A CONSTRUCTIVE LOOK AT POSITIVE LINEAR FUNCTIONALS ON $\mathscr{L}(H)$

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The main results of this paper characterize countably additive, positive linear functionals on the algebra of bounded operators on a separable Hilbert space, and show how such functionals can be viewed as analogues of positive measures of Bishop-Cheng type. The paper is written entirely within the framework of constructive mathematics.

Since the first appearance of Bishop's book [1], constructive techniques have been applied in several areas of analysis, algebra and topology. In this paper, we turn our attention to positive linear functionals on $\mathscr{L}(H)$, the algebra of bounded operators on a separable Hilbert space H.

One reason why a constructive study of these functionals may be of interest is that they are of considerable importance in the vast theory of C^*-and W^*-algebras [6], a theory which, in its present form, appears to be highly nonconstructive. Another, and perhaps more important, reason lies in the role played by certain of these functionals in the foundations of physics. Indeed, it is our belief that the development of constructive mathematical foundations for physics may illuminate some of the darker philosophical recesses which present theories seem unable to explore with much success.

Our discussion will rely on some preliminary material on operators of trace class. The constructive problems with this material, which is described in §1, usually arise from the nonconstructive nature of the least upper bound principle. For example, the distance from a point of H to the range of an operator in $\mathscr{L}(H)$ may not be computable; this means that the classical theory of polar decomposition of an operator has to be modified to one of approximate polar decomposition. Another place where our inability to use the least upper bound principle causes trouble is the proof of completeness of the set of trace class operators with respect to the trace class norm.

In §2, we give various characterizations of countable additivity of a positive linear functional on the set $\mathscr{A}(H)$ of bounded operators on H for which the adjoint is computable. In §3, we discuss two constructive expressions of the analogy between countably additive, positive linear functionals on $\mathscr{A}(H)$ and positive measures on the set of continuous, real-valued functions of compact support

on a locally compact metric space. The work of §3 can be regard-
ed as a first step towards a constructive theory of noncommutative
integration [5].

We shall assume familiarity with those features which disting-
uish constructive mathematics from its classical (that is, traditional)
counterpart, and with the foundations of constructive analysis de-
scribed in [1], [2] and [3]. Throughout our paper, a result stated
without proof should be regarded as an invitation to a little extra
effort on the part of the reader.

1. Let H be a separable, complex Hilbert space with scalar
product $\langle , \rangle$, $(a_n)_{n \geq 1}$ an orthonormal basis of H, and $\mathscr{L}(H)$ the linear
space of bounded operators on H. Define the *strong operator norm*
$\| \ \|_s$ and the *weak operator norm* $\| \ \|_w$ on $\mathscr{L}(H)$ by setting

$$\|T\|_s = \sum_{n=1}^{\infty} 2^{-n} \|Ta_n\| \ ,$$

$$\|T\|_w = \sum_{m,n=1}^{\infty} 2^{-m-n} |\langle Ta_m, a_n \rangle|$$

for each T in $\mathscr{L}(H)$. Strong operator norms defined in terms of
different orthonormal bases of H induce equivalent metrics on the
unit ball $\mathscr{L}_1(H) = \{T \in \mathscr{L}(H): \forall x \in H \ (\|Tx\| \leq \|x\|)\}$ of $\mathscr{L}(H)$; the
same holds for weak operator norms. Note that $\mathscr{L}_1(H)$ is a closed,
totally bounded subset of $\mathscr{L}(H)$ in the metric induced by the weak
operator norm; but that the weak operator compactness of $\mathscr{L}_1(H)$
is an essentially nonconstructive proposition [4].

A constructive proof that every element of $\mathscr{L}(H)$ has an ad-
joint would comprise a finite routine which, applied to any bounded
operator T on H, constructs the adjoint of T. It seems unlikely
that such a routine will ever be produced. However, if $\mathscr{A}(H)$ de-
notes the set of all elements of $\mathscr{L}(H)$ for which the adjoint is
computable, an examination of the proof of the theorem in [4] shows
that $\mathscr{A}_1(H) = \mathscr{A}(H) \cap \mathscr{L}_1(H)$ is weak operator norm dense in $\mathscr{L}_1(H)$.
Moreover, if H is finite dimensional, then $\mathscr{A}(H) = \mathscr{L}(H)$.

An element A of $\mathscr{L}(H)$ is said to be *positive* if it is selfadjoint
and $\langle Ax, x \rangle \geq 0$ for each x in H. If A, B are selfadjoint and $A - B$
is positive, we write $A \geq B$. If $A \in \mathscr{A}(H)$, then $A^*A \geq 0$; we write
$|A|$ for the positive square root of A^*A. The existence of $|A|$ is a
consequence of the functional calculus for A^*A [3, Ch. 6].

An element U of $\mathscr{A}(H)$ is said to be a *partial isometry* if
there exists a projection P, called the *initial projection* of U, such
that $\|UPx\| = \|Px\|$ and $U(I - P)x = 0$ for each x in H. (As usual,
I denotes the *identity operator* $x \rightarrow x$ on H.) An element U of
$\mathscr{A}(H)$ is a partial isometry if and only if U^*U is a projection; in

which case U^*U is the initial projection of U, and U^* is a partial isometry with initial projection UU^*.

Of considerable importance in classical operator theory is the polar decomposition of an operator S: $S = U|S|$, with U a partial isometry. Such a polar decomposition is not always possible in constructive mathematics, as we may be unable to construct the projection on the closure of the range of S [cf. 6, p. 15]. Fortunately, we can get by with the following constructive substitute.

THEOREM 1.1. (*Approximate Polar Decomposition*).   Let $A \in \mathscr{A}(H)$, and let $\varepsilon > 0$.   Then there exists a partial isometry U such that ε is a bound for both $A - U|A|$ and $|A| - U^*A$.

