

SMOOTH ACTIONS OF THE CIRCLE GROUP ON EXOTIC SPHERES

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Recent work of Schultz translates the question of which exotic spheres S^n admit semifree circle actions with k -dimensional fixed point set entirely to problems in homotopy theory provided the spheres bound spin manifolds. In this article we study circle actions on homotopy spheres not bounding spin manifolds and prove, in particular, that the spin boundary hypothesis can be dropped if $(n-k)$ is not divisible by 128. It is also proved that any ordinary sphere can be realized as the fixed point set of such a circle action on a homotopy sphere which is not a spin boundary; some of these actions are not necessarily semi-free. This extends earlier results obtained by Bredon and Schultz. The Adams conjecture, its consequences regarding splittings of certain classifying spaces and standard results of simply-connected surgery are used to construct the actions. The computations involved relate to showing that certain surgery obstructions vanish.

1. Introduction. Results due to Schultz give a purely homotopy theoretic characterization of those homotopy $(n + 2k)$ -spheres admitting smooth semi-free circle actions with n -dimensional fixed point sets provided one limits attention to exotic spheres bounding spin manifolds. The method of proof is similar to that described in [14] for actions of prime order cyclic groups; a detailed account will appear in [21]. Since the premise of this article relates directly to [21], we outline some of the results contained there.

Given nonnegative integers $m < n$, let CP_m^n denote the quotient complex CP^n/CP^m . CP_m^n is also the Thom space of m copies of the canonical line bundle over CP^{n-m} [10, 11]. For some integer A depending only on $n - m$, the complexes $\Sigma^A CP_m^n$ and CP_{m+A}^{n+A} are stably equivalent [4]. Using this periodicity one can define a spectrum (finite) CP_m^n for all integers $m < n$. In the same manner, it is also possible to form a limit spectrum CP_m^∞ for all finite integers m . If m and n are positive integers, there are elementary coexact sequences of the form

$$S^{2m-2} \longrightarrow CP_{m-1}^n \longrightarrow CP_m^n \xrightarrow{\delta} S^{2m-1};$$

and by periodicity and limit arguments there are similar sequences when m and n are arbitrary integers or $n = \infty$.

Let $F_{S^1}(C^q)$ be the topological monoid of S^1 equivariant self-maps

of the unit sphere S^{2q-1} contained in C^q (action via scalar multiplication), and let F_{S^1} be the corresponding stabilization. There are natural inclusions of the unitary group U_q in $F_{S^1}(C^q)$ and of $U = \lim U_q$ in F_{S^1} , and therefore one has quotient spaces and a stabilization map of the form $\sigma: F_{S^1}(C^q)/U_q \rightarrow F_{S^1}/U$. From [9], we know that the homotopy groups of F_{S^1} are isomorphic to those of the free infinite loop space $\Omega^\infty \Sigma^\infty(\Sigma CP_0^\infty)$. Now we can state the following:

THEOREM 1.1 [21]. *Let S_7^{n+2k} be a homotopy $(n + 2k)$ -sphere bounding a spin manifold, and let $P(\gamma) \subseteq \Pi_{n+2k}^S$ be its Pontryagin-Thom invariant. Then there is a smooth semi-free S^1 action on S_7^{n+2k} with n -dimensional fixed point set if and only if:*

- (1) *There is a class $\beta \in \Pi_n^S(\Sigma CP_{-k}^\infty) \cong \Pi_{n+2k}^S(\Sigma^{2k+1} CP_{-k}^\infty)$ such that $\delta_* \Sigma^{2k} \beta \in P(\gamma) \subseteq \Pi_{n+2k}^S$.*
- (2) *There is a class $\alpha \in \Pi_n(F_{S^1}(C^k)/U_k)$ such that $\sigma_* \alpha$ equals the image of β under the composition*

$$\Pi_n^S(\Sigma CP_{-k}^\infty) \xrightarrow{K_*} \Pi_n^S(\Sigma CP_0^\infty) \cong \Pi_n(F_{S^1}) \longrightarrow \Pi_n(F_{S^1}/U),$$

in which K is formally the map collapsing the subspectrum ΣCP_{-k}^{-1} to a point.

If $b \text{Spin}_{m+1}$ is the group of homotopy m -spheres bounding spin manifolds, $\theta_m/b \text{Spin}_{m+1} = Z_2$ if $m = 1, 2 \pmod 8, m > 8$ and zero otherwise [5]. Therefore the question of which homotopy spheres, not necessarily in $b \text{Spin}_{m+1}$, admit semi-free circle actions as in the theorem reduces to whether any such actions at all exist on exotic spheres not bounding spin manifolds of the appropriate dimensions. We list below the known results and the results obtained in this article.

