible* if $H_\varphi$, $\{0\}$ are the only closed linear subspaces of $H_\varphi$ left invariant by $\phi(E)$. Moreover, $\phi$ is called *non-degenerate* if $\{\phi(x)(\xi): x \in E,\ \xi \in H_\varphi\}^- = H_\varphi$ where "$-$" means norm-closure. On the other hand, a vector $\xi \in H_\varphi$ is called *cyclic* for $\phi$ if $\{\phi(x)(\xi): x \in E\}^- = H_\varphi$; in that case $\phi$ is called cyclic. Now, the representations $\phi$, $\psi$ of $E$ are *equivalent*, we write $\phi \sim \psi$ (cf. [7]), if there exists a Hilbert space isomorphism $U: H_\varphi \to H_\psi$ such that $\psi(x) \circ U = U \circ \phi(x)$, $x \in E$.

A *positive linear form* on a *-algebra $E$ is a complex linear form $f$ on $E$ with $f(x^*x) \geqq 0$, $x \in E$. If $E$ has an identity $e$, then we also suppose that $f(e) = 1$. The set of positive linear forms on $E$ is denoted by $P(E)$. Now, if $f, g \in P(E)$ we write $f \geqq g$, and we say that $f$ *bounds* $g$, if $f - g \geqq 0$. Thus, an element $f \in P(E)$ is an *extreme point* if $g \in P(E)$ and $f \geqq g$ implies $g = \lambda f$ with $\lambda \in [0, 1]$ (cf. also [7]).

A topological algebra $E$ (:topological vector space with a separately continuous multiplication) is called *locally m-convex* (l.m.c.) if it has a local basis $\mathscr{U}$ consisting of $m$-barrels, (cf. [11] and [9; Chapt. 1, Th. 1.1]), where by an *m-barrel* we mean a subset of $E$ which is closed, convex, balanced, absorbing and idempotent. We may always suppose that such a local basis is directed.

Given a l.m.c. algebra $E$ with a directed local basis $\mathscr{U} = \{U_\alpha, \alpha \in A\}$, $\{p_\alpha, \alpha \in A\}$ will denote the family of submultiplicative semi-norms (:gauges) corresponding to $\mathscr{U}$. Then, $U_\alpha = \{x \in E: p_\alpha(x) \leqq 1\}$, $\alpha \in A$, [9; Chapt. 1, Lemma 2.3].

Now, by a l.m.c. **-algebra* we mean a l.m.c. algebra $E$ with an involution * such that $p_\alpha(x^*) = p_\alpha(x)$, $\alpha \in A$, $x \in E$ (cf. also [5; p.p. 6, 7]). If moreover, $p_\alpha(x^*x) = p_\alpha(x)^2$, $\alpha \in A$, $x \in E$, $E$ is called l.m.c. *$C^*$-algebra*. Note that if $E$ is a l.m.c. algebra with an involution * such that $p_\alpha(x)^2 \leqq p_\alpha(x^*x)$, $\alpha \in A$, $x \in E$, $E$ is a l.m.c. $C^*$-algebra. By a Fréchet l.m.c. *-algebra, we mean a l.m.c. *-algebra whose underlying locally convex space is Fréchet.

Furthermore, if $N(p_\alpha) = \ker(p_\alpha)$, $\alpha \in A$, $(E/N(p_\alpha))$, $(E_\alpha)$ denote the projective systems of normed and Banach *-algebras corresponding to $E$, where $E_\alpha$ is the completion of $E/N(p_\alpha)$, $\alpha \in A$ (cf. [1], [11]). The topology of $E_\alpha$ is defined by the norm $\dot{p}_\alpha$, with $\dot{p}_\alpha(x_\alpha) = p_\alpha(x)$, $x_\alpha = \pi_\alpha(x) = x + N(p_\alpha) \in E/N(p_\alpha)$, $\alpha \in A$, where $\pi_\alpha$ is the quotient map of $E$ onto $E/N(p_\alpha)$. If $E$ is a l.m.c. $C^*$-algebra, each $E_\alpha$, $\alpha \in A$, is a $C^*$-algebra.

Now, $E_1$ will denote the respective unital l.m.c. *-algebra of $E$, with corresponding family of semi-norms $(p_\alpha^1)$ and involution* defined respectively by $p_\alpha^1(x, \lambda) = p_\alpha(x) + |\lambda|$, $(x, \lambda)^* = (x^*, \bar{\lambda})$, $(x, \lambda) \in E_1 = E \oplus \boldsymbol{C}$.

On the other hand, *a bounded approximate identity* (:b.$\alpha$.i.) on $E$ will be a net $(e_i)_{i \in I}$, with $p_\alpha(e_i) \leqq 1$, $\alpha \in A$, $i \in I$ and $\lim p_\alpha(e_i x - x) = 0 = \lim p_\alpha(x e_i - x)$, $x \in E$, $\alpha \in A$.

**3. Space of representations of a l.m.c. *-algebra.** Let $E$ be a topological *-algebra (: *-algebra, which is also topological). Then, by a continuous representation of $E$ we shall mean a *-morphism $\phi$ of $E$ into $\mathscr{L}(H_\varphi)$, continuous relative to the uniform topology on $\mathscr{L}(H_\varphi)$. In the sequel, $R(E)$ (resp. $R'(E)$) will denote the set of all continuous (resp. continuous, topologically irreducible) representations of $E$. Note that "equivalence of representations" defines an equivalence relation "$\sim$" on $R(E)$ (and hence on $R'(E)$ too). In this respect, $(\phi, \phi')$ in $R(E) \times R'(E)$ with $\phi \sim \phi'$ implies $(\phi, \phi')$ in $R'(E) \times R'(E)$.

