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COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

M. S. KHAN

Some results on common fixed points for a pair of multivalued mappings defined on a closed subset of a complete metric space are obtained. Our work extends some of the known results due to Itoh; Iséki; and Rus.

1. Introduction. There have been several extensions of known fixed point theorems for multivalued mappings which take each point of a metric space (X, d) into a closed subset K of X. However, in many applications, the mapping involved is not a self-mapping of K. Assad and Kirk [1] gave sufficient conditions for such mappings to have a fixed point by proving a fixed point theorem for multivalued contraction mappings on a complete metrically convex metric space and by putting certain boundary conditions on the mappings. Similar results for multivalued contractive mappings were obtained by Assad [2]. Itoh [4] extended the results given in [1] and [2] for more general types of contraction and contractive mappings.

In this note, we shall extend the results of Itoh [4] for a pair of generalized contraction and contractive mappings. We also prove some other results for multivalued mappings which are partial generalizations of fixed point theorems due to Iséki [3] and Rus [9].

2. Preliminaries. Let (X, d) be a metric space. Then following Nadler [6], we define

(i) $CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$.

$C(X) = \{A : A$ is a nonempty compact subset of $X\}$.

$BN(X) = \{A : A$ is a nonempty bounded subset of $X\}$.

(ii) For nonempty subsets of A and B of X, and $x \in X$

$D(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$.

$H(A, B) = \max(\{\sup D(a, B) : a \in A\},\ \{\sup D(A, b) : b \in B\})$.

$d(x, A) = \inf\{d(x, a) : a \in A\}$.

$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}$.

It is known (Kuratowski [5]), that $CB(X)$ is a metric space with the distance function H. We call H the Hausdorff metric on $CB(X)$.

We shall make frequent use of the following lemmas.

LEMMA 2.1 (*Nadler* [6]). *Let A, B be in $CB(X)$. Then for all*

$\varepsilon > 0$ and $a \in A$, there exists $b \in B$ such that $d(a, b) \leqq H(A, B) + \varepsilon$. If A, B are in $C(X)$, then one can choose $b \in B$ such that $d(a, b) \leqq H(A, B)$.

LEMMA 2.2 (*Rus* [10]). *Let $A \in CB(X)$ and $0 < \theta < 1$ be given. Then for every $x \in A$, there exists $a \in A$ such that $d(x, a) \geqq \theta\delta(x, A)$, and $d(x, a) \geqq \theta H(x, A)$.*

Next two lemmas can be easily proved.

LEMMA 2.3. *For any $x \in X$, and any A, B in $CB(X)$,*

$$|d(x, A) - d(x, B)| \leqq H(A, B) .$$

LEMMA 2.4. *For any x and y in X, $A \subset X$.*

$$|d(x, A) - d(y, A)| \leqq d(x, y) .$$

DEFINITION 2.5. A metric space (X, d) is said to be *metrically convex* if for any $x, y \in X$ with $x \neq y$, there exists $z \in X$, $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y) .$$

Following result is borrowed from Assad and Kirk [1].

LEMMA 2.6. *If K is a nonempty closed subset of a complete and metrically convex metric space (X, d), then for any $x \in K$, $y \notin K$, there exists a $z \in \partial K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y) .$$

DEFINITION 2.7. Let K be a nonempty closed subset of a metric space (X, d). A mapping $T: K \to CB(X)$ is said to be continuous at $x_0 \in K$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $H(Tx, Tx_0) < \varepsilon$, whenever $d(x, x_0) < \delta$. If T is continuous at every point of K, we say that T is continuous at K.

Motivated from Park [7], we introduce the following:

DEFINITION 2.8. Let K be a nonempty closed subset of a metric space (X, d) and S, T be mappings of K into $CB(X)$. Then (S, T) is said to be a *generalized contraction pair* of K into $CB(X)$ if there exist nonnegative reals α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that for any $x, y \in K$,

$$\begin{aligned} H(Sx, Ty) \leqq\ & \alpha d(x, y) \\ & + \beta\{D(x, Sx) + D(y, Ty)\} + \gamma\{D(x, Ty) + D(y, Sx)\} . \end{aligned}$$

Similarly, we define generalized contraction pair of K into $C(X)$.

