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ON THE RELATION $PQ - QP = -iI$

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There is a variety of literature on the relationship between the two equations

(1.1) $$PQ - QP \subset -iI$$

(1.2) $$\exp(itP)\exp(isQ) = \exp(ist)\exp(isQ)\exp(itP)\ , \quad s, t \in R$$

where P and Q are self-adjoint operators on a Hilbert space. Von Neumann has characterized the solutions of (1.2) as those pairs (P, Q) which are unitarily equivalent to a direct sum of a number of copies of the Schrödinger pair (p, q) where p is $-i(d/dx)$ and q is multiplication by x on $L^2(R)$. Hence any pair which satisfies (1.2) and is irreducible in the obvious sense is unitarily equivalent to the Schrödinger pair. It is well known that any pair satisfying (1.2) satisfies (1.1) in the following strong sense:

(1.3) **there is a dense subspace Ω which is a core for both P and Q, is invariant under P and Q and $PQf - QPf = -if$ for all f in Ω.**

In this paper we construct an uncoutable family of irreducible, unitarily inequivalent pairs satisfying (1.3) but not (1.2).

A reducible pair satisfying (1.3) but not (1.2) is given in [4, page 275]. The construction is due to Nelson. It was a detaild examination of this pair which led us to our uncountable family of pairs. In [1], Fuglede constructs a pair (P_F, Q_F) satisfying a stronger version of (1.3) but not (1.2), but the irreducibility of the pair is left as an open question. The techniques in the analysis of our examples are applicable to (P_F, Q_F) and we show that this pair is irreducible. For a related example due to Fuglede, see [2, Example 2].

The operators constructed in [4] and [1] are obtained from pairs (X, Y) of self-adjoint operators satisfying

(1.4) there is a dense subspace Ω which is a core for both. X and Y, is invariant under X and Y and

$$XYf = YXf \quad \text{for all } f \text{ in } \Omega$$

but not

(1.5) the spectral projections of X and Y commute.

We also construct an uncountable family of unitarily inequivalent pairs of operators satisfying (1.4) but not (1.5). For other such pairs see [4, page 273, Example 1], [1] and [2] and [3, Example 3].

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2. The operators. Consider the Hilbert space $H = L^2(R^2)$. The Schrödinger 2-system $\{p_1, p_2; q_1, q_2\}$ is defined by the following strongly continuous unitary groups.

$$(2.1)\qquad \begin{cases}(\exp(itq_j)f)(x_1, x_2) = \exp(itx_j)f(x_1, x_2) & j = 1, 2\\ (\exp(itp_1)f)(x_1, x_2) = f(x_1 + t, x_2) & \\ (\exp(itp_2)f)(x_1, x_2) = f(x_1, x_2 + t)\ . & \end{cases}$$

For $\alpha \in A = \{z \in C: |z| = 1\}$ define $\phi_\alpha: R^2 \to C$ by

$$\phi_\alpha(x_1, x_2) = \begin{cases}\alpha & x_1 \leqq 0,\ x_2 \geqq 0\\ 1 & \text{elsewhere.}\end{cases}$$

Define the self-adjoint operators P_α, Q_α by the following strongly continuous unitary groups.

$$(2.2)\qquad \begin{cases}\exp(itP_\alpha) = \exp(itp_1)\\ \exp(itQ_\alpha) = \bar{\phi}_\alpha(q_1, q_2)\exp(itq_1)\exp(itp_2)\phi_\alpha(q_1, q_2)\ .\end{cases}$$

THEOREM 1. *The family $\{(P_\alpha, Q_\alpha): \alpha \in A\}$ satisfies the following properties*:

(a) *For each α there is a dense subspace*

$$\Omega_\alpha \subset \operatorname{dom} P_\alpha \cap \operatorname{dom} Q_\alpha \quad (\operatorname{dom} X \textit{ is the domain of } X)$$

with $P_\alpha\Omega_\alpha \subset \Omega_\alpha$, $Q_\alpha\Omega_\alpha \subset \Omega_\alpha$,

$$P_\alpha|_{\Omega_\alpha} \textit{ and } Q_\alpha|_{\Omega_\alpha} \textit{ are essentially self-adjoint.}$$

(b) *For each $\alpha \neq 1$, the pair (P_α, Q_α) is irreducible.*

(c) *For various α, the pairs (P_α, Q_α) are unitarily inequivalent.*

(d) *For each α, the pair (P_α, Q_α) satisfies* (1.1); *that is,*

$$P_\alpha Q_\alpha f - Q_\alpha P_\alpha f = -if \quad f \in \operatorname{dom} P_\alpha Q_\alpha \cap \operatorname{dom} Q_\alpha P_\alpha\ .$$

(e) *For each $\alpha \neq 1$ the pair (P_α, Q_α) does not satisfy* (1.2).