Now, set $\mathscr{R}(E) = R'(E)/\sim$, and denote by $[\phi]$ the respective class of $\phi \in R'(E)$ in $\mathscr{R}(E)$. In the rest of this section we work out the appropriate material for defining $\mathscr{R}(E)$ as a topological space.

Let $E$ be a l.m.c. *-algebra, and $E_s'$ its weak topological dual. Then, $E_s' = \bigcup_\alpha U_\alpha^0$, where $U_\alpha^0$ is the polar of the neighborhood $U_\alpha = \{x \in E: p_\alpha(x) \leqq 1\}$, $\alpha \in A$. Thus, if $\mathscr{P}(E)$ denotes the set of all continuous positive linear forms on $E$, and $\mathscr{B}(E)$ the non-zero extreme points of $\mathscr{P}(E)$, we obtain

$$\mathscr{P}(E) = \bigcup_\alpha \mathscr{P}_\alpha(E),\ \mathscr{B}(E) = \bigcup_\alpha \mathscr{B}_\alpha(E) \tag{3.1}$$

with $\mathscr{P}_\alpha(E) = \{f \in \mathscr{P}(E): |f(x)| \leqq 1,\ x \in U_\alpha\}$ and $\mathscr{B}_\alpha(E)$ the extreme points of $\mathscr{P}_\alpha(E)$, $\alpha \in A$. The preceding sets being subsets of $E_s'$ are considered endowed with the relative topology; moreover, since $\mathscr{P}_\alpha(E) = \mathscr{P}(E) \cap U_\alpha^0 \subset U_\alpha^0$, $\mathscr{P}_\alpha(E)$ (and therefore $\mathscr{B}_\alpha(E)$), $\alpha \in A$ is an equicontinuous subset of $\mathscr{P}(E)$.

Furthermore, note that a consequence of (3.1) and [9; Chapt. 1, Lemma 1.2] is that for each $f \in \mathscr{P}(E)$ there exists $\alpha \in A$ with $|f(x)| \leqq p_\alpha(x)$ for every $x \in E$. The next theorem extends an analogous result of [5; Th. 4.1].

THEOREM 3.1. *Let $E$ be a* l.m.c. *$^*$-algebra. Then, for each $\alpha \in A$*

$$\mathscr{P}(E/N(p_\alpha)) = \mathscr{P}_\alpha(E) = \mathscr{P}(E_\alpha) ,$$

*within homeomorphisms.*

*Proof.* Let $\alpha \in A$ and $\mathscr{P}_\alpha(E)$ the corresponding subspace of $\mathscr{P}(E)$. Then, for each $f \in \mathscr{P}_\alpha(E)$, $N(p_\alpha) \subset N(f)$, so that we define $f_\alpha \in \mathscr{P}(E/N(p_\alpha))$ by $f_\alpha(x_\alpha) = f(x)$, $x_\alpha \in E/N(p_\alpha)$, and we denote its extension to $E_\alpha$ also by $f_\alpha$. Thus, the map

$$\mathscr{P}_\alpha(E) \longrightarrow \mathscr{P}(E/N(p_\alpha))(\text{resp. } \mathscr{P}(E_\alpha)): f \longmapsto f_\alpha$$

is a homeomorphism, the continuity being a consequence of the equicontinuity of $\mathscr{P}(E_\alpha)$, since then the weak topologies $\sigma((E_\alpha)'_s, E/N(p_\alpha))$, $\sigma((E_\alpha)'_s, E_\alpha)$ coincide on $\mathscr{P}(E_\alpha)$, $\alpha \in A$ [3; p. 23, Prop. 5]. □

By Theorem 3.1 it is clear that $\mathscr{P}(E_\alpha)$ consists of all continuous positive linear forms on $E_\alpha$ with norm $\leqq 1$.

COROLLARY 3.1. *Let $E$ be as in Theorem 3.1. Then, for each $\alpha \in A$*

$$\mathscr{B}(E/N(p_\alpha)) = \mathscr{B}_\alpha(E) = \mathscr{B}(E_\alpha) ,$$

*within homeomorphisms.* □

LEMMA 3.2. *Let $E$ be a topological algebra with a* b.a.i. $(e_i)_{i \in I}$. *Then,*

(i) *If $E$ has a continuous multiplication, $(e_i^2)_{i \in I}$ is a* b.a.i. *for $E$.*

(ii) *If $E$ has a continuous involution $^*$, $(e_i^*)_{i \in I}$ is a* b.a.i. *for $E$.*

(iii) *If in particular $E$ is a* l.m.c. *$^*$-algebra, then $(e_\alpha^i)_{i \in I} = (e_i + N(p_\alpha))_{i \in I}$ $\alpha \in A$ is a* b.a.i. *for both $E/N(p_\alpha)$ and $E_\alpha$, $\alpha \in A$.*

*Proof.* For (i) cf. [9; Chapt. 6, Lemma 11.1]. (ii) $(e_i^*)_{i \in I}$ is a bounded net in $E$, since $^*$ is continuous. Moreover, for each $x \in E$ $\lim (e_i^* x - x) = \lim (x^* e_i - x^*)^* = 0^* = 0$, and similarly $\lim (x e_i^*) = x$, $x \in E$. (iii) For each $\alpha \in A$ define $e_\alpha^i = \pi_\alpha(e_i) = e_i + N(p_\alpha)$, then $\dot{p}_\alpha(e_\alpha^i) = p_\alpha(e_i) \leqq 1$, $i \in I$, $\alpha \in A$. Furthermore, $\lim \dot{p}_\alpha(x_\alpha e_\alpha^i - x_\alpha) = \lim p_\alpha(x e_i - x) = 0$, $x_\alpha \in E/N(p_\alpha)$, $\alpha \in A$; by the same way $x_\alpha = \lim (e_\alpha^i x_\alpha)$, $x_\alpha \in E/N(p_\alpha)$, $\alpha \in A$. Hence, $(e_\alpha^i)_{i \in I}$ is a b.a.i. for $E/N(p_\alpha)$, $\alpha \in A$ while this net is also a b.a.i. for $E_\alpha$, $\alpha \in A$ (ibid.). □