DEFINITION 2.9. Let K be a nonempty closed subset of metric space (X, d). Let S and T be mappings of K into $CB(X)$. Then (S, T) is said to be a *generalized contractive pair* of K into $CB(X)$ if there exist nonnegative reals α, β, γ such that for any $x, y \in X$ with $x \neq y$,

$$H(Sx, Ty) < \alpha d(x, y) + \beta\{D(x, Sx) + D(y, Ty)\} + \gamma\{D(x, Ty) + D(y, Sx)\} ,$$

where $0 < 2\alpha + 2\beta + 4\gamma \leqq 1$.

REMARK. When S and T are singlevalued mappings then we simply say that (S, T) is a generalized contraction (contractive) pair of K into X.

3. Results.

THEOREM 3.1. *Let (X, d) be a complete and metrically convex metric space, K a nonempty closed subset of X. Let (S, T) be a generalized contraction pair of K into $CB(X)$. If for any $x \in \partial K$, $S(x) \subset K$, $T(x) \subset K$ and $(\alpha + \beta + \gamma)(1 + \beta + \gamma)/(1 - \beta - \gamma)^2 < 1$, then there exists $z \in K$ such that $z \in S(z)$ and $z \in T(z)$.*

Proof. Put $\theta = (\alpha + \beta + \gamma)(1 + \beta + \gamma)/(1 - \beta - \gamma)^2$. Then $0 \leqq \theta < 1$. Without loss of generality we may take $\theta > 0$ since for $\theta = 0$, the conclusion of Theorem 3.1 trivially holds. We shall construct sequences $\{x_n\}$ and $\{y_n\}$ in K and X, respectively, as follows:

Let $x_0 \in \partial K$ and $x_1 = y_1 \in S(x_0)$. Then by Lemma 2.1 we can choose a $y_2 \in T(x_1)$ such that

$$d(y_1, y_2) \leqq H(Sx_0, Tx_1) + \left(\frac{1 - \beta - \gamma}{1 + \beta + \gamma}\right)\theta .$$

If $y_2 \in K$, put $x_2 = y_2$. If $y_2 \notin K$, use Lemma 2.6 to choose an element $x_2 \in \partial K$ such that $d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2)$. Continuing in this manner, we obtain sequences $\{x_n\}$ and $\{y_n\}$ satisfying:

(i) $y_n \in S(x_{n-1})$, for an odd n, and
$y_n \in T(x_{n-1})$, for an even n.

(ii) $d(y_n, y_{n+1}) \leqq H(S(x_{n-1}), T(x_n)) + (1 - \beta - \gamma/1 + \beta + \gamma)\theta^n$; if n is odd and
$d(y_n, y_{n+1}) \leqq H(T(x_{n-1}), S(x_n)) + (1 - \beta - \gamma/1 + \beta + \gamma)\theta^n$; if n is even.

(iii) $y_{n+1} = x_{n+1}$ if $y_{n+1} \in K$, for all n, or

(iv) $d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1})$, if $y_{n+1} \notin K$ for all n and $x_{n+1} \in \partial K$.

We wish to estimate the distance $d(x_n, x_{n+1})$ for $n \geqq 2$. Let us write $P = \{x_i \in \{x_n\}: x_i = y_i\}$ and $Q = \{x_i \in \{x_n\}: x_i \neq y_i\}$. Note that if $x_n \in Q$ then x_{n-1} and x_{n+1} will be in P by boundary condition.

Case I. Let $x_n, x_{n+1} \in P$. Then for an odd n we have,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(y_n, y_{n+1}) \\ &\leqq H(S(x_{n-1}), T(x_n)) + \left(\frac{1-\beta-\gamma}{1+\beta+\gamma}\right)\theta^n \\ &\leqq \alpha d(x_{n-1}, x_n) + \beta\{D(x_{n-1}, S(x_{n-1})) + D(x_n, Tx_n)\} \\ &\quad + \gamma\{D(x_{n-1}, T(x_n)) + D(x_n, S(x_{n-1}))\} + \left(\frac{1-\beta-\gamma}{1+\beta+\gamma}\right)\theta^n \\ &\leqq \alpha d(x_{n-1}, x_n) + \beta\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \\ &\quad + \gamma\{d(x_{-1}, x_{n+1}) + d(x_n, x_n)\} + \left(\frac{1-\beta-\gamma}{1+\beta+\gamma}\right)\theta^n . \end{aligned}$$

So

$$d(x_n, x_{n+1}) \leqq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)d(x_{n-1}, x_n) + \left(\frac{\theta^n}{1+\beta+\gamma}\right) .$$

A similar inequality can be obtained when n is even.