- (1) If k is odd no such examples exist by results of Schultz [15].
- (2) If n and k are both even, Bredon has examples [6].
- (3) If n is odd and $k \neq 0 \pmod 8, k > 2$ being even, Schultz has constructed examples [16]. In general, the problem of finding examples with n odd and k even reduces to the following problem about vector bundles:

Problem. Suppose we are given an odd integer n and an integer k such that $n + 2k = 8s + 1 > 8$. Is there a real vector bundle ζ over $S^{n+1} \times CP^{k-1}$, which is stably trivial over S^{n+1} and CP^{k-1} such that the Pontryagin classes of $\psi^3 \zeta - \zeta$ satisfy

$$\langle \hat{A}_{2s}(\psi^3 \zeta - \zeta), [S^{n+2} \times CP^{k-1}] \rangle \equiv 2Z + 1; \langle L_{2s}(\psi^3 \zeta - \zeta), [S^{n+2} \times CP^{k-1}] \rangle = 0?$$

We discuss this reduction in § 2 and show how a positive solution leads to circle actions. In § 3, the problem is answered positively for $k \neq 0(64)$.

(4) If n is odd and $k = 2$, then no examples exist according to Levine [12]. In this case the methods of this paper can be adapted to yield actions which are nearly semi-free. These actions, discussed in § 2, have three types of orbits; namely fixed points, free orbits and orbits of type Z_{2^n} .

2. Fixed point sets of co-dimension 2 mod 4. We study smooth effective circle group actions on homotopy spheres with standard spheres as fixed point sets. The homotopy spheres do not bound spin manifolds and the fixed point sets have co-dimension 2 mod 4. It is known that such actions cannot be semi-free [15]. The actions we construct have three types of orbits: fixed points, free orbits and orbits of type Z_{2^n} for given n .

THEOREM 2.1. *Let $k > 0$. There is a smooth effective action of the circle group on a homotopy $(8k + 1)$ -sphere not bounding a spin manifold with fixed point set a standard sphere of co-dimension $4q + 2$, $q > 1$.*

We prove this result by constructing a homotopy sphere Σ^{8k+1} and an action on it with the desired properties. In order to do this, it is necessary to show that certain normal maps into the space $M = S^{8k-4q+2} \times_{S^1} S^{4q-1}$ are normally co-bordant to homotopy equivalences. Here, M is the orbit space of the S^1 action (trivial $\oplus \psi^{2n_C}$) $\times (2qC)$ on $S^{8k-4q+2} \times S^{4q-1}$. $2qC$ is the standard free action of S^1 on S^{4q-1} ; ψ^{2n_C} is given by $zz_1 = z^{2^n}z_1$ for z in S^1 and z_1 in C . Since M is a simply connected $8k$ -dimensional manifold, the obstruction to finding a normal cobordism of a normal map to a homotopy equivalence is an index difference. The normal maps we consider are obtained via fiber homotopically trivial bundles over M . Hence, it is sufficient, by Hirzebruch's index theorem, to calculate the rational Pontryagin classes of fiber homotopically trivial bundles over M . All such bundles arise from elements of $[M, F/0]$, [7]. For our purposes it is enough to consider elements of $[M, F/0]$ obtained from a simpler set as pullbacks in the following manner:

$$\begin{aligned} [S^{8k-4q}(CP^{2q-1})^{\eta^{2^n}}, F/0] &= [(CP^{2q-1})^{\text{trivial}} \oplus \eta^{2^n}, F/0] \\ &= [D^{8k-4q+2} \times_{S^1} S^{4q-1} / S^{8k-4q+1} \times_{S^1} S^{4q-1}, F/0] \\ \text{(By excision)} &= [S^{8k-4q+2} \times_{S^1} S^{4q-1} / D^{8k-4q+1} \times_{S^1} S^{4q-1}, F/0] \\ &\quad \downarrow \text{(Collapse)} \\ &= [M, F/0]. \end{aligned}$$

Here η is the Hopf bundle $S^{4q-1} \rightarrow CP^{2q-1}$ so that η^{2n} is the bundle

$$\begin{array}{c} S^1 \times_{S^1} S^{4q-1} \\ \downarrow \\ CP^{2q-1} \end{array}$$

where $S^1 \times_{S^1} S^{4q-1}$ is the orbit space of the action $\psi^{2n} C \times 2qC$ on $S^1 \times S^{4q-1}$. D^n denotes the lower hemisphere of S^n . This means that we are, in fact, interested in studying bundles over the space $S^{8k-4q}(CP^{2q-1})^{\eta^{2n}}$.