LEMMA 3.3. *Let $E$ be a* l.m.c. *$^*$-algebra with a* b.a.i. $(e_i)_{i \in I}$, *and* $f \in \mathscr{P}(E)$. *Then,*

(i) $f(x^*) = \overline{f(x)}$, $x \in E$ (i.e., $f$ *is real or hermitian*).

(ii) $|f(x)|^2 \leqq \|f_\alpha\| f(x^*x)$, $x \in E$.

*Proof.* (i) $f(x^*) = \lim_i f(x^*e_i) =$ [7; p. 27, (1)] $\lim_i \overline{f(e_i^*x)} = \overline{f(\lim_i e_i^*x)} =$ (Lemma 3.2, (ii)) $\overline{f(x)}$, $x \in E$.

(ii) $|f(x)|^2 =$ (Lemma 3.2, (ii)) $\lim_i |f(e_i^*x)|^2 \leqq$ [7; p. 27, (2)] $\lim_i f(e_i^*e_i) f(x^*x)$, $x \in E$. Now, if $f_\alpha$ is the element of $\mathscr{P}(E_\alpha)$ defined by $f$ as in Theorem 3.1, $\lim_i f(e_i^*e_i) =$ (Lemma 3.2, (iii)) $\lim_i f_\alpha((e_\alpha^i)^* e_\alpha^i) =$ [7; Prop. 2.1.5, (v)] $\|f_\alpha\|$. Actually, $\|f_\alpha\| \leqq 1$, since $|f_\alpha(x_\alpha)| = |f(x)| \leqq 1$, $x \in U_\alpha$. □

The above assertion (i) is actually valid for any topological algebra with continuous involution and a not necessarily bounded a.i. Every element $f \in \mathscr{P}(E)$ satisfying conditions (i), (ii) of Lemma 3.3 is called *extendable.*

PROPOSITION 3.4. *Let $E$ be a* l.m.c. **-algebra with a* b.a.i. $(e_i)_{i \in I}$. *Then,*

(i) *Each* $f \in \mathscr{P}(E)$ *is uniquely extended to an element* $f_1 \in \mathscr{P}(E_1)$ *with* $f_1(0, 1) = \|f_\alpha\|$, *where* $(0, 1)$ *denotes the identity element of* $E_1$.

(ii) *Each element of* $\mathscr{P}(E_1)$ *extending* $f$ *bounds* $f_1$.

(iii) *If* $Q(E_1) = \{h \in \mathscr{P}(E_1)\colon h(0, 1) = \|(h|_E)_\alpha\|\}$ *and an element of* $\mathscr{P}(E_1)$ *is bounded by an element of* $Q(E_1)$, *it must itself belong to* $Q(E_1)$.

(iv) $f \in \mathscr{B}(E) \Leftrightarrow f_1 \in \mathscr{B}(E_1) \Leftrightarrow \tilde{f}_1 \in \mathscr{B}(\tilde{E}_1)$, *where* $\tilde{E}_1$ *is the completion of* $E_1$ *and* $\tilde{f}_1$ *the extension of* $f_1$ *to* $\tilde{E}_1$.

*Proof.* (i) For each $f \in \mathscr{P}(E)$ define $f_1\colon E_1 \to \boldsymbol{C}\colon (x, \lambda) \mapsto f_1(x, \lambda) = f(x) + \lambda \|f_\alpha\|$, where $f_\alpha \in \mathscr{P}(E_\alpha)$ (cf. Th. 3.1). Then, $f_1 \in P(E_1)$ with $f_1(0, 1) = \|f_\alpha\|$. Moreover, $|f_1(x, \lambda)| \leqq |f(x)| + |\lambda| \leqq p_\alpha(x) + |\lambda| = p_\alpha^1(x, \lambda)$, $(x, \lambda) \in E_1$, hence $f_1 \in \mathscr{P}(E_1)$.

(ii) Suppose that $g \in \mathscr{P}(E_1)$ extends $f \in \mathscr{P}(E)$. Then, there exists $\gamma \in A$ with $g \in \mathscr{P}_\gamma(E_1)$ and $f \in \mathscr{P}_\gamma(E)$, hence $\|g_\gamma\| \geqq \|f_\gamma\|$ which yields $g \geqq f_1$.

(iii) Let $g = h + k$ with $g \in Q(E_1)$ and $h, k \in \mathscr{P}(E_1)$. Then, $g \geqq h, k$ and $h + k = g = (g|_E)_1 = (h|_E)_1 + (k|_E)_1$. Moreover, $h(0, 1) \geqq (h|_E)_1(0, 1)$, $k(0, 1) \geqq (k|_E)_1(0, 1)$, which implies $h(0, 1) = (h|_E)_1(0, 1)$, $k(0, 1) = (k|_E)_1(0, 1)$, that is $h, k \in Q(E_1)$.

(iv) Let $f \in \mathscr{B}(E)$ and $g \in \mathscr{P}(E_1)$ with $f_1 \geqq g$. Then, $f \geqq g|_E$, i.e., $g|_E = \lambda f$, $\lambda \in [0, 1]$ and since $g(0, 1) = \lambda f_1(0, 1)$ by (iii), we conclude $g = \lambda f_1$, $\lambda \in [0, 1]$.