Proof. Without loss of generality, we assume that $A \in \mathscr{A}_1(H)$. Let $(\mu, f \to f(A^*A))$ be the functional calculus for A^*A [3, Ch. 6, § 6], and compute a strictly decreasing sequence $(\alpha_n)_{n \geq 1}$ of positive numbers converging to 0 such that the complemented sets

$$S(n) = (\{x \in X: \mathrm{pr}_1(x) \leq -\alpha\} , \quad \{x \in X: \mathrm{pr}_1(x) > -\alpha\})$$

are all integrable [3, Ch. 5, 6.5]. (Here, pr_1 is the mapping $(x_n)_{n \geq 1} \to x_1$ on the compact product metric space $X = \prod_{n=1}^\infty [-1, 1]$ underlying the integral μ.)   Were $\mu S(n) > 0$, it would follow from the inequality $(|\mathrm{pr}_1| - \mathrm{pr}_1 - 2\alpha_n)\chi_{S(n)} \geq 0$ and [3, Ch. 6, 6.3] that there exists $x \in H$ with $\|x\| = 1$ and $\langle(|A^*A| - A^*A)x, x\rangle \geq 2\alpha_n > 0$.   As this contradicts the fact that $|A^*A| = A^*A$, we conclude that $\mu S(n) = 0$. It follows that the complemented set $(\mathrm{pr}_1 < 0)$, with characteristic function $\sup_{n \geq 1}\chi_{S(n)}$, has measure zero.   Thus, by [3, Ch. 5, 5.1], $F = \{x \in X: \mathrm{pr}_1(x) \geq 0\}$ is a full set.   As $|\mathrm{pr}_1| = \mathrm{pr}_1$ throughout F, we have

$$A^*A = |A^*A| = |\mathrm{pr}_1|(A^*A) = \mathrm{pr}_1(A^*A) .$$

Now choose α in $(0, \varepsilon)$ so that the complemented set

$$S = (\mathrm{pr}_1 \geq \alpha) = (\{x \in X: \mathrm{pr}_1(x) \geq \alpha\}, \{x \in X: \mathrm{pr}_1(x) < \alpha\})$$

is integrable, and let $\chi = \chi_S$, $\phi = (\mathrm{pr}_1 \vee \alpha)^{-1}(\mathrm{pr}_1 \vee 0)^{1/2}\chi$ and $U = A\phi(A^*A)$.   As the domain G of χ is a full set and $\mathrm{pr}_1\phi^2 = \chi = \chi^2$ on G, we have

$$\begin{aligned}
U^*U &= \phi(A^*A)A^*A\phi(A^*A) \\
&= A^*A\phi(A^*A)^2 \\
&= (\mathrm{pr}_1\phi^2)(A^*A) \\
&= \chi(A^*A) ,
\end{aligned}$$

and $\chi(A^*A) = \chi(A^*A)^2$. (Note that A^*A commutes with every oper-
ator $f(A^*A)$ defined by the functional calculus for A^*A.) Hence
$\chi(A^*A)$ is a projection, and U is a partial isometry.

Throughout the full set $F \cap G$, we have $|\mathrm{pr}_1(1 - \phi\mathrm{pr}_1^{1/2})| \leqq \alpha$ and
$\mathrm{pr}_1^{1/2}\phi = \chi$. Hence, by [3, Ch. 6, 6.1], α is a bound for $A^*A(I - \phi(A^*A)|A|)$, and $|A|\phi(A^*A) = \chi(A^*A)$. For each $x \in H$ with $\|x\| \leqq 1$, we therefore have

$$\|(A - U|A|)x\|^2$$
$$= \langle (A^* - |A| U^*)(A - U|A|)x, x \rangle$$
$$\leqq |\langle A^*A(I - \phi(A^*A)|A|)x, x \rangle| + |\langle (|A|\phi(A^*A)A^*A - U^*U|A|^2)x, x \rangle|$$
$$\leqq \alpha + |\langle (|A|\phi(A^*A) - \chi(A^*A))A^*Ax, x \rangle|$$
$$< \varepsilon.$$

Thus ε is a bound for $A - U|A|$. Similar considerations, the details
of which we omit, enable us to show that ε is a bound for $|A| - U^*A$. $\square$

An element A of $\mathscr{A}(H)$ is said to be a *Hilbert-Schmidt operator*
if $\sum_{n=1}^{\infty} \|Aa_n\|^2$ converges; in which case the sum of this series is
independent of the orthonormal basis (a_n), and we write $\|A\|_2$ for
$(\sum_{n=1}^{\infty} \|Aa_n\|^2)^{1/2}$. If A is a Hilbert-Schmidt operator, then so is A^*;
if also $B \in \mathscr{A}(H)$ and β is a bound for B, then AB and BA are
Hilbert-Schmidt operators, $\|AB\|_2 \leqq \beta\|A\|_2$ and $\|BA\|_2 \leqq \beta\|A\|_2$.

An operator $A \in \mathscr{A}(H)$ is said to be of *trace class* if $\|A\|_1 = \sum_{n=1}^{\infty} \langle |A|a_n, a_n \rangle$ converges. In that case, $\|A\|_1 = \sum_{n=1}^{\infty} \||A|^{1/2}a_n\|^2$; so
that $|A|^{1/2}$ is a Hilbert-Schmidt operator and $\|A\|_1$ is independent of
the orthonormal basis (a_n). Linear combinations of trace class
operators are also of trace class.

Before discussing trace class operators any further, it is con-
venient to state the following simple consequence of the Cauchy-
Schwarz inequalities in H and C^n.