The operators P_α and Q_α are related to the self-adjoint operators X_α and Y_α defined by

$$(2.3)\qquad \begin{cases}\exp(itX_\alpha) = \exp(itp_1)\\ \exp(itY_\alpha) = \bar{\phi}_\alpha(q_1, q_2)\exp(itp_2)\phi_\alpha(q_1, q_2)\end{cases}$$

and we have the following result:

THEOREM 2. *The family* $\{(X_\alpha, Y_\alpha): \alpha \in A\}$ *satisfies properties* (a), (b) *and* (c) *of Theorem* 1. *In addition, for each* $f \in (\operatorname{dom} X_\alpha Y_\alpha) \cap (\operatorname{dom} Y_\alpha X_\alpha)$

(d)′ $X_\alpha Y_\alpha f = Y_\alpha X_\alpha f$.

REMARKS. Since the pair (X_α, Y_α) is irreducible for $\alpha \neq 1$ it is clear that their spectral projections cannot commute.

Proof of Theorem 1. Let Ω_α be the set of all $f \in L^2(R^2)$ such that

(1) $f \in C^\infty(R^2 \backslash \{(x_1, x_2): x_2 = 0, x_1 \leqq 0\})$

(2) $\alpha(D_1^j D_2^k f)(x_1, 0^+) = (D_1^j D_2^k f)(x_1, 0^-)$ $x_1 < 0,\ j, k = 0, 1, 2, \cdots$

(3) f is compactly supported.

It can be shown that:

$$\operatorname{dom} p_j = \{f \in L^2(R^2): \text{for almost all } x_k\ (k \neq j),\ (x_j \to f(x_1, x_2)) \text{ is absolutely continuous with } D_j f \in L^2(R^2)\}$$

$$(p_j f)(x_1, x_2) = -i(D_j f)(x_1, x_2)\ , \quad f \in \operatorname{dom} p_j$$

$$\operatorname{dom} q_j = \{f \in L^2(R^2): ((x_1, x_2) \longrightarrow x_j f(x_1, x_2)) \in L^2(R^2)\}$$

$$(q_j f)(x_1, x_2) = x_j f(x_1, x_2)\ , \quad f \in \operatorname{dom} q_j\ .$$

From these facts it is easily deduced that $\Omega_\alpha \subset \operatorname{dom} P_\alpha \cap \operatorname{dom} Q_\alpha$ and for $f \in \Omega_\alpha$ we have the following formulas for a.e. (x_1, x_2):

$$(P_\alpha f)(x_1, x_2) = -i(D_1 f)(x_1, x_2)$$

$$(Q_\alpha f)(x_1, x_2) = x_1 f(x_1, x_2) - i(D_2 f)(x_1, x_2)\ .$$

This shows that $P_\alpha \Omega_\alpha \subset \Omega_\alpha$ and $Q_\alpha \Omega_\alpha \subset \Omega_\alpha$. To prove the rest of part (a) we introduce the dense subspaces

$$\Omega'_\alpha = \{f \in \Omega_\alpha: (D_1^j D_2^k f)(0, x_2) = 0\ \forall x_2, \forall j, k = 0, 1, \cdots\}$$

$$\Omega''_\alpha = \{f \in \Omega_\alpha: (D_1^j D_2^k f)(x_1, 0^\pm) = 0\ \forall x_1, \forall j, k = 0, 1, \cdots\}\ .$$

These subspaces are dense for they contain all C^∞ compactly supported functions vanishing in a neighborhood of the axes. Furthermore Ω'_α is invariant under the group $(\exp(itQ_\alpha): t \in Q)$ and Ω''_α is invariant under the group $(\exp(itP_\alpha): t \in R)$. It follows from well-known properties of strongly continuous semigroups (see [1] Theorem VIII.10) that

$$(P_\alpha|_{\Omega''_\alpha})^{**} = P_\alpha \quad \text{and} \quad (Q_\alpha|_{\Omega'_\alpha})^{**} = Q_\alpha\ ,$$

so the same result holds when Ω_α is substituted for Ω'_α and Ω''_α. This proves part (a).