Conversely, let $f \in \mathscr{P}(E)$ with $f_1 \in \mathscr{B}(E_1)$ and $g \in \mathscr{P}(E)$ such that $f \geqq g$. Then, $f - g \in \mathscr{P}(E)$, so that $(f-g)_1 = f_1 - g_1 \in \mathscr{P}(E_1)$, i.e., $f_1 \geqq g_1$, $g_1 \in \mathscr{P}(E_1)$; but then, $g_1 = \lambda f_1$, $\lambda \in [0, 1]$, hence also $g = \lambda f$, $\lambda \in [0, 1]$. The second equivalence of (iv) is clear. □

REMARK 3.4. For $E$ as in Proposition 3.4 and $\phi \in R(E)$ we define $\phi_1 \colon E_1 \to \mathscr{L}(H_\varphi) \colon (x, \lambda) \mapsto \phi_1(x, \lambda) = \phi(x) + \lambda id_{H_\varphi}$. Then, $\phi_1 \in R(E_1)$ and particularly $\phi \in R'(E) \Leftrightarrow \phi_1 \in R'(E_1) \Leftrightarrow \tilde{\phi}_1 \in R'(\tilde{E}_1)$, where $\tilde{\phi}_1$ is the extension of $\phi_1$ to $\tilde{E}_1$.

Now, if $f$, $\tilde{f}_1$ are as in Proposition 3.4, $L_{\tilde{f}_1} = \{z \in \tilde{E}_1 \colon \tilde{f}_1(z^*z) = 0\}$ is a left ideal of $\tilde{E}_1$ and $H_1 = \tilde{E}_1/L_{\tilde{f}_1}$ is a pre-Hilbert space with inner product $\langle z + L_{\tilde{f}_1}, w + L_{\tilde{f}_1}\rangle = \tilde{f}_1(w^*z)$, $w, z \in \tilde{E}_1$. Denote by $H$ the respective Hilbert space, completion of $H_1$. Then, one obtains

$$\overline{E/L_{\tilde{f}_1}} = E_1/L_{\tilde{f}_1}$$

since $\|(e_i, 0) + L_{\tilde{f}_1} - (0, 1) + L_{\tilde{f}_1}\|^2 = f_1((e_i, -1)^*(e_i, -1)) = f(e_i^*e_i) - f(e_i) - \overline{f(e_i)} + \|f_\alpha\| \to 0$ (cf. proof of Lemma 3.3 and note that $\lim_i f(e_i) =$(Th. 3.1, Lemma 3.2) $\lim_i f_\alpha(e_\alpha^i) =$[7; Prop. 2.1.5, (v)] $\|f_\alpha\|$).

On the other hand,

$$\overline{E_1/L_{\tilde{f}_1}} = H_1 ,$$

hence one finally obtains

$$\overline{E/L_{\tilde{f}_1}} = H . \tag{3.2}$$

In this respect, the following extends [5; Th. 6.1], being actually the analogue in our case of the standard *Gel'fand-Naimark-Segal* construction.

THEOREM 3.4. *Let $E$ be a* l.m.c. **-algebra with a* b.a.i., *and $f \in \mathscr{P}(E)$. Then, there exists a continuous representation $\phi_f$ of $E$ and a cyclic vector $\xi_f$ of $\phi_f$ such that $f(x) = \langle \phi_f(x)(\xi_f), \xi_f\rangle$, $x \in E$.*

*Proof.* For each $f \in \mathscr{P}(E)$, $\tilde{f}_1$ belongs to $\mathscr{P}(\tilde{E}_1)$ (Prop. 3.4), so that [5; Th. 6.1] there exists a continuous representation $\phi_{\tilde{f}_1}$ of $\tilde{E}_1$ into $\mathscr{L}(H)$ and a cyclic vector $\xi_{\tilde{f}_1}$ of $\phi_{\tilde{f}_1}$ in $H$ such that

$$\tilde{f}_1(z) = \langle \phi_{\tilde{f}_1}(z)(\xi_{\tilde{f}_1}), \xi_{\tilde{f}_1}\rangle , \; z \in \tilde{E}_1 .$$

Thus, if $\phi_f = \phi_{\tilde{f}_1}|_E$ and $\xi_f = \xi_{\tilde{f}_1} \in H$, one obtains

$$f(x) = \langle \phi_f(x)(\xi_f), \xi_f\rangle , \; x \in E ,$$

where $\xi_f$ is cyclic for $\phi_f$ as this follows by (3.2) and $\phi(E)(\xi_f) = E/L_{\tilde{f}_1}$. □

Now, given a l.m.c. *-algebra $E$ let, for each $\alpha \in A$

$$(3.3) \qquad R_\alpha(E) = \{\phi \in R(E): \|\phi(x)\| \leqq kp_\alpha(x),\ x \in E\},\ k > 0\ ,$$

so that $R(E) = \bigcup_\alpha R_\alpha(E)$. Thus, we can define $\phi_\alpha \in R(E/N(p_\alpha))$ with $\phi_\alpha(x_\alpha) = \phi(x)$, $x_\alpha \in E/N(p_\alpha)$, so that if $\phi_\alpha$ denotes also the extension of $\phi_\alpha$ to $E_\alpha$, one has $\|\phi_\alpha(z)\| \leqq \hat{p}_\alpha(z)$, $z \in E_\alpha$ [7; Prop. 1.3.7]; hence $\|\phi(x)\| \leqq p_\alpha(x)$, $x \in E$ in such a way that one may assume $k \leqq 1$ in (3.3), for each $\phi \in R_\alpha(E)$. Besides, if $R'_\alpha(E) = \{\phi \in R'(E): \phi \in R_\alpha(E)\}$ and $\mathscr{R}_\alpha(E) = R'_\alpha(E)/\sim$, we get

$$(3.4) \quad R(E) = \lim_{\overrightarrow{\alpha}} R_\alpha(E),\ R'(E) = \lim_{\overrightarrow{\alpha}} R'_\alpha(E,)\ \mathscr{R}(E) = \lim_{\overrightarrow{\alpha}} \mathscr{R}_\alpha(E)\ ,$$

within bijections [4; p. 92].