Case II. $x_n \in P$ and $x_{n+1} \in Q$. Then by (iv) we see that

$$d(x_n, x_{n+1}) \leqq d(x_n, y_{n+1}) = d(y_n, y_{n+1}) .$$

By method similar to Case I, we have for even and odd n

$$d(x_n, x_{n+1}) \leqq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)d(x_{n-1}, x_n) + \left(\frac{\theta^n}{1+\beta+\gamma}\right) .$$

Case III. $x_n \in Q$ and $x_{n+1} \in P$. Then $x_{n-1} = y_{n-1}$ holds. So we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leqq d(x_n, y_n) + d(y_n, x_{n+1}) \\ &= d(x_n, y_n) + d(y_n, y_{n+1}) . \end{aligned}$$

Then for an odd n, we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leqq H(Sx_{n-1}, Tx_n) + \left(\frac{1-\beta-\gamma}{1+\beta+\gamma}\right)\theta^n \\ &\leqq \alpha d(x_{n-1}, x_n) + \beta\{D(x_{n-1}, Sx_{n-1}) + D(x_n, Tx_n)\} \\ &\quad + \gamma\{D(x_{n-1}, Tx_n) + D(x_n, Sx_{n-1})\} + \left(\frac{1-\beta-\gamma}{1+\beta+\gamma}\right)\theta^n \end{aligned}$$

$$\begin{aligned}
&\leqq \alpha d(x_{n-1}, x_n) + \beta\{d(x_{n-1}, y_n) + d(x_n, y_{n+1})\} \\
&\quad + \gamma\{d(x_{n-1}, y_{n+1}) + d(x_n, y_n)\} + \Big(\frac{1-\beta-\gamma}{1+\beta+\gamma}\Big)\theta^n \\
&= \alpha d(x_{n-1}, x_n) + \beta\{d(x_{n-1}, y_n) + d(x_n, x_{n+1})\} \\
&\quad + \gamma\{d(x_{n-1}, x_{n+1}) + d(x_n, y_n)\} + \Big(\frac{1-\beta-\gamma}{1+\beta+\gamma}\Big)\theta^n \\
&\leqq \alpha d(x_{n-1}, x_n) + \beta\{d(x_{n-1}, y_n) + d(x_n, x_{n+1})\} \\
&\quad + \gamma\{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, y_n)\} + \Big(\frac{1-\beta-\gamma}{1+\beta+\gamma}\Big)\theta^n .
\end{aligned}$$

As $0 \leqq \theta < 1$, and $d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n)$, we obtain

$$\begin{aligned}
d(x_n, x_{n+1}) &\leqq (1+\gamma)d(x_n, y_n) + (\alpha+\gamma)d(x_{n-1}, x_n) + \beta d(x_{n-1}, y_n) \\
&\quad + (\beta+\gamma)d(x_n, x_{n+1}) + \Big(\frac{1-\beta-\gamma}{1+\beta+\gamma}\Big)\theta^n \\
&\leqq (1+\gamma)d(x_{n-1}, y_n) + \beta d(x_{n-1}, y_n) \\
&\quad + (\beta+\gamma)d(x_n, x_{n+1}) + \Big(\frac{1-\beta-\gamma}{1+\beta+\gamma}\Big)\theta^n .
\end{aligned}$$

Therefore,

$$d(x_n, x_{n+1}) \leqq \Big(\frac{1+\beta+\gamma}{1-\beta-\gamma}\Big)d(x_{n-1}, y_n) + \Big(\frac{1}{1+\beta+\gamma}\Big)\theta^n .$$