Parts (b), (c) and (e) can be proven by computing the "group commutator"

$$C_\alpha(s, t) = \exp(isP_\alpha)\exp(itQ_\alpha)\exp(-isP_\alpha)\exp(-itQ_\alpha)\ .$$

Using the definitions of P_α and Q_α we have

$$(2.4)\quad \begin{aligned} C_\alpha(s,t) &= \exp(isp_1)\bar{\phi}_\alpha(q_1,q_2)\exp(itq_1)\exp(itp_2)\phi_\alpha(q_1,q_2) \\ &\quad\times \exp(-isp_1)\bar{\phi}_\alpha(q_1,q_2)\exp(-itq_1)\exp(-itp_2)\phi_\alpha(q_1,q_2)\,. \end{aligned}$$

It follows from (2.1) that if $\phi\colon R^2 \to C$ is a bounded Borel function then

$$(2.5)\quad \begin{cases} \exp(itp_1)\phi(q_1,q_2) = \phi(q_1+tI,q_2)\exp(itp_1) \\ \exp(itp_2)\phi(q_1,q_2) = \phi(q_1,q_2+tI)\exp(itp_2) \\ \exp(isp_1)\exp(itq_1) = \exp(ist)\exp(itq_1)\exp(isp_1)\,. \end{cases}$$

Using (2.5) we can simplify (2.4)

$$\begin{aligned} C_\alpha(s,t) &= \exp(ist)\bar{\phi}_\alpha(q_1+sI,q_2)\phi_\alpha(q_1+sI,q_2+tI) \\ &\quad\times \bar{\phi}_\alpha(q_1,q_2+tI)\phi_\alpha(q_1,q_2)\,. \end{aligned}$$

Now for $s, t \geqq 0$ define

$$\psi_\alpha(s,t)(x_1,x_2) = \begin{cases} \alpha & -s < x_1 < 0,\ 0 < x_2 < t \\ 1 & \text{elsewhere.} \end{cases}$$

Then for $s, t \geqq 0$ we have

$$(2.6)\quad C_\alpha(s,-t) = \exp(-ist)\psi_\alpha(s,t)(q_1,q_2)\,.$$

Part (c) follows immediately from (2.6) for the operators $\psi_\alpha(s,t)(q_1,q_2)$ have different eigenvalues for different values of α.

Part (e) follows immediately as well.

To prove (b), let $\mathfrak{a}_\alpha$ be the von Neumann algebra generated by the spectral projections of P_α and Q_α. We must show that the commutant of $\mathfrak{a}_\alpha$, is $\{\lambda I\colon \lambda \in C\}$. From (2.6) it is clear that

$$(2.7)\quad \psi_\alpha(s,t)(q_1,q_2) \in \mathfrak{a}_\alpha \quad s, t \geqq 0\,.$$

Since $\phi_\alpha(q_1,q_2)$ is the strong limit of $\psi_\alpha(s,t)(q_1,q_2)$ as $s, t \to \infty$ we have

$$(2.8)\quad \phi_\alpha(q_1,q_2) \in \mathfrak{a}_\alpha\,.$$

Using (2.8) and (2.2) we obtain

$$(2.9)\quad \begin{cases} \exp(itp_1) \in \mathfrak{a}_\alpha & \forall\, t \in R \\ \exp(itq_1)\exp(itp_2) \in \mathfrak{a}_\alpha & \forall\, t \in R\,. \end{cases}$$

If $\chi(E)$ denotes the characteristic function of a set $E \subset R^2$ then (2.7) implies that

$$(2.10)\quad \chi([-t,0]\times[0,s])(q_1,q_2) = \frac{1}{\alpha-1}(\psi_\alpha(s,t)(q_1,q_2) - I)\mathfrak{a}_\alpha\,.$$