Now, if $\phi_\alpha \in R'(E_\alpha)$ and $M$ is a closed linear subspace of $H_\varphi(=H_{\varphi_\alpha})$ with $\phi(E)(M) \subset M$, then $\phi_\alpha(E_\alpha)(M) \subset M$. Hence, $\phi \in R'_\alpha(E) \Leftrightarrow \phi_\alpha \in R'(E/N(p_\alpha))$ (resp. $R'(E_\alpha)$). Finally, notice that $\phi \sim \psi$ in $R'_\alpha(E)$ implies $\phi_\alpha \sim \psi_\alpha$ in $R'(E_\alpha)$. The above yields the following

PROPOSITION 3.5. *Let $E$ be a* l.m.c. **-algebra. Then,*
( i ) $R(E/N(p_\alpha)) = R_\alpha(E) = R(E_\alpha)$, $\alpha \in A$,
( ii ) $R'(E/N(p_\alpha)) = R'_\alpha(E) = R'(E_\alpha)$, $\alpha \in A$,
(iii) $\mathscr{R}(E/N(p_\alpha)) = \mathscr{R}_\alpha(E) = \mathscr{R}(E_\alpha)$, $\alpha \in A$, *within bijections.* □

The following Banach *-algebras analogue [7; Prop. 2.5.4] extends also Corollary 6.4 of [5].

PROPOSITION 3.6. *Let $E$ be a* l.m.c. **-algebra with a* b.a.i. *Let also $f \in \mathscr{P}(E)$ and $\phi_f$ the respective element of $R(E)$* (cf. Th. 3.4). *Then, $f \in \mathscr{B}(E) \Leftrightarrow \phi_f \in R'(E)$.*

*Proof.* $f \in \mathscr{B}(E)$ implies $\tilde{f}_1 \in \mathscr{B}(\tilde{E}_1)$ (Prop. 3.4, (iv)), so that [5; Cor. 6.4] $\phi_{\tilde{f}_1} \in R'(\tilde{E}_1)$, which implies $\phi_{f_1} = \phi_{\tilde{f}_1}|_{E_1} \in R'(E_1)$ and since $\phi_{f_1} = (\phi_f)_1$, $\phi_f \in R'(E)$ by Rem. 3.4.

Conversely, let $f \in \mathscr{P}(E)$ with $\phi_f \in R'(E)$. Then, $\phi_{f_1} = (\phi_f)_1 \in R'(E_1)$ (Remark 3.4), so that $\phi_{\tilde{f}_1} \in R'(\tilde{E}_1)$, which yields $\tilde{f}_1 \in \mathscr{B}(\tilde{E}_1)$ [5; Cor. 6.4]; hence $f \in \mathscr{B}(E)$ by Proposition 3.4, (iv). □

Furthermore, one gets the next (cf. also [7; Prop. 2.4.1, (ii)].

LEMMA 3.7. *Let $E$ be a *-algebra and $\phi$, $\psi$ representations of $E$ into $\mathscr{L}(H_\varphi)$, $\mathscr{L}(H_\psi)$ respectively. Let also $\xi$ (resp. $\eta$) be a cyclic vector of $\phi$ (resp. $\psi$), with $\langle\phi(x)(\xi), \xi\rangle = \langle\psi(x)(\eta), \eta\rangle$, $x \in E$. Then, $\phi \sim \psi$ such that there exists a Hilbert space isomorphism $U: H_\varphi \rightarrow H_\psi$*

*with* $U\circ\phi(x) = \psi(x)\circ U$, $x \in E$ *and* $U(\xi) = \eta$. □

Now, regarding Proposition 3.6 we notice that for each $\phi \in R'(E)$ there exists $f \in \mathscr{B}(E)$ such that $\phi \sim \phi_f$: Indeed, if $\xi$ is a cyclic vector of $\phi$, the formula $f(x) = \langle \phi(x)(\xi), \xi \rangle$, $x \in E$ defines an element $f$ of $\mathscr{P}(E)$. Hence, (Th. 3.4) there exists $\phi_f \in R(E)$ and a cyclic vector $\xi_f$ of $\phi_f$ with $f(x) = \langle \phi_f(x)(\xi_f), \xi_f \rangle$, $x \in E$, so that (Lemma 3.7) $\phi \sim \phi_f$ in $R(E)$, i.e., $\phi_f \in R'(E)$, which by Proposition 3.6 implies $f \in \mathscr{B}(E)$. Hence, by Theorem 3.4 and Proposition 3.6 we now define an onto map

$$\delta_E : \mathscr{B}(E) \longrightarrow \mathscr{R}(E) : f \longmapsto \delta_E(f) = [\phi_f] \,. \tag{3.5}$$

The set $\mathscr{R}(E)$ equipped with the final topology $\tau_{\delta_E}$ induced on it by $\delta_E$, is called the *space of representations* of $E$.

In the next § 4, under additional conditions for $E$ we prove the openess of the map (3.5).