A similar inequality is obtained for an even n. Since $x_{n-1} = y_{n-1}$ and $y_n \neq x_n$, as in the Case II we have for an odd n,

$$d(x_{n-1}, y_n) \leqq \Big(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\Big)d(x_{n-2}, x_{n-1}) + \Big(\frac{1}{1+\beta+\gamma}\Big)\theta^{n-1} .$$

Similarly, we can obtain an inequality for even n. Combining the above two inequalities we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leqq \Big(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\Big)\Big(\frac{1+\beta+\gamma}{1-\beta-\gamma}\Big)d(x_{n-2}, x_{n-1}) \\
&\quad + \Big(\frac{\theta^{n-1}}{1-\beta-\gamma}\Big) + \Big(\frac{\theta^n}{1+\beta+\gamma}\Big) .
\end{aligned}$$

Then, as noted in Itoh [4], it can be shown that $\{x_n\}$ is a Cauchy sequence, hence convergent. Call the limit z. By the way of choosing $\{x_n\}$, there exists an infinite subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \in P$. Then for an even n_i, we have

$$\begin{aligned}
D(x_{n_i}, Sz) &\leqq H(Tx_{n_i-1}, Sz) \\
&\leqq \alpha d(x_{n_i-1}, z) + \beta\{D(x_{n_i-1}, Tx_{n_i-1}) + D(z, Sz)\} \\
&\quad + \gamma\{D(x_{n_i-1}, Sz) + D(z, Tx_{n_i-1})\}
\end{aligned}$$

$$\leqq \alpha\{d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, z)\} + \beta\{d(x_{n_i-1}, x_{n_i}) + d(x_{n_i-1}, x_{n_i}) + d(z, x_{n_i}) + D(x_{n_i}, Sz)\} + \gamma\{d(x_{n_i-1}, x_{n_i}) + D(x_{n_i}, Sz) + d(z, x_{n_i})\} .$$

So

$$D(x_{n_i}, Sz) \leqq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)(d(x_{n_i}, x_{n_i}) + d(x_{n_i}, z)) .$$

Using this and the inequality

$$D(z, Sz) \leqq d(z, x_{n_i}) + D(x_{n_i}, Sz) ,$$

we see that $D(z, Sz) = 0$. As Sz is closed, $z \in Sz$. Similarly, we can show that $z \in Tz$. Thus z is a common fixed point of S and T. This finishes the proof.

We can also prove the following result:

THEOREM 3.2. *Let (X, d) be a complete and metrically convex metric space, K a nonempty closed subset of X. Let (S, T) be a generalized contraction pair of K into $C(X)$. If for any $x \in \partial K$, $S(x) \subset K$ and $T(x) \subset K$ and $(\alpha + \beta + \gamma)(1 + \beta + \gamma)/(1 - \beta - \gamma)^2 < 1$, then S and T have a common fixed point in K.*

Proof. As in the proof of Theorem 3.1, we shall construct two sequences $\{x_n\}$ and $\{y_n\}$ which satisfy (i), (iii) and (iv). The condition (ii) is replaced by the following:

(ii)′ $\qquad d(y_n, y_{n+1}) \leqq H(Sx_{n-1}, Tx_n)$, if n is odd

and

$$d(y_n, y_{n+1}) \leqq H(Tx_{n-1}, Sx_n) , \quad \text{if } n \text{ is even} .$$

These relations are possible due to Lemma 2.1. The rest of the proof is identical with Theorem 3.1.

As every Banach space is metrically convex, we have the following corollaries for singlevalued mappings:

COROLLARY 3.3. *Let X be a Banach space and K be a nonempty closed subset of X. Let (S, T) be a generalized contraction pair of K into X. If $S(\partial K) \subset K$ and $T(\partial K) \subset K$ and $(\alpha + \beta + \gamma)(1 + \beta + \gamma)/(1 - \beta - \gamma)^2 < 1$, then S and T have a unique common fixed point in K.*

REMARK. The technique of the proof of Theorem 3.1 and Theorem 3.2 can be used to extend a result of Rhoades [8] for a pair of singlevalued mappings.

Next theorem extends Theorem 2 of Itoh [4] for a pair of multivalued mappings, and hence generalizes a fixed point theorem of Assad [2].