Let $a \leqq b$, $c \leqq d$, then using (2.9) and (2.10) we have

$$\begin{aligned}(2.11)\quad & \chi([a, b] \times [c, d])(q_1, q_2) \\ &= \exp(-ibp_1)\exp(-icq_1)\exp(-icp_2)\chi([a-b, 0] \times [0, d-c]) \\ &\quad \times (q_1, q_2)\exp(icq_1)\exp(icp_2)\exp(ibp_1) \in \mathfrak{a}_\alpha .\end{aligned}$$

So $\mathfrak{a}_\alpha$ must contain the maximal abelian von Neumann algebra $\mathfrak{m} = \{\mathfrak{n}(q_1, q_2): \mathfrak{n} \in L^\infty(R^2)\}$, since linear combinations of operators of the form (2.11) are weakly dense in $\mathfrak{m}$. Now let $B \in \mathfrak{a}'_\alpha$ (commutant of $\mathfrak{a}_\alpha$) then $B \in \mathfrak{m}' = \mathfrak{m}$. So there is an $\mathfrak{n} \in L^\infty(R^2)$ such that $B = \mathfrak{n}(q_1, q_2)$. Using (2.9) and the method employed in (2.11) we obtain

$$\mathfrak{n}(q_1 + sI, q_2 + tI) = \mathfrak{n}(q_1, q_2) \quad \forall\, s, t .$$

This mean that $\mathfrak{n}$ is a constant almost everywhere. That is $B = \lambda I$ for some $\lambda \in C$. This proves part (b).

For part (d) we first show that if $f, g \in \operatorname{dom} P_\alpha \cap \operatorname{dom} Q_\alpha$ with either f or g vanishing in a neighborhood of (0, 0) then

$$(2.12)\qquad (Q_\alpha f, P_\alpha g) - (P_\alpha f, Q_\alpha g) = -i(f, g) .$$

To prove (2.12) we use (2.6) to write

$$\begin{aligned}(2.13)\quad & (\exp(itQ_\alpha)f, \exp(isP_\alpha)g) \\ &= \exp(-ist)(\exp(-isP_\alpha)\psi_\alpha(s, t)(q_1, q_2)f, \exp(-itQ_\alpha)g) .\end{aligned}$$

If f vanishes in a neighborhood of (0, 0) then we can choose s and t small enough so that (2.13) takes the form

$$\begin{aligned}(2.14)\quad & (\exp(itQ_\alpha)f, \exp(isP_\alpha)g) \\ &= \exp(-ist)(\exp(-isP_\alpha)f, \exp(-itQ_\alpha)g) .\end{aligned}$$

If $f, g \in \operatorname{dom} P_\alpha \cap \operatorname{dom} Q_\alpha$ we can differentiate both sides of (2.13) with respect to s and t. The result is (2.12). A similar proof shows that (2.12) holds when g vanishes in a neighborhood of (0, 0) and $f, g \in \operatorname{dom} P_\alpha \cap \operatorname{dom} Q_\alpha$. In particular (2.12) holds when $f \in \operatorname{dom} P_\alpha Q_\alpha \cap \operatorname{dom} Q_\alpha P_\alpha$ and g is C^∞, compactly supported and vanishes in a neighborhood of the axes. In this case we can write (2.12) as

$$(2.15)\qquad (P_\alpha Q_\alpha f - Q_\alpha P_\alpha f + if, g) = 0 .$$

Since (2.15) must hold for all such g, a dense subspace of $L^2(R^2)$, part (d) is proven. □

The proof of Theorem 1 can be slightly modified to give a proof of Theorem 2.

3. The pair (P_F, Q_F). Let H be the Hilbert space $L^2(R)$ and $\mathscr{F}$ the Fourier transform $\mathscr{F}: L^2(R) \to L^2(R)$. The Schrödinger 1-system $\{p, q\}$ is defined by the following strongly continuous unitary groups

$$(3.1)\qquad \begin{cases}(\exp(itq)f)(x) = \exp(itx)f(x) \\ (\exp(itp)f)(x) = f(x+t)\ .\end{cases}$$

It is well known that the pair (p, q) satisfies (1.2), is irreducible and $\mathscr{F}^{-1}q\mathscr{F} = p$, $\mathscr{F}q\mathscr{F}^{-1} = -p$.

