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EQUIDISCONTINUITY OF BORSUK-ULAM FUNCTIONS

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No idempotent function on the unit disc onto its boundary is continuous. The stronger fact that no such function has a modulus of discontinuity smaller than $\sqrt{3}$ is a prototype of the contents of this paper.

A principal purpose of this paper is to report this fact:

THEOREM 1. *Let g be a function on a closed ball B^n in Euclidean n-space into the boundary S^{n-1} of B^n such that g maps each pair of antipodal points of S^{n-1} onto a pair of antipodal points. Then the modulus of discontinuity of g is at least d_n, the diameter of a regular n-simplex inscribed in S^{n-1}. Moreover, there is a g whose modulus attains the bound d_n.*

The *modulus (of discontinuity)* $\delta(g)$ of a function g from a topological space into a metric space is the infimum of all numbers d such that every point in the domain of g has a neighborhood whose image has a diameter of at most d.

Plainly, Theorem 1 strengthens a well-known result conjectured by Ulam and proved by Borsuk (1933). Rather than provide an independent proof, we find it considerably simpler to use Borsuk's result as a principal stepping stone to Theorem 1. However, self-contained constructive demonstrations are provided first for special cases of Theorem 1, including the classical one in which only idempotent functions g are treated (Corollaries 1 and 2 of Proposition 1). The conclusion that idempotent g's have a uniform modulus of discontinuity which depends only on the metrization of the boundary is extended to triangulable manifolds with boundary (Corollary 4) and somewhat more generally to g's that are not quite idempotent (Corollory 5).

Some standard terms and facts facilitate the formulation of Proposition 1, our principal constructive tool.

Though actually a triangulation of a space X consists of a simplicial complex K and a homeomorphism t of the polyhedron $|K|$ onto X, in this paper t is suppressed, and $|K|$ and X are identified.

A function g that maps the vertices of a triangulation K of a polyhedron $|K|$ into a Euclidean space determines a continuous mapping ξ of $|K|$ which is linear on each simplex of K, and coincides with g on the vertices. If g assumes its values in the unit sphere S^{n-1} and ξ is never zero, then φ, the *spherolinear extension* of g, is defined for $q \in |K|$ by letting $\varphi(q)$ be the (unique) point where the ray from the origin through $\xi(q)$ intersects S^{n-1}. Of course, if q is represented as a barycenter of vertices q_i of a simplex in K, say $q = \sum a_i q_i$, then $\xi(q) = \sum a_i g(q_i)$ and $\varphi(q) = \xi(q)/\|\xi(q)\|$. When it is necessary to indicate the dependence of ξ and φ on g, K and q, the notation $\xi(g, K)$ and $\varphi(g, K)$, or $\xi(g, K, q)$ and $\varphi(g, K, q)$ is used.

PROPOSITION 1. *Let (K^n, K_s^{n-1}) be a triangulation of a manifold M^n with boundary N^{n-1} and g a functian defined on the vertices of K^n into S^{n-1}. Then for some simplex $\sigma^n = (q_0, \cdots, q_n)$ in K^n, the convex hull of $g(q_0), \cdots, g(q_n)$ contains the origin if $\xi(g, K_s^{n-1})$ vanishes somewhere on N^{n-1}, or else if one of these two conditions holds.*