Bott periodicity and Thom isomorphisms give all the information we need for calculating the rational Pontryagin classes in this case. We denote the Bott periodicity isomorphism by β , the K -theory Thom isomorphism $K^*(CP^{2q-1}) \rightarrow K((CP^{2q-1})^{\eta^{2n}})$ by Th . We also let $\mu = \eta - 1$ in $K^*(CP^n)$. Then μ generates $K^*(CP^n)$ multiplicatively [4]. Let ψ_R^3 denote an Adams' operation in real K -theory. With these notations, we can state the following:

LEMMA 2.2. *Let $\zeta = [\beta^{4k-2q} \text{Th}(\mu^{2q-1})]$. Then $\zeta = \sigma \otimes C$, where σ belongs to $KO[S^{8k-4q}(CP^{2q-1})]$ and η^{2n} is the pullback of the generator of $KO(S^{8k})$ under the collapsing map*

$$\begin{array}{c} S^{8k-4q}(CP^{2q-1})^{\eta^{2n}} \\ \downarrow \\ S^{8k-4q}(CP^{2q-1}/CP^{2q-2})^{\eta^{2n}} = S^{8k-4q}(S^{4q-2})^{\eta^{2n}} \\ = S^{8k} \end{array}$$

Let $\zeta'' = \beta^{4k-2q} \text{Th}(\mu^{2q-3})$, $\zeta' = re \zeta''$, where $re: K \rightarrow KO$ is realification

$$\alpha = \psi_R^3 \zeta' - \zeta', \quad \gamma = \psi_R^3 \sigma - \sigma.$$

$3^e \alpha$ and $3^e \gamma$ have fiberhomotopically trivial associated sphere bundles by Adams conjecture [13]. We denote the pullbacks of α, γ to M by the same symbols. The computations necessary to prove (2.1) are contained in the following

LEMMA 2.3. *There is an integer pair (x, y) such that y is odd and $\langle L_{2k}(\tau_M \oplus 3^e x \alpha + 3^e y \gamma), [M] \rangle = \langle L_{2k}(\tau_M), [M] \rangle$. Here L_{2k} denotes the Hirzebruch L class and τ_M is the tangent bundle of M . $[M]$ is the fundamental class of M .*

Proofs of Lemmas 2.2 and 2.3 are postponed.

Proof of Theorem 2.1. The elements $3^e \alpha$ and $3^e \gamma$ lift canonically

to $[M, F/0]$ for large e . Therefore, for any pair of integers (x, y) $3^e x \alpha \oplus 3^e y \gamma$ gives a normal map $f_{x,y}: X^{8k} \rightarrow M$ so that

$$\tau_x + f^* \gamma_M = f^*(3^e x \alpha \oplus 3^e y \gamma) \quad [7].$$

Here γ_M is the stable normal bundle of M . Since the bundle $3^e x \alpha \oplus 3^e y \gamma$ is induced from

$[S^{8k-4q+2} \times_{S^1} S^{4q-1} / D_-^{8k-4q+2} \times_{S^1} S^{4q-1}, F/0]$, we may choose f so that $f_{x,y}$ restricted to $f_{x,y}^{-1}(D_-^{8k-4q+2} \times_{S^1} S^{4q-1})$ is a diffeomorphism. Excise the open manifold over which f is a diffeomorphism. We get a relative map of pairs

$$f'_{x,y}: (X, \partial X) \longrightarrow (D^{8k-4q+2}, S^{8k-4q+1}) X_{S^1} S^{4q-1}$$

such that $f'/\partial X'$ is a diffeomorphism.

The relative surgery obstruction of f' is the same as the surgery obstruction of f since all the spaces involved are simply connected [20]. So f' is normally cobordant to a homotopy equivalence which is a diffeomorphism on the boundary if and only if the surgery obstruction of f is 0. That is,

$$1/8[\text{Index } X - \text{Index } M] = 0.$$

But $\text{Index } X = \langle L_{2k}(\tau_X), [X] \rangle$
 $= \langle L_{2k}(f^*(\tau_M \oplus 3^e x \alpha \oplus 3^e y \gamma)), [X] \rangle$
 $= \langle L_{2k}(\tau_M \oplus 3^e x \alpha \oplus 3^e y \gamma), [X] \rangle$
 $= \langle L_{2k}(\tau_M \oplus 3^e x \alpha \oplus 3^e y \gamma), [M] \rangle$
 $= \text{Index } M$, for some choice of (x, y) , y odd

such that

$$f'_{x,y}: (X, \partial X) \longrightarrow (D^{8k-4q+2}, S^{8k-4q+1}) \times_{S^1} S^{4q-1}$$

is normally cobordant to a homotopy equivalence, which is a diffeomorphism on the boundary. Consider the pullback diagram

$$\begin{array}{ccc} (Y, \partial Y) & \xrightarrow{f''} & (D^{8k-4q+2}, S^{8k-4q+1}) \times S^{4q-1} \\ \downarrow \Pi & & \downarrow \\ (X, \partial X) & \xrightarrow{f'} & (D^{8k-4q+2}, S^{8k-4q+1}) \times_{S^1} S^{4q-1}. \end{array}$$

The orbit map Π is a principal bundle projection, since the action (trivial $\oplus \psi^{2n} C) \times 2qC$ is free. So there is a free action on $(Y, \partial Y)$ such that f'' is an S^1 equivariant homotopy equivalence. Further, $f''/\partial Y$ is a diffeomorphism. Let $\Sigma_{(x,y)}^{8k+1} = Y \mathbf{U}_{f''/\partial Y} S^{8k-4q+1} \times D^{4q}$. Then $\Sigma_{(x,y)}^{8k+1}$ has a smooth effective circle action given by the circle actions on the two component pieces. Clearly, the fixed point set is $4q + 2$.