LEMMA 1.2. *Let A be a positive operator, R and S elements of
$\mathscr{L}(H)$, and p, q positive integers with $q > p$. Then*

$$\sum_{p+1}^{q} |\langle A^{1/2}Ra_n, A^{1/2}Sa_n \rangle|$$
$$\leqq \left(\sum_{p+1}^{q} \|A^{1/2}Ra_n\|^2 \right)^{1/2} \left(\sum_{p+1}^{q} \|A^{1/2}Sa_n\|^2 \right)^{1/2}. \qquad \square$$

Let A be of trace class, let $B \in \mathscr{A}(H)$, and let β be a bound
for B. Given $\varepsilon > 0$, construct partial isometries U, V so that ε is
a bound for both $|AB| - U^*AB$ and $A - V|A|$. For $q > p$, we
have

$$\sum_{p+1}^{q} \langle |AB| a_n, a_n \rangle$$

$$\leqq \sum_{p+1}^{q} |\langle (|AB| - U^*AB)a_n, a_n \rangle| + \sum_{p+1}^{q} |\langle U^*(A - V|A|)Ba_n, a_n \rangle|$$

$$+ \sum_{p+1}^{q} |\langle U^*V|A|Ba_n, a_n \rangle|$$

$$\leqq \varepsilon(1 + \beta)(q - p) + \sum_{p+1}^{q} |\langle |A|^{1/2}Ba_n, |A|^{1/2}V^*Ua_n \rangle|$$

$$\leqq \varepsilon(1 + \beta)(q - p) + \Big(\sum_{n=p+1}^{q} \||A|^{1/2}Ba_n\|^2 \Big)^{1/2} \Big(\sum_{n=p+1}^{q} \||A|^{1/2}V^*Ua_n\|^2 \Big)^{1/2}$$

$$\leqq \varepsilon(1 + \beta)(q - p) + \Big(\sum_{p+1}^{q} \||A|^{1/2}Ba_n\|^2 \Big)^{1/2} \||A|^{1/2}V^*U\|_2 .$$

As $\||A|^{1/2}V^*U\|_2 \leqq \||A|^{1/2}\|_2 = \|A\|_1^{1/2}$, and as $\varepsilon > 0$ is arbitrary, it follows that

$$(1.3) \qquad \sum_{p+1}^{q} \langle |AB| a_n, a_n \rangle \leqq \|A\|_1^{1/2} \Big(\sum_{p+1}^{q} \||A|^{1/2}Ba_n\|^2 \Big)^{1/2} \qquad (q > p) .$$

As $|A|^{1/2}B$ is a Hilbert-Schmidt operator, we see that $\sum_1^{\infty} \langle |AB| a_n, a_n \rangle$ converges, and hence that AB is of trace class. Moreover, $\|AB\|_1 \leqq \|A\|_1^{1/2} \||A|^{1/2}B\|_2 \leqq \beta \|A\|_1$. A similar argument using approximate polar decompositions shows that

$$(1.4) \qquad \sum_{p+1}^{q} \langle |BA| a_n, a_n \rangle \leqq \beta \|A\|_1^{1/2} \Big(\sum_{p+1}^{q} \||A|^{1/2}a_n\|^2 \Big)^{1/2} \qquad (q > p) .$$

Hence BA is of trace class, and $\|BA\|_1 \leqq \beta \|A\|_1$.

Taking $B = I$ in (1.3), we obtain

$$(1.5) \quad \sum_{p+1}^{q} |\langle Aa_n, a_n \rangle| \leqq \sum_{p+1}^{q} \langle |A| a_n, a_n \rangle \leqq \|A\|_1^{1/2} \Big(\sum_{p+1}^{q} \langle |A| a_n, a_n \rangle \Big)^{1/2}$$

$$(q > p) .$$

It follows that $\sum_1^{\infty} \langle Aa_n, a_n \rangle$ converges absolutely and that $\sum_{n=1}^{\infty} |\langle Aa_n, a_n \rangle| \leqq \|A\|_1$. Writing $A = (|A| + A)/2 - (|A| - A)/2$, a linear combination of positive operators of trace class, we see that

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} \langle Aa_n, a_n \rangle ,$$

the *trace* of A, is independent of the orthonormal basis (a_n). If A is of trace class and $B \in \mathscr{L}(H)$, then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ [6, p. 37].

If A, B are operators of Hilbert-Schmidt type, then an argument using the approximate polar decompositions of AB and A shows that

$$\sum_{p+1}^{q} \langle |AB| a_n, a_n \rangle \leqq \Big(\sum_1^{\infty} \|A^*a_n\|^2 \Big)^{1/2} \Big(\sum_{p+1}^{q} \|Ba_n\|^2 \Big)^{1/2} \qquad (q > p) ;$$

whence AB is of trace class. Similarly, BA is of trace class. Writing $Ba_n = \sum_{k=1}^{\infty} \langle Ba_n, a_k \rangle a_k$, we easily prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

If A is of trace class, then $B \to \mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ is a linear mapping of $\mathscr{A}(H)$ into C. This mapping is uniformly continuous on $\mathscr{A}_1(H)$ with respect to the weak operator norm, and so extends to a linear functional ϕ_A on $\mathscr{L}(H)$ that is weak operator uniformly continuous on $\mathscr{L}_1(H)$. As $\mathscr{L}_1(H)$ is weak operator totally bounded, ϕ_A is *normable*: that is, $\|\phi_A\| = \sup\{|\phi_A(B)|: B \in \mathscr{L}_1(H)\}$ is computable.

For each B in $\mathscr{A}_1(H)$,

$$|\phi_A(B)| \leqq \sum_1^{\infty} |\langle ABa_n, a_n \rangle| \leqq \|AB\|_1 \leqq \|A\|_1 .$$

On the other hand, given ε in $(0, 1)$, and computing in turn a positive integer ν and a partial isometry U such that $\sum_{\nu+1}^{\infty} \langle |A|a_n, a_n \rangle < \varepsilon^2$ and $\varepsilon\nu^{-1}$ is a bound for $|A| - U^*A$, we have (by (1.4))

$$|\phi_A(U^*)| = \left| \sum_{n=1}^{\infty} \langle U^*Aa_n, a_n \rangle \right|$$

$$\geqq \|A\|_1 - \sum_{n=1}^{\nu} |\langle (|A| - U^*A)a_n, a_n \rangle|$$

$$- \sum_{n=\nu+1}^{\infty} \langle |A|a_n, a_n \rangle - \sum_{n=\nu+1}^{\infty} |\langle U^*Aa_n, a_n \rangle|$$

$$\geqq \|A\|_1 - \varepsilon - \varepsilon^2 - \|A\|_1^{1/2} \left(\sum_{\nu+1}^{\infty} \| |A|^{1/2}a_n \|^2 \right)^{1/2}$$

$$\geqq \|A\|_1 - 2\varepsilon - \|A\|_1^{1/2} \left(\sum_{\nu+1}^{\infty} \langle |A|a_n, a_n \rangle \right)^{1/2}$$

$$\geqq \|A\|_1 - \varepsilon(2 + \|A\|_1^{1/2}) .$$

It now follows that $\|\phi_A\| = \|A\|_1$.