(1) *The modulo-2 degree of $\varphi(g, K_s^{n-1})$ is not zero.*

(2) *M^n is orientable and the degree of $\varphi(g, K_s^{n-1})$ is not zero.*

Proof. If ξ vanishes at $q \in \sigma^n \in K^n$, then σ^n fulfills the conclusion of the theorem. If ξ vanishes nowhere on N^{n-1}, consider first case (1). For $\sigma^{n-1} \in K^n$, let $F(\sigma^{n-1})$ be the image of σ^{n-1} under φ. Defining the (*modulo* 2-) *sum* of a finite collection of subsets of S^{n-1} to be the closure of the set of points that belong to oddly many of the subsets of the collection, let $F'(\sigma^n)$ be defined for each σ^n in K^n as the sum of the sets $F(\sigma^{n-1})$ as σ^{n-1} ranges over the faces of σ^n. The sum of F' over all σ^n in K^n is clearly equal to the sum of F over all σ^{n-1} in K_s^{n-1}. Since the degree of $\varphi(g, K_s^{n-1})$ is odd, the latter sum is all of S^{n-1}. This implies that there is a σ^n in K^n with the asserted property, as becomes evident by the following observation. If an n-simplex in E^n excludes the origin, its boundary is intersected in precisely two points by any ray from the origin that intersects it but none of its $(n-2)$-faces; so, if $\sigma^n = (q_0, \cdots, q_n) \in K^n$, and the convex hull of $g(q_0), \cdots, g(q_n)$ does not contain the origin, $F'(\sigma^n)$ is a set of dimension $n-2$ or less. For the remaining case in which ξ vanishes nowhere on N^{n-1} and condition (2) holds, replace the set-valued F by the real-valued cochain F^* where $F^*(\sigma^{n-1})$ is the signed $(n-1)$-volume of $F(\sigma^{n-1})$, note that the sum of the new F' over all σ^n in K^n is a nonzero multiple of the volume of S^{n-1}, and verify that $F'(\sigma^n) = 0$ if the convex hull of $g(q_0), \cdots, g(q_n)$ does not contain the origin. □

REMARK. If M^n is not orientable, condition (1) cannot be replaced by the weaker condition that the degree be not zero even when N^{n-1} is orientable. For example, for a Möbius strip realized in the complex plane as the anulus $1 \leqq |Z| \leqq 2$ with Z and $-Z$

identified when $|Z| = 2$, let $g(Z) = Z/\bar{Z}$. As is easily verified, g is a continuous mapping of the strip onto its boundary S^1, and its restriction to S^1 is of degree 2. For fine-meshed triangulations (K^2, K_s^1) of the strip, the degree of $\varphi(g, K_s^1)$ is also 2, yet, by continuity of g, $g(q_0)$, $g(q_1)$ and $g(q_2)$ are too close together to fulfill the conclusion of Proposition 1.

In Proposition 1 and its proof, the unit sphere in any Minkowski-space can be substituted for S^{n-1}. In Lemma 1, however, which provides the link with the metric character of the corollaries below, it is essential that E^n be Euclidean.

LEMMA 1. *Every subset of the unit sphere in E^n whose convex hull contains the origin has a diameter of at least d_n, where $d_n = (2 + 2n^{-1})^{1/2}$ is the diameter of a regular n-simplex inscribed in the sphere.*

Proof. As is well-known, the subset must contain $n + 1$ points $v_0, \cdots, v_n$, not necessarily distinct, whose convex hull contains the origin. Let $a_0, \cdots, a_n$ be nonnegative numbers, not all zero, such that $\sum a_i v_i = 0$. Denote inner products by $\langle \cdot, \cdot \rangle$, and obtain the equality

$$0 = \|\sum a_i v_i\|^2 = \sum_{i \neq j} a_i a_j \langle v_i, v_j \rangle + \sum a_i^2$$

which, together with the inequality

$$0 \leqq \sum_{i \neq j} (a_i - a_j)^2 = 2n \sum a_i^2 - 2 \sum_{i \neq j} a_i a_j$$

implies

$$\sum_{i \neq j} a_i a_j (\langle v_i, v_j \rangle + n^{-1}) \leqq 0 \ .$$

Therefore, for some $i \neq j$,

$$\langle v_i, v_j \rangle \leqq -n^{-1} \ .$$

For these i, j,

$$\|v_i - v_j\|^2 \geqq 2 + 2n^{-1} = d_n^2 \ . \qquad \square$$

COROLLARY 1. *Let L^n be any manifold whose boundary is S^{n-1}. The moduli of all functions of L^n to its boundary, which leave each point on the boundary fixed, are no less than d_n.*

Proof. As is not difficult to verify, there are arbitrarily fine-meshed triangulations K^n of L^n such that the corresponding sphero-linear extension of the identity mapping on the boundary vertices of K^n is simply the identity mapping on the boundary of L^n. Proposition 1 and Lemma 1 now apply. □

COROLLARY 2. *Let f be a mapping of S^2 into S^1. If f maps every pair of antipodal points of S^2 onto a pair of antipodes of S^1, then its modulus $\delta(f)$ is at last $\sqrt{3}$.*