By a Mayer-Vietoris argument $\sum_{(x,y)}^{8k+1}$ is a homology sphere. It is simply connected by van Kampen's theorem. Hence $\sum_{(x,y)}^{8k+1}$ is a homotopy $(8k + 1)$ -dimensional sphere.

It remains to show that this sphere does not bound a spin manifold. This is done by identifying the Pontryagin-Thom invariant of $\sum_{(x,y)}^{8k+1}$.

Consider the map on Thom spaces obtained via the following diagram of bundle maps:

$$\begin{CD} S^1 \times S^{4q-1} @>p_2>> S^{4q-1} \\ @VVHVV @VVHV \\ S^1 \times_{S^1} S^{4q-1} @>\gamma^{2n}>> CP^{2q-1} . \end{CD}$$

Here H is the orbit map for the action $\psi^{2n}C \times 2qC$, H is the Hopf map and p_2 is the projection. Let $X_{k,q} = S^{8k-4q}(CP^{2q-1})$ and $\Pi^*: [X_{k,q}, F/O_{(2)}] \rightarrow [S^{8k-4q+2}(S^{4q-1}VS^0), F/O_{(2)}]$ the map induced by this Thom space map. Here $X_{(p)}$ is the localization of X at p [18, 19].

We wish to calculate $\Pi^*(3^e x\alpha \oplus 3^e y\gamma)$. It is enough to calculate the image of $3^e(x\zeta' + y\sigma)$ under the map analogous to the above with $BSO_{(2)}$ replacing $F/O_{(2)}$.

$$\begin{aligned} [S^{8k-4q+2}(S^{4q-1}VS^0), BSO_{(2)}] &= \Pi_{8k+1}(BSO_{(2)}) \times \Pi_{8k-4q+2}(BSO_{(2)}) \\ \text{(by Bott periodicity)} \quad &= [\Pi_{8k+1} BSO \otimes Z_{(2)}] \times [\Pi_{8k-4q+2}(BSO) \\ &\quad \otimes Z_{(2)}] . \end{aligned}$$

So the group has exponent two. Therefore, the factor 3^e acts as identity and may be suppressed. The bundle ζ has a complex structure; in fact $\zeta' = re\zeta''$ where $\zeta'' = \beta^{4k-2q} \text{Th}(\mu^{2q-3})$. Since $H^* = 0$ on $K(CP^{2q-1})$, it follows by naturality of Thom isomorphism and the Bott periodicity that

$$(2.4) \quad \Pi^* \zeta'' = 0 \text{ and hence } \Pi^* \zeta' = 0 .$$

By Lemma 2.2, σ is the pullback of the generator of $\Pi_{8k}(BSO)$ under the collapsing map

$$S^{8k-4q}(CP^{2q-1})^{\gamma^{2n}} \longrightarrow S^{8k-4q}(CP^{2q-1}/CP^{2q-2})^{\gamma^{2n}} = S^{8k} .$$

By definition of Π^* , it follows that $\Pi^* \sigma$ is the composite of this generator with the map

$$S^{8k-4q}(S^{4q-1})^{\text{trivial}} \longrightarrow S^{8k-4q}(CP^{2q-1})^{\gamma^{2n}}$$

induced by the suspended Hopf map H on the Thom spaces. Looking at this map on the top cells, it is seen that Π^* has the form $M_k \gamma$ and therefore

$$(2.5) \quad \Pi^*(3^e x \alpha \oplus 3^e y \gamma) = M_k \eta \text{ since } y \text{ is odd.}$$

Elementary attaching considerations [17, 20] tell us that $\Pi^*(3^e x \alpha \oplus 3^e y \gamma)$ is the Pontryagin-Thom invariant of $\sum_{(x,y)}^{8k+1}$. By the results in §1 and (2.4) and (2.5) this completes our proof.

Now, we return to the proofs of computational Lemmas 2.2-2.3.