THEOREM 3.4. *Let (X, d) be a complete and metrically convex metric space and K be a nonempty compact subset of X. Let (S, T) be a generalized contractive pair of K into $CB(X)$, and S, T are continuous on K. If for any $x \in \partial K$, $S(x) \subset K$, $T(x) \subset K$; $S(x) \cap T(x) \neq \emptyset$ for all $x \in K$ and $(\alpha + \beta + \gamma)(1 + \beta + \gamma)/(1 - \beta - \gamma)^2 \leqq 1$, then there exists a common fixed point of S and T in K.*

Proof. Consider $f: K \to R^+$ (the nonnegative reals) defined by $f(x) = d(x, Tx)$, $x \in K$. Then using Lemma 2.3, Lemma 2.4 and the continuity of T we have for $x, y \in K$

$$\begin{aligned} |f(x) - f(y)| &\leqq |d(x, Tx) - d(y, Tx)| + |d(y, Tx) - d(y, Ty)| \\ &\leqq d(x, y) + H(Tx, Ty) . \end{aligned}$$

Thus f is continuous on the compact set K. Let $z \in K$ such that $f(z) = \inf\{f(x): x \in K\}$. Suppose that $f(z) > 0$. Then for each $n = 1, 2, 3, \cdots$, we can choose $x_n \in T(z)$ such that

$$d(x_n, z) \leqq f(z) + \frac{1}{n} .$$

As K is compact, if $x_n \in K$ for very large n, then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to an element $x_0 \in K$. We may assume that $x_0 \neq z$. Then

$$\begin{aligned} f(x_0) &= d(x_0, Tx_0) \\ &\leqq H(Tz, Tx_0) \\ &\leqq H(Tz, Sz) + H(Sz, Tx_0) \\ &< \alpha d(z, x_0) + \beta\{D(z, Sz) + D(x_0, Tx_0)\} \\ &\quad + \gamma\{D(z, Tx_0) + D(x_0, Sz)\} \\ &< \alpha\{d(z, Tz) + H(Tz, Tx_0)\} + \beta\{d(z, Tz) \\ &\quad + H(Tz, Sz) + f(x_0)\} + \gamma\{d(z, Tz) + H(Tz, Tx_0) \\ &\quad + f(x_0) + H(Tx_0, Tz) + H(Tz, Sz)\} . \end{aligned}$$

Then we get

$$f(x_0) < \left(\frac{\alpha + \beta + \gamma}{1 - \alpha - \beta - 3\gamma}\right) f(z) .$$

Since $((\alpha + \beta + \gamma)/(1 - \alpha - \beta - 3\gamma)) \leqq 1$, we have $f(x_0) < f(z)$ which contradicts the minimality of z. Therefore $f(z) = 0$.

If some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is such that $x_{n_i} \notin K$, then $z \notin \partial K$. For the sake of convenience, we may assume that $x_n \notin K$, $n = 1, 2, 3, \cdots$. Then by applying Lemma 2.6 we see that for each n there is a $y_n \in \partial K$ such that $d(x_n, y_n) + d(y_n, z) = d(x_n, z)$. As K is compact and $S(y_n) \subset K$ there exists $w_n \in S(y_n)$ such that $d(x_n, w_n) \leqq H(Tz, Sy_n) + \varepsilon$ by Lemma 2.1. We may further assume that $\{y_n\}$ converges to some $y_0 \in \partial K$. Let

$$8\varepsilon = \alpha d(y_0, z) + \beta\{d(z, Tz) + d(y_0, Sy_0)\} + \gamma\{d(z, Sy_0) + d(y_0, Tz)\} - H(Tz, Sy_0) .$$

Then $\varepsilon > 0$ as $y_0 \neq z$. For this choice of ε, we can find a positive integer N such that for all $n \geqq N$,

(a) $d(y_0, z) - d(y_n, z) < 2\varepsilon$,

(b) $f(y_0) - \varepsilon < f(y_n)$,

(c) $d(x_n, z) < f(z) + 2\varepsilon$,

(d) $H(Tz, Sy_n) < H(Tz, Sy_0) + \varepsilon$, (here continuity of S is used).