## 4. Enveloping algebra of a l.m.c. *-algebra.

We define below the enveloping algebra $\mathscr{E}(E)$ of a l.m.c. *-algebra $E$ with a b.a.i. It is proved that the representation theory of $E$ is actually reduced to that of $\mathscr{E}(E)$ (Th. 4.1), the last algebra having the important "$C^*$-property", hence its significance for the latter theory. On the other hand, by further obtaining under appropriate conditions the openess of the map $\delta_{\mathscr{E}(E)}$, we finally get the same property for the map (3.5) (Th. 4.2). Further applications, concerning topological tensor product algebras, will be given elsewhere.

LEMMA 4.1. *Let $E$ be a* l.m.c. **-algebra with a* a.b.i. *Then, for any $x \in E$ and $\alpha \in A$, the following hold true*:

( i ) $a = b = c = d$, *where*

$$a = \sup\{\|\phi(x)\| : \phi \in R_\alpha(E)\}, \; b = \sup\{\|\phi(x)\| : \phi \in R'_\alpha(E)\} \,,$$
$$c = (\sup\{f(x^*x) : f \in \mathscr{P}_\alpha(E)\})^{1/2}, \; d = (\sup\{f(x^*x) : f \in \mathscr{B}_\alpha(E)\})^{1/2} \,,$$
$$x \in E \,.$$

(ii) *For each $\alpha \in A$, the map $r_\alpha : E \to \boldsymbol{R}^+ : x \mapsto r_\alpha(x) = d$, defines a submultiplicative semi-norm on $E$, which is *-preserving and has the $C^*$-property.*

*Proof*. The proof is an immediate consequence of [7; Prop. 2.7.1] since by Theorem 3.1, Corollary 3.1 and Proposition 3.5, one concludes that

$a = \sup\{\|\phi_\alpha(x_\alpha)\|: \phi_\alpha \in R(E_\alpha)\}$, $b = \sup\{\|\phi_\alpha(x_\alpha)\|: \phi_\alpha \in R'(E_\alpha)\}$,
$c = (\sup\{f_\alpha(x_\alpha^* x_\alpha): f_\alpha \in \mathscr{P}(E_\alpha)\})^{1/2}$, $d = (\sup\{f_\alpha(x_\alpha^* x_\alpha): f_\alpha \in \mathscr{B}(E_\alpha)\})^{1/2}$.

□

Regarding Lemma 4.1, note that $b$ also coincides with

$$\sup\{\|\phi(x)\|: [\phi] \in \mathscr{R}_\alpha(E)\}\ .$$

Furthermore, since $\|\phi(x)\| \leqq p_\alpha(x)$, $x \in E$ for each $\phi \in R_\alpha(E)$, one obtains $r_\alpha(x) \leqq p_\alpha(x)$ for any $\alpha \in A$, $x \in E$, that is each $r_\alpha(\alpha \in A)$ is continuous with respect to the given topology of $E$.

DEFINITION 4.1. Let $E$ be a l.m.c. *-algebra with a b.a.i., and $(E, (r_\alpha))$ the respective l.m.c. $C^*$-algebra defined by Lemma 4.1. Then, the "Hausdorff completion" of the latter, that is the algebra

$$\mathscr{E}(E) = \widetilde{(E, (r_\alpha))/I} \tag{4.1}$$

with $I = \cap\{N(r_\alpha): \alpha \in A\}$ a closed 2-sided self-adjoint ideal of $E$, is called the *enveloping algebra of* $E$.

In this regard, cf. also [6; p. 65] concerning Fréchet l.m.c. *-algebras with identity. It is clear that (4.1) provides a complete l.m.c. $C^*$-algebra, whose topology is defined by the family $(\tilde{q}_\alpha)$ of submultiplicative semi-norms, extensions of $q_\alpha$, $\alpha \in A$ to $\mathscr{E}(E)$, where $q_\alpha(x + I) = \inf\{r_\alpha(x + i): i \in I\}$, $x + I \in (E, (r_\alpha))/I$. Moreover, if $(e_j)$ is a b.a.i. for $E$, the net $(e_j + I)$ is a b.a.i. for $\mathscr{E}(E)$.

REMARK 4.1. A given l.m.c. *-algebra $E$ with a b.a.i. has the $C^*$-property iff $r_\alpha = p_\alpha$ for each $\alpha \in A$, that is one has then $p_\alpha(x) \leqq r_\alpha(x)$, with $\alpha \in A$, $x \in E$ : In fact, since $E$ has the $C^*$-property, each $E_\alpha$ is a $C^*$-algebra, therefore $E_\alpha$, $\alpha \in A$ has an isometric representation, say $\phi_\alpha$, that is $\|\phi_\alpha(z)\| = \dot{p}_\alpha(z)$, $z \in E_\alpha$ (cf. [7; Th. 2.6.1]). But then, $\|\phi(x)\| = p_\alpha(x)$, $x \in E$ with $\phi \in R_\alpha(E)$ (Prop. 3.5).

Now, it is clear that every complete l.m.c. $C^*$-algebra coincides with its enveloping algebra. In the sequel $E/I$ will stand for $(E, (r_\alpha))/I$.