Proof of Lemma 2.2. Look at the following diagram

$$\begin{array}{ccccccc} 0 & \longleftarrow & K(\mathbb{C}P^{2q-2}) & \xleftarrow{i^*} & K(\mathbb{C}P^{2q-1}) & \xleftarrow{j^*} & K(\mathbb{C}P^{2q-1}/\mathbb{C}P^{2q-2}) \longleftarrow 0 \\ & & \downarrow \text{Thom} & & \downarrow \text{Thom} & & \begin{array}{c} \parallel \\ K(S^{4q-2}) \\ \downarrow \text{Thom} \end{array} \\ 0 & \longleftarrow & K(\mathbb{C}P^{2q-2}) & \xleftarrow{i'^*} & K(\mathbb{C}P^{2q-1})^{2n} & \xleftarrow{j'^*} & K(S^{4q}) \end{array}$$

where i is the inclusion and j is the collapse. j^* is a split monomorphism identifying the generator of $K(S^{4q-2})$ with μ^{2q-1} . $(j')^*$ identifies the generator $K(S^{4q})$ with $\text{Th}(\mu^{2q-1})\eta^{2n}$. Suspend this diagram $(8k - 4q)$ times and look at the map of the corresponding KO groups into the above diagram under the complexification map. Lemma 2.2 is now obvious.

Proof of Lemma 2.3.

$$\begin{aligned} \alpha &= \psi_R^3 r e \zeta'' - r e \zeta'', \quad \gamma = \psi_R^3 \sigma - \sigma, \\ \sigma \times C &= \zeta. \quad \text{Also } \zeta = \beta^{4k-2q} \text{Th}(\mu^{2q-1}) \\ \zeta'' &= \beta^{2k-2q} \text{Th}(\mu^{2q-3}). \end{aligned}$$

We can calculate $\text{ch}(\)$ and $\text{ch}(\)$ as follows;

$$\begin{aligned} \text{ch Th}(\mu^{2q-1}) &= \text{ch}(\mu^{2q-1}) \cup [\text{Td}(\eta^{2n})]^{-1} \cdot U \eta^{2n} \\ \text{ch Th}(\mu^{2q-3}) &= \text{ch}(\mu^{2q-3}) \cup [\text{Td}(\eta^{2n})]^{-1} \cdot U \eta^{2n}. \end{aligned}$$

We have $\text{ch}(\mu) = \text{ch}(\eta - 1) = e^c - 1$, where c is the generator of $H^2(\mathbb{C}P^{2q-1})$, [4]. Hence $\text{ch}(\mu^{2q-1}) = e^{2q-1}$. Further $\text{Td}(\eta^{2n}) = Q[c_1(\eta^{2n})]$ where $Q(x) = x/1 - e^{-x}$. So $[\text{Td}(\eta^{2n})]^{-1} = 1 - e^{-x}/x$ and $x = c_1(\eta^{2n}) = X(\eta^{2n}) = 2nc$. (Here X denotes the Euler class.) Therefore

$$(2.6) \quad \text{Td}[(\eta^{2n})]^{-1} = 1 - cd + 2n^2 \frac{c^2}{3} + \dots$$

$$\text{ch Th}(\mu^{2q-1}) = e^{2q-1}$$

$$(2.7) \quad \text{ch Th}(\mu^{2q-3}) = e^{2q-3} [1 + Ac + Bc^2] \cdot \left[1 - nc + \frac{2}{3} n^2 c^2 \right]$$

where $A = 2q - 3/2$, $B = 2q - 3/6 + (2q - 3)(2q - 4)/8$

$$(2.8) \quad \text{ch Th}(\mu^{2q-3}) = e^{2q-3} \left[1 + A'c + \left(\frac{2}{3}n^2 - nA + B \right) c^2 \right].$$

We don't need A' for further calculations.

$$(2.9) \quad \begin{aligned} \frac{2}{3}n^2 - nA + B &= 2n^2 + \frac{2q-3}{2} \left[-n + \frac{1}{3} + \frac{q-2}{2} \right] \\ &= \frac{(2q-3)(3q-6n+4) + 8n^2}{12}. \end{aligned}$$

We can obtain the Pontryagin classes of α and γ from (2.7)-(2.9).

$$\begin{aligned} p_j(\alpha) &= (-1)^j c_{2j}(\alpha \otimes c) \text{ by definition} \\ &= (-1)^j \cdot 2 \cdot c_{2j}(\psi_C^3 - 1)\zeta'' . \end{aligned}$$