Then for any $n \geqq N$, we get

$$\begin{aligned} f(y_0) - \varepsilon < f(y_n) &= D(y_n, Ty_n) \\ &\leqq d(y_n, x_n) + d(x_n, w_n) + d(w_n, Ty_n) \\ &\leqq d(y_n, x_n) + H(Tz, Sy_n) + \varepsilon + H(Sy_n, Ty_n) . \end{aligned}$$

Here last term vanishes and $x_n \in Tz$. Then we have

$$\begin{aligned} f(y_0) - \varepsilon &< d(y_n, x_n) + H(Tz, Sy_0) + 2\varepsilon \\ &< d(x_n, y_n) + \alpha d(z, y_0) + \beta\{D(z, Tz) + D(y_0, Sy_0)\} \\ &\quad + \gamma\{D(z, Sy_0) + D(y_0, Tz)\} - 6\varepsilon \\ &< d(x_n, y_n) + \alpha d(y_0, z) + \beta\{D(z, Tz) + D(y_0, z) + D(z, Tz) \\ &\quad + H(Tz, Sy_0)\} + \gamma\{D(z, Tz) + H(Tz, Sy_0) + d(y_0, z) \\ &\quad + D(z, Tz)\} - 6\varepsilon . \end{aligned}$$

Then this yields

$$\begin{aligned} f(y_0) - \varepsilon &< d(x_n, y_n) + \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right) d(y_0, z) + \left(\frac{2\beta + 2\gamma}{1 - \beta - \gamma}\right) f(z) - 6\varepsilon \\ &< d(x_n, y_n) + d(y_0, z) + \left(\frac{2\beta + 2\gamma}{1 - \beta - \gamma}\right) f(z) - 6\varepsilon \\ &< d(x_n, y_n) + d(y_n, z) + \left(\frac{2\beta + 2\gamma}{1 - \beta - \gamma}\right) f(z) - 4\varepsilon \\ &= d(x_n, z) + \left(\frac{2\beta + 2\gamma}{1 - \beta - \gamma}\right) f(z) - 4\varepsilon \end{aligned}$$

$$< f(z) + \left(\frac{2\beta + 2\gamma}{1 - \beta - \gamma}\right)f(z) - 2\varepsilon .$$

So

$$f(y_0) < \left(\frac{1 + \beta + \gamma}{1 - \beta - \gamma}\right)f(z) - 2\varepsilon .$$

Now choose $u \in S(y_0) \cap T(y_0)$ such that $d(y_0, Ty_0) = d(y_0, u)$. As $f(z) > 0$, we see that $u \neq y_0$. Then

$$\begin{aligned} f(u) &= D(u, Tu) \leqq H(Sy_0, Tu) \\ &< \alpha d(y_0, u) + \beta\{D(y_0, Sy_0) + D(u, Tu)\} \\ &\quad + \gamma\{D(y_0, Tu) + D(u, Sy_0)\} \\ &< \alpha\{D(y_0, Ty_0) + D(Ty_0, u)\} + \beta\{D(y_0, Ty_0) \\ &\quad + H(Ty_0, Sy_0) + D(u, Tu)\} + \gamma\{D(y_0, Ty_0) + H(Ty_0, Sy_0) \\ &\quad + H(Sy_0, Tu) + D(u, Sy_0)\} . \end{aligned}$$

Then using the facts $D(u, Ty_0) = 0$ and $D(u, Sy_0) = 0$, we have

$$f(u) < \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)f(y_0) .$$

Now using previous relation between $f(y_0)$ and $f(z)$ we have

$$\begin{aligned} f(u) &< \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)\left(\frac{1 + \beta + \gamma}{1 - \beta - \gamma}\right)f(z) - \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)\varepsilon \\ &< f(z) - \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)\varepsilon . \end{aligned}$$

This contradicts the minimality of z. Hence $f(z) = 0$ and as Tz is a closed subset of X, we find that $z \in Tz$. Further, $D(z, Sz) \leqq D(z, Tz) + H(Tz, Sz)$, implies that $z \in Sz$. Therefore z is a common fixed point of S and T. This completes the proof.