The set of trace class operators on H, taken with the *trace class norm* $\| \|_1$, is a normed linear space over C. We prove that this space is $\| \|_1$-complete. Let $(A_k)_{k \geqq 1}$ be a $\| \|_1$-Cauchy sequence of trace class operators. As $\|A_j - A_k\|_1$ is a bound for $A_j - A_k$, $(A_k)_{k \geqq 1}$ converges in bound (that is, uniformly on the unit ball of H) to an element A of $\mathscr{L}(H)$; as $\|A_j - A_k\|_1$ is also a bound for $A_j^* - A_k^*$, $(A_k^*)_{k \geqq 1}$ converges in bound to an element of $\mathscr{L}(H)$ which is clearly the adjoint of A. Given $\varepsilon > 0$, compute $k(\varepsilon)$ so that $\|A_j - A_k\|_1 \leqq \varepsilon$ whenever $j, k \geqq k(\varepsilon)$. If U is a partial isometry and $q > p$, we have

$$\sum_{p+1}^{q} |\langle U^*A_ka_n, a_n \rangle| \leqq \|A_k - A_{k(\varepsilon)}\|_1 + \sum_{p+1}^{q} |\langle U^*A_{k(\varepsilon)}a_n, a_n \rangle|$$

for each $k \geqq 1$, and therefore

$$\sum_{\nu+1}^{q} \langle |A|a_n, a_n \rangle$$

$$\leqq \sum_{p+1}^{q} |(\langle |A| - U^*A)a_n, a_n\rangle| + \lim_{k\to\infty} \sum_{p+1}^{q} |\langle U^*A_k a_n, a_n\rangle|$$

$$\leqq \sum_{p+1}^{q} |\langle (|A| - U^*A)a_n, a_n\rangle| + \varepsilon + \sum_{p+1}^{q''} |\langle U^*A_{k(\varepsilon)}a_n, a_n\rangle|$$

$$\leqq \sum_{p+1}^{q} |\langle (|A| - U^*A)a_n, a_n\rangle| + \varepsilon + \sum_{p+1}^{q} \langle U^*A_{k(\varepsilon)}| a_n, a_n\rangle$$

$$\leqq \sum_{p+1}^{q} |\langle (|A| - U^*A)a_n, a_n\rangle| + \varepsilon + \|A_{k(\varepsilon)}\|_1^{1/2} \left(\sum_{p+1}^{q} \||A_{k(\varepsilon)}|^{1/2}a_n\|^2 \right)^{1/2}$$

the last step following from (1.4). In view of (1.1), we have

$$\sum_{p+1}^{q} \langle |A| a_n, a_n\rangle \leqq \varepsilon + \|A_{k(\varepsilon)}\|_1^{1/2} \left(\sum_{p+1}^{q} \| |A_{k(\varepsilon)}|^{1/2}a_n\|^2 \right)^{1/2} \quad (q > p).$$

It follows that A is of trace class. On the other hand, if $\nu \geqq 1$ and $k \geqq k(\varepsilon)$, we have

$$\sum_{1}^{\nu} |\langle U^*(A_j - A_k)a_n, a_n\rangle| \leqq \|A_j - A_k\|_1 \leqq \varepsilon \quad (j \geqq k(\varepsilon)),$$

and therefore

$$\sum_{1}^{\nu} \langle |A - A_k| a_n, a_n\rangle$$

$$\leqq \sum_{1}^{\nu} |\langle (|A - A_k| - U^*(A - A_k))a_n, a_n\rangle|$$

$$+ \lim_{j\to\infty} \sum_{1}^{\nu} |\langle U^*(A_j - A_k)a_n, a_n\rangle|$$

$$\leqq \sum_{1}^{\nu} |\langle (|A - A_k| - U^*(A - A_k))a_n, a_n\rangle| + \varepsilon.$$

It follows from (1.1) that $\sum_{1}^{\nu} \langle |A - A_k| a_n, a_n\rangle \leqq \varepsilon$ whenever $k \geqq k(\varepsilon)$ and $\nu \geqq 1$, and therefore that $\lim_{k\to\infty} \|A - A_k\|_1 = 0$.

2. Let $\mathscr{R}$ be a linear subset of $\mathscr{L}(H)$, and ϕ a linear mapping of $\mathscr{R}$ into C. We say that ϕ is *ultraweakly* (resp. *ultrastrongly*) *continuous* on $\mathscr{R}$ if ϕ is uniformly continuous on $\mathscr{R} \cap \mathscr{L}_1(H)$ with respect to the weak (resp. strong) operator norm. If ϕ is ultraweakly continuous and $\mathscr{A}(H) \subset \mathscr{R}$, then ϕ extends to an ultraweakly continuous linear functional on $\mathscr{L}(H)$, and this extended functional is normable.

Classically, our condition of ultraweak continuity is equivalent to the pointwise continuity of ϕ with respect to the weak operator topology on $\mathscr{R}$. As uniform, rather than pointwise, continuity seems to be necessary for successful computation, and as there is no known constructive proof of the Uniform Continuity Theorem, we are obliged to build uniform continuity on $\mathscr{R} \cap \mathscr{L}_1(H)$ into our definition.

We say that a linear mapping $\phi\colon \mathscr{R} \to C$ is *positive*, and write $\phi \geqq 0$, if $\phi(A) \geqq 0$ for each positive A in $\mathscr{R}$. We then have $\phi(A^*) = \phi(A)^*$ for each A in $\mathscr{R} \cap \mathscr{A}(H)$. If ϕ is positive and A, B belong to $\mathscr{R} \cap \mathscr{A}(H)$, we have the following *Cauchy-Schwarz inequality*: $|\phi(A^*B)|^2 \leqq \phi(A^*A)\phi(B^*B)$. Moreover, if $\mathscr{A}(H) \subset \mathscr{R}$ then ϕ is also bounded; in fact, it is normable, and $\|\phi\| = \phi(I)$.