Since our calculations are over a suspension, namely, $S^{8k-4q}(CP^{2q-1})^{\gamma^{2n}}$, products of cohomology classes vanish. It follows from (2.7)-(2.9) that the possible nonzero classes are $p_{2k}(\alpha)$, $p_{2k-1}(\alpha)$ and $p_{2k}(\gamma)$ only. These are easily written down using the fact that

$$c_i \psi_C^3 () = 3^i c_i () [1]$$

and

$$(2.10) \quad \begin{aligned} c_{2j} () &= (-1)(2j-1)! \text{ch}^{2j} () \\ \left[\begin{aligned} p_{2k}(\alpha) &= (-1)(4k-1)! (3^{4k}-1) \left[\frac{(2q-3-6n-4) + 8n^2}{6} \right] \\ &\quad \beta^{4k-2q}(c^{2q-1}) \cdot U_\gamma 2n \\ p_{2k-1}(\alpha) &= (4k-3)! (3^{4k-2}-1) \cdot 2 \cdot \beta^{4k-2q}(c^{2q-2}) \cdot U_\gamma^{2n} \\ p_{2k}(\gamma) &= (-1)(4k-1)! (3^{4k}-1) \beta^{4k-2q}(c^{2q-1}) \cdot U_\gamma^{2n} . \end{aligned} \right. \end{aligned}$$

Here β^{4k-2q} denotes the $(8k-4q)$ -fold suspension in cohomology. We may drop β^{2k-2q} and U_γ^{2n} from (2.10) with no loss of clarity.

$$\begin{aligned} \langle L_{2k}(\tau_M \oplus 3^e x\alpha \oplus 3^e y\gamma), [M] \rangle &= \langle L_{2k}(\tau_M), [M] \rangle \\ &+ \langle L_1(\tau_M) L_{2k-1}(3^e \times \alpha \oplus 3^e y\gamma), [M] \rangle \\ &+ \langle L_{2k}(3^e x\alpha \oplus 3^e y\gamma), [M] \rangle . \end{aligned}$$

So in order to prove the lemma, we must be able to choose x, y (with y odd) such that $\langle L_1(\tau_M) L_{2k-1}(3^e x\alpha \oplus 3^e y\gamma), [M] \rangle = -\langle L_{2k}(3^e x\alpha \oplus 3^e y\gamma), [M] \rangle$. Again, since α, γ are defined on a suspension, we have $L_m = 1_m p_m$ where $1_m = 2^{2m+1}(2^{2m-1}-1)/(2m-1)! j_{4m} \text{num}(B_m/4m)$ [8, 16] $j_{4m} = \text{order } J(S^{4m-1}) = \text{odd/odd} \cdot (3^{2m}-1)$ and $\text{num}(B_m/4m)$ is odd (B_m is the n th Bernoulli number [3]). Since $L_1(\tau_M) = p_1(M)/3$, we want

$$\left\langle p_1(M) \cdot \frac{1}{3} \cdot 1_{2k-1} p_{2k-1}(x\alpha), [M] \right\rangle = \langle -1_{2k}(p_{2k}(x\alpha) + p_{2k}(y\gamma), [M] \rangle .$$

This relation can be written as

$$y\alpha = xb$$

where

$$(2.11) \quad a = -\langle 1_{2k} p_{2k}(\gamma), [M] \rangle = \left\langle 2^{4k+1} \cdot \frac{\text{odd}}{\text{odd}} \cdot c^{2q-1}, [M] \right\rangle \text{ from (1.10) .}$$

We write $b = b' - b''$; $b'' = \langle L_{2k} p_{2k}(\alpha), [M] \rangle$

$$\begin{aligned} &= 2^{4k+1} \cdot \frac{\text{odd}}{\text{odd}} \cdot \left[\frac{(2q-3)(3q-6n+4) + 8n^2}{6} \right] \cdot \langle c^{2q-1}, [M] \rangle \\ &= 2^{4k+1} \cdot \frac{\text{odd}}{\text{odd}} \cdot \frac{1}{2} \cdot \frac{\text{odd}}{\text{odd}} [(3q-6n+4) + 8n^2] \cdot \langle c^{2q-1}, [M] \rangle . \end{aligned}$$

Therefore

$$(2.12) \quad \begin{aligned} b'' &= 2^{4k} \cdot \frac{\text{odd}}{\text{odd}} \cdot \langle c^{2q-1}, [M] \rangle \text{ if } q \text{ is nonzero mod } 4 \text{ and } n \text{ is} \\ &\text{even or } q \text{ is zero mod } 4 \text{ and } n \text{ is odd.} \\ b' &= \left\langle \frac{p_1(M)}{3} 1_{2k-1} p_{2k-1}(\alpha), [M] \right\rangle . \text{ To calculate } b' \text{ we need } p_1(M) . \end{aligned}$$

$M = S^{8k-4q+2} \times_{S^1} S^{4q-1} \xrightarrow{\Pi} CP^{2q-1}$ be the bundle (trivial $\oplus \gamma^{2n}$).