For Banach space we have the following:

COROLLARY 3.5. *Let K be a nonempty compact subset of a Banach space X and (S, T) be the generalized contractive pair of K into X and S, T are continuous on K. If $S(\partial K) \subset K$, $T(\partial K) \subset K$, and $(\alpha + \beta + \gamma)(1 + \beta + \gamma)/(1 - \beta - \gamma)^2 \leqq 1$, then there exists a unique common fixed point of S and T in K.*

Now we prove two results concerning unique common fixed points of a pair of multivalued mappings defined on a nonempty complete subset of a metric space which are not necessarily metrically convex, and also the mappings involved do not satisfy any boundary conditions.

THEOREM 3.6. *Let (X, d) be a metric space and K a nonempty complete subset of X. Suppose that $S, T: K \to CB(X)$ are multivalued mappings such that for all x, y in K:*

$$\begin{aligned}\delta(Sx, Ty) \leqq \alpha d(x, y) &+ \beta\{\delta(x, Sx) + \delta(y, Ty)\} \\ &+ \gamma\{\delta(x, Ty) + \delta(y, Sx)\} ,\end{aligned}$$

where $\alpha, \beta, \gamma \geqq 0$ and $\alpha + 2\beta + 2\gamma < 1$.

Then S and T have a unique common fixed point in K.

Proof. Put $\theta = (\alpha + 2\beta + 2\gamma)^{1/2}$. Then θ is positive. We shall now define singlevalued mappings S_1 and T_1 of K into itself such that $S_1(x) \in S(x)$, $T_1(x) \in T(x)$ for all $x, y \in K$, and

$$\begin{aligned}d(x, S_1(x)) &\geqq \theta\delta(x, S(x)) , \\ d(x, T_1(x)) &\geqq \theta\delta(x, T(x)) ,\end{aligned}$$

for all $x \in K$.
Lemma 2.2 justifies our choice of $S_1(x)$ and $T_1(x)$. Then one gets

$$\begin{aligned}d(S_1x, T_1y) &\leqq \delta(S(x), T(y)) \\ &\leqq \alpha d(x, y) + \beta\{\delta(x, S(x)) + \delta(y, T(y))\} \\ &\quad + \gamma\{\delta(x, T(y)) + \delta(y, S(x))\} \\ &\leqq \alpha d(x, y) + \beta\{\theta^{-1}d(x, S_1x) + \theta^{-1}d(y, T_1y)\} \\ &\quad + \gamma\{\theta^{-1}d(x, T_1y) + \theta^{-1}d(y, S_1x)\} .\end{aligned}$$

As $\theta^{-1}(2\beta + 2\gamma) + \alpha \leqq \theta^{-1}(2\beta + 2\gamma + \alpha) = \alpha + 2\beta + 2\gamma < 1$ and K is complete, it follows from Theorem 1 of Wong [11] that S_1 and T_1 have a unique common fixed point, say z in K. Consider

$$0 = d(z, S_1z) \geqq \theta\delta(z, S(z)) .$$

This shows that $\delta(z, S(z)) = 0$ giving thereby that $z \in S(z)$, as $S(z)$ is closed. Similarly, we have $z \in T(z)$. This ends the proof.

The method of proof of Theorem 3.6 can be used to prove the following as well:

THEOREM 3.7. *Let (X, d) be a metric space and K a nonempty complete subset of X. Suppose that $S, T: K \to BN(X)$ are multivalued mappings such that for all x, y in K:*

$$\delta(S(x), T(y)) \leqq \alpha d(x, y) + \beta\{H(x, Sx) + H(y, Ty)\} \\ + \gamma\{H(x, Ty) + H(y, Sx)\}\ ,$$

where $\alpha + 2\beta + 2\gamma < 1$, $\alpha, \beta, \gamma \geqq 0$.
Then S and T have a unique common fixed point in K.

REMARKS. Theorem 3.6 and Theorem 3.7 are slight extensions of results obtained by Iséki [3] and Rus [9].

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LA TROBE UNIVERSITY
BUNDOORA, VICTORIA, 3083
AUSTRALIA

Current address: Aligarh Muslim University
Aligarh-202001
India