If $I \in \mathscr{R}$, and $\phi(I) = \sum_{n=1}^{\infty} \phi(P_n)$ whenever $(P_n)_{n \geqq 1}$ is a sequence of pairwise orthogonal projections in $\mathscr{R}$ whose sum in the strong operator norm is I, then we say that ϕ is *countably additive*.

We now give constructive proofs of certain relations between the various types of linear mapping defined above.

THEOREM 2.1. *The following are equivalent conditions on a positive linear functional ϕ on $\mathscr{A}(H)$:*
  (i) *ϕ is countably additive.*
  (ii) *ϕ is ultrastrongly continuous.*
  (iii) *ϕ is ultraweakly continuous.*
  (iv) *there exists a positive operator A of trace class such that $\phi(S) = \mathrm{Tr}(SA)$ for each S in $\mathscr{A}(H)$.*
  (v) *there exists a sequence $(x_n)_{n \geqq 1}$ in H such that $\sum_{n=1}^{\infty} \|x_n\|^2$ converges, and $\phi(S) = \sum_{n=1}^{\infty} \langle Sx_n, x_n \rangle$ for each S in $\mathscr{A}(H)$.*

Proof. Although several parts of the following proof are based on fairly well-known classical arguments, we shall give most of the details of the proof, for the sake of completeness. For each $n \geqq 1$, let P_n be the projection of H on Ca_n and let $Q_n = \sum_{k=1}^{n} P_k$. Then Q_n is the projection of H on the span of $\{a_1, \cdots, a_n\}$. Define also elements $S_{j,k}$ of $\mathscr{A}(H)$ by setting $S_{j,k}x = \langle x, a_j \rangle a_k$ whenever $x \in H$, $j \geqq 1$ and $k \geqq 1$. Note that, if $A \in \mathscr{L}(H)$ and $n \geqq 1$, then $P_n A P_n = \langle Aa_n, a_n \rangle P_n$ and $Q_n A Q_n = \sum_{j,k=1}^{n} \langle Aa_j, a_k \rangle S_{j,k}$.

(i) $\Rightarrow$ (ii) Suppose that ϕ is countably additive. Define linear mappings $\phi_n \colon \mathscr{A}(H) \to C$ by setting $\phi_n(S) = \sum_{k=1}^{n} \phi(SP_k)$ for each S in $\mathscr{A}(H)$ and each $n \geqq 1$. Given S in $\mathscr{A}(H)$ and using the Cauchy-Schwarz inequality for positive linear functionals, we have

$$|\phi_n(S)| \leqq \sum_{k=1}^{n} \phi(I)^{1/2} \phi(P_k S^* S P_k)^{1/2}$$

$$= \phi(I)^{1/2} \sum_{k=1}^{n} \phi(P_k)^{1/2} \langle S^* S a_k, a_k \rangle^{1/2}$$

$$\leqq \phi(I)^{1/2} \sum_{k=1}^{n} \phi(I)^{1/2} \|Sa_k\|$$

$$= \phi(I) \sum_{k=1}^{n} \|Sa_k\| .$$

It readily follows from this that ϕ_n is ultrastrongly continuous. Moreover, for $S \in \mathscr{A}_1(H)$

$$|\phi(S) - \phi_n(S)| = \left| \phi\left(S\left(I - \sum_{k=1}^{n} P_k \right) \right) \right|$$

$$\leqq \phi(SS^*)^{1/2} \left(\phi(I) - \sum_{k=1}^{n} \phi(P_k) \right)^{1/2}$$

$$\leqq \phi(I)^{1/2} \left(\phi(I) - \sum_{k=1}^{n} \phi(P_k) \right)^{1/2} .$$

As ϕ is countably additive, it follows that $\phi_n(S)$ converges to $\phi(S)$ uniformly on $\mathscr{A}_1(H)$; whence ϕ is ultrastrongly continuous.

(ii) $\Rightarrow$ (iii)   Suppose ϕ is ultrastrongly continuous. Let δ be a modulus of continuity for ϕ on $(\mathscr{A}_1(H), \| \ \|_s)$, and let $\varepsilon > 0$. Compute $\nu \geqq 1$ so that $\phi(I - Q_\nu) \leqq \varepsilon^2/9(1 + \phi(I))$ and $\sum_{k=\nu+1}^{\infty} 2^{-k} \leqq \delta(\varepsilon/3)$. Given S in $\mathscr{A}_1(H)$, we have $\| S(I - Q_\nu) \|_s \leqq \delta(\varepsilon/3)$, and therefore $|\phi(S(I - Q_\nu))| \leqq \varepsilon/3$. Also,

$$|\phi((I - Q_\nu)SQ_\nu)|^2 \leqq \phi(I - Q_\nu)\phi(Q_\nu S^* S Q_\nu)$$

$$\leqq \phi(I - Q_\nu)\phi(I)$$

$$\leqq \varepsilon^2/9 ,$$

and so $|\phi((I - Q_\nu)SQ_\nu)| \leqq \varepsilon/3$. Finally, as all norms on the finite-dimensional space $\mathscr{L}(Q_\nu(H))$ are equivalent, there exists $\kappa > 0$ such that, for all S in $\mathscr{L}(H)$,

$$\| Q_\nu S Q_\nu \|_s = \sum_{k=1}^{\nu} 2^{-k} \| Q_\nu S a_k \| \leqq \kappa \sum_{j,k=1}^{\nu} 2^{-j-k} |\langle Q_\nu S a_j, a_k \rangle| \leqq \kappa \| S \|_w .$$

Taken with the identity

$$\phi(S) = \phi(Q_\nu S Q_\nu) + \phi(S(I - Q_\nu)) + \phi((I - Q_\nu)SQ_\nu) ,$$

all this shows that $|\phi(S)| \leqq \varepsilon$ whenever $S \in \mathscr{A}_1(H)$ and $\| S \|_w \leqq \kappa^{-1}\delta(\varepsilon/3)$; whence ϕ is ultraweakly continuous.