Proof. Embed the range-space S^1 of f as a great circle on S^2, and let M^2 be one of the closed hemispheres bounded by S^1. If K^2 is a triangulation of M^2 whose induced triangulation K_s^1 of S^1 is symmetric around the origin, Proposition 1 will apply to the restriction g of f to M^2, once it is shown that $\hat{g} = \varphi(g, K_s^1)$ is of odd degree. Since $\hat{g}$ is a continuous mapping of S^1 into itself, its degree is the winding number ω of the point $\hat{g}(t)$ as t goes once around S^1. To evaluate ω, fix some $t_0 \in S^1$, and let $\alpha(t)$, for $t \in S^1$, be the angle accumulated by $\hat{g}(s)$ as s varies continuously from t_0 to t. Since g, and hence $\hat{g}$, preserve antipodality, $\hat{g}(-t_0) = \hat{g}(t_0)$, and therefore $\alpha(-t_0)$ is an odd multiple of π, say πr. Using the antipodality once again, the total change in $\alpha(t)$ is twice as much, that is, $2\pi r$, when t goes once around the circle. Hence $\omega = r$ is odd, and by Proposition 1, and Lemma 1 with $n = 2$, $\delta(g) \geqq d_2 = \sqrt{3}$. Since g is a restriction of f, $\delta(f)$ is not less. □

Plainly, Corollaries 1 and 2 are special cases of Theorem 1. A tool for inferring the lower boundedness of the moduli for the family of functions treated in Theorem 1 from the discontinuity of its members is provided by the following proposition, which possibly has applications elsewhere.

PROPOSITION 2. *Let $\mathscr{R}$ be a set of functions defined on a subpolyhedron N of a polyhedron M into the Euclidean sphere S^{n-1} that satisfies these two conditions:*

(1) *For each $f \in \mathscr{R}$ and $\varepsilon > 0$ there is a triangulation (K, K') of (M, N) with mesh less than ε such that either $\xi(f, K', q) = 0$ for some $q \in N$ or $\varphi(f, K') \in \mathscr{R}$;*

(2) *No extension g of any $f \in \mathscr{R}$ to M is continuous.*

Then every extension of each $f \in \mathscr{R}$ to M has a modulus no less than the diameter of a regular n-simplex inscribed in S^{n-1}.

Proof. Let g be an extension of an $f \in \mathscr{R}$ to M and, for $\varepsilon > 0$, let (K, K') be a triangulation of (M, N) as in (1). If $\xi(g, K)$ were never zero on M, $\varphi(g, K)$ would be a continuous extension of $\varphi(f, K')$ to M. But by (1), $\varphi(f, K') \in \mathscr{R}$, and, hence by (2), it has no continuous extension to M. Consequently, $\xi(g, K, q) = 0$ for some

$q \in M$. If $\sum_0^m a_i q_i$ is the barycentric representation of q, then the convex full of $g(q_0), \cdots, g(q_m)$ contains the origin. Now Lemma 1 applies. □

Proof of the inequality in Theorem 1. Let $M = B^n$ be identified with a closed hemisphere of S^n, and let N be its boundary S^{n-1}. For $\mathscr{R}$ the set of all antipodality-preserving functions on S^{n-1} into itself, condition (1) of Proposition 2 holds for any ε-meshed triangulations that are invariant under the map $q \to -q$ on S^{n-1}. Clearly, every extension g of every $f \in \mathscr{R}$ to all of B^n has in turn a unique antipodality-preserving extension G to the entire S^n. If g were continuous, G would be too. But, by Satz II of Borsuk (1933), there is no such G. Consequently, condition (2) holds as well, and Proposition 2 applies. □

COROLLARY 3. *The modulus of each mapping g of S^n into S^{n-1} that maps every pair of antipodal points of S^n onto antipodal points of S^{n-1} is no less than d_n.*

Proof. Theorem 1 applied to the restriction g' of g to any closed hemisphere of S^n yields $\delta(g) \geqq \delta(g') \geqq d_n$. □

Scholium 1. The bounds in Theorem 1 and in Corollaries 1, 2, and 3 are attained.