Let $\omega = \sqrt{2\pi}$ and define

$$(3.2)\qquad \begin{cases}X = \exp(\omega q)\ , \quad Y = \exp(-\omega p) \\ P_F = \exp(-i\omega^{-1}X)p\exp(i\omega^{-1}X) \\ Q_F = \exp(-i\omega^{-1}Y)q\exp(i\omega^{-1}Y)\ .\end{cases}$$

Fuglede [1] proves that both pairs, (X, Y) and (P_F, Q_F), satisfy property (a) (of Theorem 1) and the commutation relations

$$(3.3)\qquad (Xf, Yg) = (Yf, Xg) \quad f, g \in \operatorname{dom} X \cap \operatorname{dom} Y$$

$$(3.4)\qquad (Q_F f, P_F g) - (P_F f, Q_F g) = -i(f, g) \quad f, g \in \operatorname{dom} P_F \cap \operatorname{dom} Q_F\ .$$

The pair (X, Y) is shown to be irreducible but the question of irreducibility of (P_F, Q_F) is lef open.

THEOREM 3. *The pair* (P_F, Q_F) *is irreducible.*

Proof. We use the group commutator

$$C(s, t) = \exp(itP_F)\exp(isQ_F)\exp(-itP_F)\exp(-isQ_F)\ .$$

To simplify $C(s, t)$ we first show

$$(3.5)\qquad \exp(itP_F) = \exp(ih(t)X)\exp(itp)$$

$$(3.6)\qquad \exp(isQ_F) = \exp(ih(s)Y)\exp(isq)$$

where $h(t) = \omega^{-1}(e^{\omega t} - 1)$.

To prove (3.5) let $f \in L^2(R)$ then for a.e. x in R.

$$\begin{aligned}(\exp(itP_F)f)(x) &= (\exp(-i\omega^{-1}X)\exp(itp)\exp(i\omega^{-1}Xf)(x) \\ &= \exp(-i\omega^{-1}\exp(\omega x))(\exp(i\omega^{-1}X)f)(x+t) \\ &= \exp(-i\omega^{-1}\exp(\omega x) + i\omega^{-1}\exp(\omega x + \omega t))f(x+t) \\ &= \exp(ih(t)\exp(\omega x))f(x+t) \\ &= (\exp(ih(t)X)\exp(itp)f)(x)\end{aligned}$$

(3.6) follows by a similar calculation. Using (3.5) and (3.6) we can simplify $C(s, t)$ to

$$C(s, t) = \exp(ist)\exp(ih(t)X)\exp(ih(s)Y)\exp(-ih(t)X)\exp(-ih(s)Y)\ .$$

Let $\mathscr{A}$ be the von Neumann algebra generated by P_F and Q_F.

Letting $s, t \to -\infty$ in $\exp(-ist)C(s, t)$ we obtain

(3.7) $\exp(-i\omega^{-1}X)\exp(-i\omega^{-1}Y)\exp(i\omega^{-1}X)\exp(i\omega^{-1}Y) \in \mathscr{A}$.

Since bounded Borel functions of P_F are in $\mathscr{A}$,

(3.8) $$\exp(i\omega^{-1}\exp(-\omega P_F)) = \exp(-i\omega^{-1}X)\exp(i\omega^{-1}Y)\exp(i\omega^{-1}X) \in \mathscr{A} .$$

Multiplying (3.7) and (3.8) we obtain

$$\exp(-i\omega^{-1}Y) \in \mathscr{A} .$$

So for all $t \in R$,

$$\exp(itq) = \exp(i\omega^{-1}Y)\exp(itQ_F)\exp(-i\omega^{-1}Y) \in \mathscr{A} .$$

Similarly $\exp(isp) \in \mathscr{A}$ for all $s \in R$. Since the pair (p, q) is irreducible we see that (P_F, Q_F) is irreducible. □

We have been unable to decide whether the relations (3.3) and (3.4) hold for the families $\{(X_\alpha, Y_\alpha): \alpha \neq 1\}$ and $\{(P_\alpha, Q_\alpha): \alpha \neq 1\}$ respectively.

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