(iii) $\Rightarrow$ (iv)   Suppose that ϕ is ultraweakly continuous. For each positive integer n, define $A(n) = \sum_{j,k=1}^{n} \phi(S_{k,j})S_{j,k}$. Then $A(n)$ is a selfadjoint, positive operator of trace class. For each S in $\mathscr{A}(H)$, $\phi(Q_n S Q_n) = \phi_{A(n)}(S)$, and so

$$|\phi(S) - \phi_{A(n)}(S)| \leqq |\phi(S(I - Q_n))| + |\phi((I - Q_n)SQ_n)| .$$

As $\| \ \|_w \leqq \| \ \|_s$, ϕ is ultrastrongly continuous, and an argument used above shows that $\| \phi - \phi_{A(n)} \| \to 0$ as $n \to \infty$. As the linear functional $\phi - \phi_{A(j)}$ is ultraweakly continuous, it is normable. Moreover,

$$\| A(j) - A(k) \|_1 = \| \phi_{A(j) - A(k)} \|$$
$$= \| \phi_{A(j)} - \phi_{A(k)} \|$$
$$\leqq \| \phi - \phi_{A(j)} \| + \| \phi - \phi_{A(k)} \| \,,$$

and so $(A(k))_{k \geqq 1}$ is a $\| \; \|_1$-Cauchy sequence. Thus there exists a trace class operator A such that $\lim_{k \to \infty} \| A - A(k) \|_1 = 0$. As $\| A - A(k) \|_1$ is a bound for $A - A(k)$, $(A(k))_{k \geqq 1}$ converges to A in bound; so that A is selfadjoint and positive. Finally, as

$$\| \phi - \phi_A \| \leqq \| \phi - \phi_{A(k)} \| + \| \phi_{A - A(k)} \|$$
$$= \| \phi - \phi_{A(k)} \| + \| A - A(k) \|_1 \,,$$

we have $\| \phi - \phi_A \| = 0$, $\phi = \phi_A$, and therefore $\phi(S) = \mathrm{Tr}(SA)$ for each S in $\mathscr{A}(H)$.

(iv) $\Rightarrow$ (v) Suppose that (iv) obtains, and let $S \in \mathscr{A}(H)$. Then $A^{1/2}$ and $SA^{1/2}$ are Hilbert-Schmidt operators, and so $A^{1/2}SA^{1/2}$ is of trace class. Moreover,

$$\phi(S) = \mathrm{Tr}(SA^{1/2}A^{1/2})$$
$$= \mathrm{Tr}(A^{1/2}SA^{1/2})$$
$$= \sum_{n=1}^{\infty} \langle A^{1/2}SA^{1/2}a_n, a_n \rangle$$
$$= \sum_{n=1}^{\infty} \langle Sx_n, x_n \rangle \,,$$

where $x_n = A^{1/2}a_n$ and $\sum_{n=1}^{\infty} \| x_n \|^2 = \mathrm{Tr}(A)$ converges.

(v) $\Rightarrow$ (i) We omit the comparatively trivial details of this part of the proof. $\square$

COROLLARY 2.2. *If ϕ is a countably additive, positive linear functional on $\mathscr{L}(H)$, then ϕ is of the form $S \to \sum_{n=1}^{\infty} \langle Sx_n, x_n \rangle$, where each x_n belongs to H and $\sum_{n=1}^{\infty} \| x_n \|^2$ converges.* $\square$

In classical mathematics, the expression of a compact operator A in diagonal form enables us to add the following to the list of equivalent conditions in (2.2):

(vi) there exists an orthonormal basis $(e_n)_{n \geqq 1}$ of H, and a sequence $(\lambda_n)_{n \geqq 1}$ of nonnegative numbers, such that $\phi(S) = \sum_{n=1}^{\infty} \lambda_n \langle Se_n, e_n \rangle$ for each S in $\mathscr{A}(H)$.

It is easy to see that, if (iv) implies (vi), then the trace class operator A in (iv) can be expressed in diagonal form. The following argument shows that, even in a two-dimensional Hilbert space, this

expression may not be possible by constructive means.

Let x and y be nonnegative numbers with $xy = 0$ and $y < 1$, and let A be the operator on C^2 with matrix $\begin{bmatrix} 1+x & y \\ y & 1+y \end{bmatrix}$. Then A is a positive, compact operator on the Hilbert space C^2. Suppose that A can be expressed in diagonal form; so that there exist complex numbers λ, u, v such that $|u|^2 + |v|^2 \neq 0$ and $\begin{bmatrix} 1+x & y \\ y & 1+y \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \lambda\begin{bmatrix} u \\ v \end{bmatrix}$. Then $(1+x)u + yv = \lambda u$, $yu + (1+y)v = \lambda v$, $(1+x)uv + yv^2 = \lambda uv = yu^2 + (1+y)uv$, and so

(*) $$xuv + yv^2 = yu^2 + yuv .$$

Multiplying (*) through by x we obtain $x^2 uv = 0$. On the other hand, multiplying (*) through by y, we obtain $y^2(u^2 + uv - v^2) = 0$. If $u \neq 0$, then we have either $v \neq 0$, in which case $x^2 = 0$ and so $x = 0$; or $|v| < |u|/2$. In the latter case, we have

$$|u^2 + uv - v^2| \geqq |u|^2 - |u||v| - |v|^2 \geqq |u|^2/4 ,$$

whence $y^2 = 0$, and therefore $y = 0$. If $v \neq 0$, then, replacing u by v in this argument, we again find that either $x = 0$ or $y = 0$. It follows from all this that if we have constructive means of expressing any positive operator on the Hilbert space C^2 in diagonal form, then we have a constructive proof of the essentially nonconstructive proposition

$$\forall x \geqq 0 \quad \forall y \geqq 0 \quad (xy = 0 \Longrightarrow (x = 0 \quad \text{or} \quad y = 0)) .$$

We conclude that a constructive proof that (iv) implies (vi) is unlikely to materialize.