Proof. For Theorem 1 and Corollary 1, inscribe a regular n-simplex in S^{n-1}, let g map each interior point of L^n to the closest (or one of the closest) of its vertices, and on the boundary, let g be the identity. The modulus of g is easily seen to be d_n. For Corollaries 2 and 3, embed S^n in E^{n+1} as the boundary of the unit ball B^{n+1}. Choose a hyperplane through the origin. It divides S^n into two open hemispheres H_1 and H_2 and intersects B^{n+1} in an n-ball B^n. Inscribe in B^n two mutually antipodal regular n-simplices σ^n and $-\sigma^n$. Let f map each point of H_1 to the vertex of σ^n closest to it, and each point of H_2 to the closest vertex of $-\sigma^n$. On the common boundary S^{n-1} of the two hemispheres let f be the identity. The modulus of f on each of the two closed hemispheres separately is clearly d_n since there, f is just the function g above for $L^n = B^n$, transferred via a homeomorphism of the domains and an isometry of the ranges. For the modulus at a point p on the common boundary of the hemispheres, note that the closest to p among the vertices $v_0, \cdots, v_n$ of σ^n, and the closest among their antipodes are never as far as d_n apart: in fact, for any p in S^n, if v_i is closest to p among the former, $\langle v_i, p\rangle > 0$, therefore $\langle -v_i, p\rangle < 0$, and

$-v_i$ is certainly not closest to p among the latter. Hence, some $-v_j$ with $j \neq i$ is closest. Since $\langle v_i, v_j \rangle = -n^{-1}$,

$$\| v_i - (-v_j) \|^2 = 2 + 2\langle v_i, v_j \rangle = 2(n-1)/n < d_n^2 .$$

Consequently, the modulus of f on all of S^n is still d_n. □

The values obtained for the minima of the moduli are, of course, contingent on the metric on the range spaces. The existence of a positive lower bound, however, is a topological fact, valid for any metrization. Since the next two corollaries of Proposition 1 deal with mappings into range spaces where no one metric seems distinguished, it is the topological fact that is asserted there. A simple lemma about the behavior of moduli under composition is used in its proof.

LEMMA 2. *Let h be a uniformly continuous mapping from a metric space Y to a metric space Z. For every $d > 0$ there is a $t > 0$, such that for any function g on any topological space X into Y, $\delta(h \circ f) \geqq d$ implies $\delta(f) \geqq t$.*

Proof. Choose $t>0$ so that $\rho_y(y_1, y_2)<t$ implies $\rho_z[h(y_1), h(y_2)]<d$. Then any set in X whose image under $h \circ f$ has a diameter at least d, has an image under g whose diameter is no less than t. □

COROLLARY 4. *Let M^n be a triangulable manifold with boundary N^{n-1}. For any metrization of N^{n-1}, there is a $t > 0$ such that the modulus $\delta(f)$ of any function f of M^n onto N^{n-1} that leaves each point of N^{n-1} fixed is at least t.*

Proof. Let V be a subset of N^{n-1} that is homeomorphic to an open $(n-1)$-ball and hence to $S^{n-1} - \{p\}$, for an arbitrary $p \in S^{n-1}$. Define a continuous mapping h of N^{n-1} onto S^{n-1} that maps V homeomorphically onto $S^{n-1} - \{p\}$, and on the rest of N^{n-1}, let $h = p$. The composition $h \circ f$ is a function on M^n onto S^{n-1}, and its restriction to N^{n-1} is h. Since h covers every point of S^{n-1}, except p, precisely once, the degree of h is 1. For all sufficiently fine-meshed triangulations (K^n, K_s^{n-1}) of (M^n, N^{n-1}) the degree of $\varphi(h, K_s^{n-1})$ is also 1, by the following lemma.

LEMMA 3. *For a continuous mapping h of any polyhedron N^{n-1} into S^{n-1}, there is a $\delta > 0$ such that, for any triangulation K_s^{n-1} of N^{n-1} with* $\mathrm{Mesh}(K_s^{n-1}) < \delta$, *$h$ and its sphero-linear extension $\varphi(h, K_s^{n-1})$ nowhere take on antipodal values and, consequently, are homotopic.*

Proof. By uniform continuity, there is a $\delta > 0$ such that, for p and q in N^{n-1}, $\rho(p, q) < \delta$ implies $\|h(p) - h(q)\|^2 < 2$ or, equivalently, the inner product of $h(p)$ and $h(q)$ is positive. If Mesh $(K_s^{n-1}) < \delta$, $p \in \sigma^{n-1} \in K_s^{n-1}$ and $q \in \sigma^{n-1}$, then $\langle h(p), h(q)\rangle > 0$. Since this inequality holds in particular for every pair of vertices p and q of σ^{n-1}, the sphero-linear extension $\varphi(h, K_s^{n-1})$ is well-defined and $\langle \varphi(h, K_s^{n-1}, p), h(q)\rangle > 0$ for all $p \in \sigma^{n-1}$ and any $q \in \sigma^{n-1}$. Since $h(p)$ and $\varphi(h, K_s^{n-1}, p)$ have positive inner products with the same vector $h(q)$, they are not antipodal. Therefore, the origin is not a covex combination of $h(p)$ and $\varphi(h, K_s^{n-1}, p)$. As is now routine to verify, $[th + (1-t)\varphi(h, K_s^{n-1})]/\|[th + (1-t)\varphi(h, K_s^{n-1})]\|$ is a homotopy of h and $\varphi(h, K_s^{n-1})$. □