THEOREM 4.1. *Let $E$ be a* l.m.c. *-*algebra with a* b.a.i., *and $\mathscr{E}(E)$ its enveloping algebra with $\mathscr{B}(\mathscr{E}(E))$ locally equicontinuous. Then, $\mathscr{B}(E) = \mathscr{B}(\mathscr{E}(E))$ and $\mathscr{R}(E) = \mathscr{R}(\mathscr{E}(E))$ within homeomorphisms.*

*Proof.* If $f \in \mathscr{B}(E)$ there exists $\alpha \in A$ with $f \in \mathscr{B}_\alpha(E)$ and $|f(x)| \leqq r_\alpha(x)$, $x \in E$ (Lemma 3.3, (ii)). Thus, we define $g \in \mathscr{B}(E/I)$

with $g(x+I)=f(x)$, $x+I\in E/I$. Denoting also by $g$ the respective element of $\mathscr{B}(\mathscr{E}(E))$ we have $g\in\mathscr{B}(\mathscr{E}(E))\Leftrightarrow f\in\mathscr{B}(E)$. Now, the map $\Psi\colon\mathscr{B}(\mathscr{E}(E))\to\mathscr{B}(E)\colon g\mapsto\Psi(g)=f$ with $f=g\circ\tau$, where $\tau\colon E\to\mathscr{E}(E)$ is the canonical continuous morphism (Def. 4.1), is a continuous bijection. Moreover, the inverse of $\Psi$ is certainly continuous for the weak topology induced on its range by $E/I$. On the other hand, let $V$ be a neighborhood of $g$ in $\mathscr{B}(\mathscr{E}(E))$ which we may always assume to be equicontinuous by hypothesis. Then, the weak topologies on $V$ from $E/I$ and $\widetilde{E/I}=\mathscr{E}(E)$ coincide [3; p. 23, Prop. 5], which proves the continuity of $\Psi^{-1}$.

Now, if $\phi\in R(E)$, there exists $\alpha\in A$ with $\phi\in R_\alpha(E)$ and $N(r_\alpha)\subset N(\phi)$, so that one gets $\phi'\in R(E/I)$ with $\phi'(x+I)=\phi(x)$, $x+I\in E/I$. Thus, preserving the same symbol for the extension of $\phi'$ to $\mathscr{E}(E)$ we have $\phi'\in R'(\mathscr{E}(E))\Leftrightarrow\phi\in R'(E)$, so that the map $r\colon\mathscr{R}(\mathscr{E}(E))\to\mathscr{R}(E)\colon[\phi']\mapsto r([\phi'])=[\phi]$ with $\phi=\phi'\circ\tau$, is a homeomorphism as this follows by the relation $r\circ\delta_{\mathscr{E}(E)}=\delta_E\circ\Psi$, since $\delta_E$, $\Psi$ are continuous and $\mathscr{R}(\mathscr{E}(E))$ has the final topology induced on it by $\delta_{\mathscr{E}(E)}$, an analogous argument being valid for the inverse of $r$. □

Concerning the above theorem, we note that $\Psi$, $r$ are always continuous bijections. Moreover, an element $\phi\in R(E)$ is non-degenerate iff the element $\phi'\in\mathscr{R}(\mathscr{E}(E))$ is non-degenerate, and for any $(\phi,\phi')\in R(E)\times R(\mathscr{E}(E))$ the set $\phi(E)$ is dense in $\phi'(\mathscr{E}(E))$.

Regarding the local equicontinuity of $\mathscr{B}(\mathscr{E}(E))$ we note that this, is equivalent with that of $\mathscr{B}(E)$ when for instance, $\mathscr{E}(E)$ is barrelled (cf., for example, [9; Chapt. III, Cor. 5.31]). In this respect (cf. also Def. 4.2 below as well as the comments following it.

Now, a topological algebra $E$ is said to be a *Q-algebra*, if the set of its quasi-regular elements is open. If $E$ is a $Q$-algebra, the same holds also true for its respective unital algebra $E_1$ [12; p. 174, I].

DEFINITION 4.2. A l.m.c. *-algebra $E$ with a b.a.i., whose enveloping algebra $\mathscr{E}(E)$ is barrelled (l.m.c.) $Q$-algebra, is called a *bQ* l.m.c. **-algebra.*

In case $E$ is a Fréchet l.m.c. *-algebra, $\mathscr{E}(E)$ is by its definition Fréchet and thus barrelled. However, we still assume that $\mathscr{E}(E)$ is a $Q$-algebra to have the situation provided by Theorem 3 of [8], hence its application to the next result.

THEOREM 4.2. *Let $E$ be a bQ* l.m.c. **-algebra with a* b.a.i. *Then,*

$$\delta_E: \mathscr{B}(E) \longrightarrow \mathscr{R}(E)$$

*is a (continuous) open map.*

*Proof.* Clearly $\delta_E$ is continuous by the definition of the final topology $\tau_{\delta_E}$ on $\mathscr{R}(E)$. Now, by [8; Th. 3] $\mathscr{E}(E)_1$ is a $C^*$-algebra (cf. also [13; Cor. 5]), and since $\mathscr{E}(E) \underset{\rightarrow}{\subset} \mathscr{E}(E)_1$ ($\underset{\rightarrow}{\subset}$ means topological algebraic imbedding) $\mathscr{E}(E)$ becomes also a $\overrightarrow{C^*}$-algebra, so that $\mathscr{B}(\mathscr{E}(E))$ is equicontinuous, and $\delta_{\mathscr{E}(E)}$ open by [7; Th. 3.4.11]. Thus the assertion follows by Theorem 4.1 and the relation $\delta_E = r \circ \delta_{\mathscr{E}(E)} \circ \Psi^{-1}$. $\square$

In the rest of this section we relate $\mathscr{E}(E)$ with the decomposition of $E$ as an inverse limit of Banach algebras [1], [11]. Namely, we give $\mathscr{E}(E)$ (Th. 4.3) as an inverse limit of the $C^*$-algebras $\mathscr{E}(E_\alpha)$, $\alpha \in A$, which are the enveloping algebras of the Banach algebras $E_\alpha$, $\alpha \in A$, corresponding to $E$. However, we still need the following.