3. In classical mathematics, there is a well-known analogy between countably additive, positive linear functionals on $\mathscr{L}(H)$ and positive integrals on $C(X)$, where X is a locally compact metric space. In this analogy, projections correspond to characteristic functions of integrable sets, positive operators to nonnegative integrable functions, and expressions of the form $\langle Ax, x \rangle$ to values of integrable functions.

In the constructive integration theory of Bishop and Cheng ([2], [4, Ch. 5]), countable additivity of an integral μ is replaced by the stronger property:

(†) *If f_0 is an integrable function, and $(f_n)_{n \geqq 1}$ is a sequence of nonnegative integrable functions such that $\sum_{n=1}^{\infty} \mu(f_n)$ is convergent and less than $\mu(f_0)$, then there exists x such that $\sum_{n=1}^{\infty} f_n(x)$ is convergent and less than $f_0(x)$.*

It is very easy to obtain the corresponding property of countably additive, positive linear functionals:

PROPOSITION 3.1. *Let ϕ be a countably additive, positive linear functional on $\mathscr{L}(H)$, A_0 an element of $\mathscr{L}(H)$, and $(A_n)_{n \geq 1}$ a sequence of positive, selfadjoint elements of $\mathscr{L}(H)$ such that $\sum_{n=1}^{\infty} \phi(A_n)$ is convergent and less than $\phi(A_0)$. Then there exists a unit vector x in H such that $\sum_{n=1}^{\infty} \langle A_n x, x \rangle$ is convergent and less than $\langle A_0 x, x \rangle$.*

Proof. By 2.2, ϕ is of the form $S \to \sum_{k=1}^{\infty} \langle S x_k, x_k \rangle$, where each x_k belongs to H and $\sum_{k=1}^{\infty} \|x_k\|^2$ converges. Thus $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle A_n x_k, x_k \rangle$ converges to $\sum_{n=1}^{\infty} \phi(A_n)$. As each term $\langle A_n x_k, x_k \rangle$ is nonnegative for $n \geq 1$, it follows that $\sum_{n=1}^{\infty} \langle A_n x_k, x_k \rangle$ converges for each k, and that $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \langle A_n x_k, x_k \rangle$ converges to $\sum_{n=1}^{\infty} \phi(A_n)$. Hence

$$\sum_{k=1}^{\infty} \left(\langle A_0 x_k, x_k \rangle - \sum_{n=1}^{\infty} \langle A_n x_k, x_k \rangle \right) = \phi(A_0) - \sum_{n=1}^{\infty} \phi(A_n) > 0 \ ,$$

and so there exists k with $\sum_{n=1}^{\infty} \langle A_n x_k, x_k \rangle < \langle A_0 x_k, x_k \rangle$. It only remains to set $x = \|x_k\|^{-1} x_k$. $\qquad \square$

In the special case where each f_n is the characteristic function of an integrable set, the Bishop-Cheng property ($\dagger$) of the integral μ enables us to construct x such that $f_0(x) = 1$, and $f_n(x) = 0$ for each $n \geq 1$. Our proof of the analogue of this property for countably additive, positive linear functionals requires the following lemma.

LEMMA 3.2. *If $P_1, \cdots, P_n$ are pairwise commuting projections on H, then $I - \prod_{k=1}^{n} (I - P_k) \leqq \sum_{k=1}^{n} P_k$.*

Proof. For $n = 1$ this is trivial. To complete an inductive proof, we observe that if $I - \prod_{k=1}^{m} (I - P_k) \leqq \sum_{k=1}^{m} P_k$ for some positive integer m, and if $x \in H$, then

$$\left\langle \left(I - \prod_{k=1}^{m+1} (I - P_k) \right) x, x \right\rangle$$

$$= \left\langle \left(I - \prod_{k=1}^{m} (I - P_k) \right) x, x \right\rangle + \left\langle P_{m+1} \left(\prod_{k=1}^{m} (I - P_k) \right) x, x \right\rangle$$

$$\leqq \left\langle \sum_{k=1}^{m} P_k x, x \right\rangle + \left\| \left(\prod_{k=1}^{m} (I - P_k) \right) P_{m+1} x \right\|^2$$

$$\leqq \left\langle \sum_{k=1}^{m} P_k x, x \right\rangle + \| P_{m+1} x \|^2$$

$$= \left\langle \sum_{k=1}^{m} P_k x, x \right\rangle + \langle P_{m+1} x, x \rangle$$

$$= \left\langle \sum_{k=1}^{m+1} P_k x, x \right\rangle . \qquad \square$$

THEOREM 3.3. *Let ϕ be a countably additive, positive linear functional on $\mathscr{L}(H)$ and $(P_n)_{n\geq 0}$ and sequence of pairwise commuting projections on H such that $\sum_{n=1}^{\infty} \phi(P_n)$ is convergent and less then $\phi(P_0)$. Then there exists a unit vector x in H such that $P_0 x = x$, and $P_n x = 0$ for each $n \geq 1$.*

Proof. We first consider the special case where there exists ξ in H such that $\phi(R) = \langle R\xi, \xi \rangle$ for each R in $\mathscr{L}(H)$. If m, n are integers with $m > n \geq 1$, then

$$\left\| \left(\prod_{k=1}^{m} (I - P_k) \right)\xi - \left(\prod_{k=1}^{n} (I - P_k) \right)\xi \right\|^2$$

$$= \left\| \left(\prod_{j=1}^{n} (I - P_j) \right)\left(I - \prod_{k=n+1}^{m} (I - P_k) \right)\xi \right\|^2$$

$$\leq \left\| \left(I - \prod_{k=n+1}^{m} (I - P_k) \right)\xi \right\|^2$$

$$= \left\langle \left(I - \prod_{k=n+1}^{m} (I - P_k) \right)\xi, \xi \right\rangle$$

$$\leq \left\langle \sum_{k=n+1}^{m} P_k\xi, \xi \right\rangle$$

$$= \sum_{k=n+1}^{m} \langle P_k\xi, \xi \rangle,$$

the second last step following from 3.1. As $\sum_{k=1}^{\infty} \langle P_k\xi, \xi \rangle = \sum_{k=1}^{\infty} \phi(P_k)$ is convergent, we see that $((\prod_{k=1}^{n} (I - P_k))\xi)_{n \geq 1}$ is a Cauchy sequence in H; whence we can compute

$$\eta \equiv \left(\prod_{k=1}^{\infty} (I - P_k) \right)\xi = \lim_{n \to \infty} \left(\prod_{k=1}^{n} (I - P_k) \right)\xi.$$