$\tau_M = \Pi^* \tau CP^{2q-1} \oplus$ the tangent along the fibers, is stably equal to $\Pi^* \tau CP^{2q-1} \oplus \Pi^* \gamma^{2n}$. Hence, $p_1(\tau_M) = 2q\Pi^*c^2 \pm 4n^2\Pi^*c^2 = (2q \pm 4n^2)\Pi^*c^2$. Substituting this value of $p_1(M)$ in b' , we see that

$$(2.13) \quad b' = 2^{4k} \cdot \frac{\text{even}}{\text{odd}} \cdot \langle c^{2q-1}, [M] \rangle$$

and

$$(2.14) \quad b = b' - b'' = 2^{4k} \cdot \frac{\text{odd}}{\text{odd}} \cdot \langle c^{2q-1}, [M] \rangle$$

if q is nonzero mod 4 and n is even or if q is zero mod 4 and n is odd. It is clear from (2.11)-(2.14) that there are integers (x, y) , y odd such that $ya = xb$. This proves the lemma.

3. Semi-free actions. We discuss semi-free circle actions on homotopy spheres not bounding spin manifolds with standard spheres as fixed point sets. The fixed point set has co-dimension $4q$ [15].

G. Bredon has shown that for every k there is a $8k + 2$ homotopy sphere not bounding a spin manifold which admits a smooth semi-free action of the circle group [6], and in [16] Schultz proved the following complementary result in dimension $8k + 1$:

THEOREM 3.1 [16]. *Let $k > 0$ and $q < 2k + 1$. Assume that q is not equal to 1 and not equal to zero mod 4. Then there is a semi-free action of the circle on some homotopy $(8k + 1)$ -sphere not bounding a spin manifold whose fixed point set is an ordinary sphere of co-dimension $4q$.*

Levine has shown that co-dimension 4 is not realizable [12]; we wish to show that the restriction on q may be relaxed in general and may be removed in low dimensional cases.

THEOREM 3.2. *Let $k > 0$ and $q < 2k + 1$. An ordinary $(8k - 4q + 1)$ -sphere can be realized as the fixed point set of a smooth semi-free circle action on a homotopy $(8k + 1)$ -sphere not bounding a spin manifold provided q is not equal to 0 mod 32.*

Our arguments closely follow those in [16] and the first part of this paper. First we establish an analogue of Lemma 2.3. Let

$$\begin{aligned}
 s &= q - 1 \\
 M &= S^{8k-4q+2} \times CP^{2s+1} \\
 \alpha &= \psi_R^3 re \beta^{4k-2s-1} (\mu^{2s-3}) \\
 &= \psi_R^3 \sigma - \sigma, \text{ where } \sigma \otimes C \text{ generates } K \left[S^{8k-4s-2} \left(\frac{CP^{2s+1}}{CP^{2s}} \right) \right] = K(S^{8k}).
 \end{aligned}$$

We can calculate the Pontryagin classes of α and γ just as in Lemma 2.3. We see that

$$(3.3) \quad p_n(\alpha) = (-1)^{n+1} \cdot 2 \cdot (2n - 1)! \cdot (3^{2n} - 1) \text{ ch}^{2n} \beta^{8k-4s-2} (\mu^{2s-3})$$

and

$$\text{ch}(\mu^{2s-3}) = (e^c - 1)^{2s-3} = c^{2s-3} + A'c^{2s-1} + B'c^{2s+1} + \text{terms}$$

of even degree in c . Here

$$\begin{aligned}
 A' &= \frac{2s - 3}{6} + \frac{1}{4} \binom{2s - 3}{2} \\
 B' &= \frac{2s - 3}{5!} + \binom{2s - 3}{2} \left[\frac{1}{4!} + \frac{1}{(3!)^2} \right] + \frac{1}{8} \binom{2s - 3}{3} + \frac{1}{16} \binom{2s - 3}{4}.
 \end{aligned}$$

Therefore, the only nonzero Pontryagin classes are $p_{2k}(\alpha)$, $p_{2k-1}(\alpha)$

and $p_{2k-2}(\alpha)$. These are given by

$$\begin{aligned}
 p_{2k}(\alpha) &= (-2)(4k-1)! (3^{4k}-1)B'c^{2s-1} \\
 p_{2k-1}(\alpha) &= (2)(4k-3)! (3^{4k-2}-1)A'c^{2s-2} \\
 p_{2k-2}(\alpha) &= (-2)(4k-5)! (3^{4k-4}-1)c^{2s-3}.
 \end{aligned}
 \tag{3.5}$$

Note that the right sides have to be suspended enough times to place them in the proper cohomology groups.