To complete the proof of Corollary 4, first note that the composition $h \circ f$ is a g to which Proposition 1 applies, then apply Lemma 1 to obtain $\delta(h \circ f) \geqq d_n$, and finally Lemma 2 with $d = d_n$. □

REMARK. The assumption that f is the identity on N^{n-1} is used in the proof of Corollary 4 only to imply:

(*) The restriction of $h \circ f$ to N^{n-1} covers S^{n-1}-$\{p\}$ precisely once.

But (*) also follows from the weaker assumption that for an open $(n-1)$-ball $V \subset N^{n-1}$, f is the identity on V and the complement of V is invariant under g. This proves the following generalization of Corollary 4.

COROLLARY 5. *Let M^n be a triangulable manifold with boundary N^{n-1}, ρ a metric for N^{n-1} and V a subset of N^{n-1} that is homemorphic to an open $(n-1)$-ball. Then there is a $t > 0$ depending only on V and ρ that satisfies this condition: If f is a function on M^n into N^{n-1} that is the identity on V, and the complement of V is invariant under f, then the modulus of f exceeds t. In particular, no such f is continuous.*

The derivation of the minimum of the moduli of idempotent functions on a manifold onto its boundary can also be extended beyond the case where the boundary is a sphere. This extension is an easy consequence of Corollary 1.

COROLLARY 6. *Let L^n be a manifold whose boundary is S^{n-1}, M an arbitrary manifold (without boundary), and ρ a metric on $S^{n-1} \times M$, such that, for $s_i \in S^{n-1}$ and $m_i \in M$,*

$$\rho[(s_1, m_1), (s_2, m_2)] \geqq \rho[(s_1, m_1), (s_2, m_1)] = \| s_1 - s_2 \| .$$

Then the minimum of the moduli of all idempotent functions on $L^n \times M$ *onto its boundary* $S^{n-1} \times M$ *is* d_n.

For example, if a solid torus is realized in E^3 by rotating a closed unit disc in the plane around a line disjoint from it ($n = 2$, $L^2 = B^2$, $M = S^1$), then each idempotent function on the solid torus onto its boundary has a modulus no less than $\sqrt{3}$, and there is at least one such function whose modulus is $\sqrt{3}$.

At a lecture where this paper was presented, Ed Spanier asked whether Proposition 2 can be applied to extensions to B^4 of the Hopf map $f_H: S^3 \to S^2$ (see e.g., Dugundji (1966) p. 408), or more generally, to the extensions of a mapping f to a superspace M of its domain N to which it has no continuous extension. An answer to his question is included in the following corollary.

COROLLARY 7. *Let* f *be a continuous mapping defined on a subpolyhedron* N *of a polydedron* M *into the Euclidean sphere* S^{n-1} *that has no continuous extension to* M. *Then the modulus of every function of that extends* f *to* M *is at least* d_n, *and there is an extension that attains the bound.*

Proof. Let $\mathscr{R}$ be the class of all continuous mappings of N into S^{n-1} that are homotopic to f. Condition (1) of Proposition 2 follows by applying Lemma 3 to f; and condition (2) is a consequence of the homotopy extension property for subpolyhedra (see e.g., Spanier (1966) p. 118). For the attainment of the bound, inscribe a regular n-simplex $[p_0, \cdots, p_n]$ in S^{n-1}, extend f to a continuous function h defined on an open neighborhood W of N in M, and define g on M by: on N, let $g = f$; for $x \in W - N$, $g(x)$ is (one of) the p_i closest to $h(x)$; for $x \in M - W$, $g(x) = p_0$. Since the image of the open set $M - N$ under g is a subset of $\{p_0, \cdots, p_n\}$, the modulus at any $x \in M - N$ is at most d_n. At any $x \in N$, the modulus is no greater, as can be seen by an argument similar to the conclusion of the proof of Scholium 1. □

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