LEMMA 4.3. *Let $E$ be a* l.m.c. **-algebra with a* b.a.i. *Then,*

$$\mathscr{E}(E_\alpha) = \mathscr{E}(E/N(p_\alpha)) = (E/I)_\alpha = \mathscr{E}(E)_\alpha,\ \alpha \in A\ , \tag{4.2}$$

*within topological algebraic isomorphisms.*

*Proof.* By Definition 4.1 $\mathscr{E}(E/N(p_\alpha)) = \widetilde{(E/N(p_\alpha),\ t_\alpha)/I_\alpha}$ with $t_\alpha(x_\alpha) = \sup\{\|\phi_\alpha(x_\alpha)\|: \phi_\alpha \in R(E/N(p_\alpha))\} = r_\alpha(x)$, $x_\alpha \in E/N(p_\alpha)$, $\alpha \in A$ (cf. Prop. 3.5 and Lemma 4.1) and $I_\alpha = N(t_\alpha)$. Moreover, $t_\alpha \leqq \dot{p}_\alpha$, $\alpha \in A$, hence $t_\alpha$ has a unique extension $\tilde{t}_\alpha$ to $E_\alpha$, $\alpha \in A$, so that if $\tilde{I}_\alpha = N(\tilde{t}_\alpha)$, $\mathscr{E}(E_\alpha) = \widetilde{(E_\alpha,\ \tilde{t}_\alpha)/\tilde{I}_\alpha}$, $\alpha \in A$. Now, for $F_\alpha = (E/N(p_\alpha), t_\alpha)/I_\alpha$ and $G_\alpha = (E_\alpha, \tilde{t}_\alpha)/\tilde{I}_\alpha$, $\alpha \in A$, consider the map

$$h_\alpha: F_\alpha \longrightarrow G_\alpha: x_\alpha + I_\alpha \longmapsto x_\alpha + \tilde{I}_\alpha,\ \alpha \in A\ ,$$

which is an algebraic isomorphism into. Then, if $Q_\alpha$, $\tilde{Q}_\alpha$, $\alpha \in A$, are the norms defining the quotient topologies of $F_\alpha$, $G_\alpha$, $\alpha \in A$ respectively, one gets

$$Q_\alpha(x_\alpha + I_\alpha) = t_\alpha(x_\alpha) = \tilde{Q}_\alpha(x_\alpha + \tilde{I}_\alpha),\ x_\alpha \in E/N(p_\alpha),\ \alpha \in A\ ,$$

which yields $h_\alpha$, $\alpha \in A$, as a topological isomorphism too. Now, since by $t_\alpha \leqq \dot{p}_\alpha$ $\mathrm{Im}\,(h_\alpha)$ is dense in $G_\alpha$, $\alpha \in A$, one obtains the first part of the assertion. The last part of the statement is similarly proved. Concerning the 2nd equality in (4.2), if $M_\alpha = (E/I)/N(q_\alpha)$, $\alpha \in A$, the map

$$k_\alpha: M_\alpha \longrightarrow F_\alpha: (x + I)_\alpha \longmapsto x_\alpha + I_\alpha,\ \alpha \in A\ ,$$

is an algebraic isomorphism. In fact, $k_\alpha$, $\alpha \in A$ is a topological isomorphism: Namely, $Q_\alpha(x_\alpha + I_\alpha) \leqq \dot{q}_\alpha((x + I)_\alpha)$, which yields the continuity of $k_\alpha$. Besides, $k_\alpha^{-1}$ is continuous iff $\rho: (E/N(p_\alpha), t_\alpha) \to M_\alpha$: $x_\alpha \mapsto (x + I)_\alpha$ is continuous, which is true since $\dot{q}_\alpha(\rho(x_\alpha)) \leqq r_\alpha(x) = t_\alpha(x_\alpha)$, $x_\alpha \in E/N(p_\alpha)$, $(\alpha \in A)$. □

THEOREM 4.3. *If $E$ is a* l.m.c. **-algebra with a* b.a.i., *and $\mathscr{E}(E)$ its enveloping algebra, then*

$$\mathscr{E}(E) = \lim_{\substack{\longleftarrow \\ \alpha}} \mathscr{E}(E_\alpha)\ ,$$

*within an isomorphism of topological algebras.*

*Proof.* $\mathscr{E}(E)$ is by its definition a complete l.m.c. $C^*$-algebra, hence

$$\mathscr{E}(E) = \lim_{\substack{\longleftarrow \\ \alpha}} \mathscr{E}(E)_\alpha \tag{4.3}$$

within a topological algebraic isomorphism, where $(\mathscr{E}(E)_\alpha)$ is the inverse system of $C^*$-algebras corresponding to $\mathscr{E}(E)$ [2], [11; Th. 5.1]. Now, (4.3) and Lemma 4.3 yield the assertion. □

Theorem 4.3 has a special bearing on a previous result in [6; Th. 4.3] referred to a Fréchet l.m.c. *-algebra with an identity. On the other hand, by applying categorical language, since $\mathscr{E}$ preserves continuous morphisms between l.m.c. *-algebras with b.a.i.'s (cf. also Th. 4.1) one may consider $\mathscr{E}$ as a *covariant functor* between the categories of the respective algebras $E$ and $\mathscr{E}(E)$. Moreover, $\mathscr{E}$ is continuous (:preserves inverse limits) by Theorem 4.3 restricted to the full subcategory of Banach *-algebras.

The technique developed hitherto is further applied to the case of topological tensor products [10], by considering $\mathscr{E}(E \hat{\otimes}_\tau F)$ and $\mathscr{R}(E \hat{\otimes}_\tau F)$ with $E, F$ suitable l.m.c. *-algebras and $\tau$ an "admissible" tensor product topology.

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