For each $n \geq 1$, we clearly have $P_n P_0 \eta = P_0 P_n \eta = 0$. On the other hand, writing

$$\alpha \equiv \left(\|P_0\xi\| + \left\| \left(I - \prod_{k=1}^{n} (I - P_k) \right)\xi \right\| \right)^{-1},$$

we have

$$\left\langle \left(\prod_{k=1}^{n} (I - P_k) \right)\xi, P_0\xi \right\rangle$$

$$= \langle \xi, P_0\xi \rangle - \left\langle \left(I - \prod_{k=1}^{n} (I - P_k) \right)\xi, P_0\xi \right\rangle$$

$$\geq \| P_0\xi \|^2 - \left\| \left(I - \prod_{k=1}^{n} (I - P_k) \right)\xi \right\| \| P_0\xi \|$$

$$= \| P_0\xi \|\left(\| P_0\xi \| - \left\| \left(I - \prod_{k=1}^{n} (I - P_k) \right)\xi \right\| \right)$$

$$= \| P_0\xi \| \, \alpha\left(\| P_0\xi \|^2 - \left\| \left(I - \prod_{k=1}^{n} (I - P_k) \right)\xi \right\|^2 \right)$$

$$= \| P_0 \xi \| \, \alpha \Big(\langle P_0 \xi, \, \xi \rangle - \Big\langle \Big(I - \prod_{k=1}^{n} (I - P_k) \Big) \xi, \, \xi \Big\rangle \Big)$$

$$\geqq \| P_0 \xi \| \, \alpha \Big(\langle P_0 \xi, \, \xi \rangle - \sum_{k=1}^{n} \langle P_k \xi, \, \xi \rangle \Big) \qquad \text{by 3.2}$$

$$\geqq \| P_0 \xi \| \, \alpha \Big(\phi(P_0) - \sum_{k=1}^{\infty} \phi(P_k) \Big) .$$

Hence

$$\langle P_0 \eta, \, \xi \rangle = \langle \eta, \, P_0 \xi \rangle = \lim_{n \to \infty} \Big\langle \Big(\prod_{k=1}^{n} (I - P_k) \Big) \xi, \, P_0 \xi \Big\rangle$$

$$\geqq \| P_0 \xi \| \, \alpha \Big(\phi(P_0) - \sum_{k=1}^{\infty} \phi(P_k) \Big)$$

$$> 0 .$$

The Cauchy-Schwarz inequality now ensures that $P_0 \eta \neq 0$. To complete the proof of our special case of the theorem, it only remains to take $x = \| P_0 \eta \|^{-1} P_0 \eta$.

To handle the general case, we first apply 2.2 to construct a sequence $(\xi_k)_{k \geqq 1}$ in H such that $\sum_{k=1}^{\infty} \| \xi_k \|^2$ is convergent and $\phi(R) = \sum_{k=1}^{\infty} \langle R \xi_k, \, \xi_k \rangle$ for each R in $\mathscr{L}(H)$. Let H_∞ be the direct sum of a sequence of copies of H, with the standard scalar product, and let $\boldsymbol{\xi} = (\xi_k)_{k \geqq 1}$. For each projection P on H and each $\boldsymbol{x} = (x_k)_{k \geqq 1}$ in H_∞, define $\boldsymbol{P} \boldsymbol{x} = (P x_k)_{k \geqq 1}$. Then $\boldsymbol{P}$ is a projection on H_∞ and $\langle \boldsymbol{P} \boldsymbol{\xi}, \, \boldsymbol{\xi} \rangle = \sum_{k=1}^{\infty} \langle P \xi_k, \, \xi_k \rangle = \phi(P)$. Thus

$$\sum_{n=1}^{\infty} \langle \boldsymbol{P}_n \boldsymbol{\xi}, \, \boldsymbol{\xi} \rangle = \sum_{n=1}^{\infty} \phi(P_n) < \phi(P_0) = \langle \boldsymbol{P}_0 \boldsymbol{\xi}, \, \boldsymbol{\xi} \rangle .$$

As $(\boldsymbol{P}_n)_{n \geqq 0}$ is a sequence of pairwise commuting projections on H_∞, the first part of the proof shows that there exists $\boldsymbol{x} = (x_k)_{k \geqq 1}$ in H_∞ such that $\sum_{k=1}^{\infty} \| x_k \|^2 = \| \boldsymbol{x} \|^2 = 1$, $\boldsymbol{P}_0 \boldsymbol{x} = \boldsymbol{x}$, and $\boldsymbol{P}_n \boldsymbol{x} = (0)$ for each $n \geqq 1$. With k chosen so that $\| x_k \| \neq 0$, we complete the proof by setting $x = \| x_k \|^{-1} x_k$. $\qquad \square$

We end with the analogue of the fourth postulate for the Bishop-Cheng integral [2, p. 2]. This leaves undiscussed only the third postulate, whose analogue is the unnecessary restriction that $\| \phi \|$ be positive.

THEOREM 3.4. *Let ϕ be a countably additive, positive linear functional on $\mathscr{L}(H)$, and let A be a selfadjoint operator on H. Then $\lim_{n \to \infty} \phi(A \wedge nI) = \phi(A)$ and $\lim_{n \to \infty} \phi(|A| \wedge n^{-1}I) = 0$.* $\qquad \square$

Here, $A \wedge nI$ and $|A| \wedge n^{-1}I$ are to be interpreted by means of the functional calculus for A. The proof of 3.7 is a fairly straight-

forward application of that calculus, and is left to the reader.

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