As in Lemma 2.2, $\sigma \otimes C = \beta^{4k-2s-1}(\mu^{2s+1})$ and hence

$$p_{2k}(\gamma) = (-1)(4k-1)! (3^{4k}-1)c^{2s+1}
 \tag{3.6}$$

and $p_i(\gamma) = 0$ if $i \neq 2k$. Also note that

$$\begin{aligned}
 L_1(CP^{2s+1}) &= \frac{1}{3}p_1(CP^{2s+1}) = \frac{2}{3}(s+1) \\
 L_2(CP^{2s+1}) &= \frac{1}{45}(7p_2 - p_1^2) = \frac{1}{45}(s+1)(10s-3)
 \end{aligned}
 \tag{3.7}$$

α, γ can be considered to be bundles over M ; $3^e\alpha, 3^e\gamma$ are therefore elements of $[M, F/0]$ for large values of e . We have the following proposition

PROPOSITION 3.8. *There are integer pairs (x, y) such that y is odd and $\langle L_{2k}(\tau_M \oplus 3^e\alpha \oplus 3^e\gamma), [M] \rangle = 0$, provided q is even and nonzero mod 32.*

Proof of Theorem 3.2. Let (x, y) be an integer pair given by Proposition 3.8. $3^e\alpha \oplus 3^e\gamma$ gives a normal map $f_{x,y}: X^{8k} \rightarrow M$ such that

$$\tau_X + f^*(\gamma_M) = f^*(3^e\alpha \oplus 3^e\gamma).$$

It is possible to choose f such that $f/f_{x,y}^{-1}(D_-^{8k-4s-2} \times CP^{2s+1})$ is a diffeomorphism. Excising the manifold on which f is a diffeomorphism, we get a relative map $f'_{x,y}: (X, \partial X) \rightarrow (D^{8k-4s-2}, S^{8k-4s-3}) \times CP^{2s+1}$ such that $f'/\partial X$ is a diffeomorphism. The relative surgery obstruction of f' is the same as the surgery obstruction of f since all spaces involved are simply connected [20]. So f' is normally co-bordant to a homotopy equivalence if and only if surgery obstruction of $f = 1/8 [\text{Index } X - \text{Index } M] = 0 = \text{Index } M$, since $\text{Index } M = 0$. But

$$\begin{aligned}
 \text{Index } X &= \langle L_{2k}\tau_X, [X] \rangle = \langle L_{2k}f^*\tau_M \oplus 3^e\alpha \oplus 3^e\gamma, [X] \rangle \\
 &= \langle L_{2k}\tau_M \oplus 3^e\alpha \oplus 3^e\gamma, [M] \rangle.
 \end{aligned}$$

This last expression is zero by Proposition 3.8 and our choice of x

if q is even. So f' is normally co-bordant to a homotopy equivalence which is a diffeomorphism on the boundary. Consider the pullback diagram

$$\begin{CD} (Y, \partial Y) @>f''>> (D^{8k-4s-2}, S^{8k-4s-3}) \times S^{4s+3} \\ @VVV @VV1 \times H V \\ (X, \partial X) @>f'>> (D^{8k-4s-2}, S^{8k-4s-3}) \times CP^{2s+1} . \end{CD}$$

Here H is the Hopf fiber map. The pullback $(Y, \partial Y) \rightarrow (X, \partial X)$ is a principal S^1 bundle map and hence there is a circle action on $(Y, \partial Y)$ such that f'' is equivariant. Further, f'' is a homotopy equivalence which is a diffeomorphism on ∂Y . Therefore $\sum_{x,y}^{8k+1} = YU_{f', \partial Y} S^{8k-4s-3} \times D^{4s+4}$ has a smooth effective circle action given by the actions on the component pieces. This action is semi-free and the fixed point set is $S^{8k-4q+1}$. By van Kampen's theorem $\sum_{x,y}^{8k+1}$ is simply connected. A Mayer-Vietoris argument shows that it is a homology sphere. A discussion similar to the one in the proof of Theorem 2.1 shows that this sphere does not bound a spin manifold.

When q is odd, the theorem is contained in [16].

REMARK 3.4. It appears from the calculations in this paper that different choices of α, γ might relax the conditions in Theorem 3.1 further. But the calculation of the surgery obstructions become unmanageably involved. Probably a transition to use of some machinery from analytical number theory is called for. The reader may be referred to [8].

4. Realizability of any ordinary sphere as a fixed point set. In this final section we assemble our results into a single main theorem. We need to state $(8k + 2)$ dimensional versions of Theorems 2.1, 3.2.

THEOREM 4.1. *Let $k > 0$. There is a smooth effective circle action on the homotopy $(8k + 2)$ -sphere not bounding a spin manifold with fixed point set a standard sphere of codimension $4q + 2, q > 1$.*

THEOREM 4.2. *Let $k > 0, q < 2k + 1$ and q not equal to 1. Any ordinary sphere $S^{8k-4q+2}$ can be realized as the fixed point set of a smooth semi-free action of the circle group on a homotopy $(8k + 2)$ -sphere not bounding a spin manifold.*

These results were known to Bredon [6]. The following, then, summarises our results.

MAIN THEOREM 4.3. *Any ordinary sphere S^n ($n > 0$) can be realized as the fixed point set of an infinite number of smooth effective actions of the circle group on a Σ^{8k+1} or Σ^{8k+2} not bounding spin manifolds, for every $k > 0$, $8k > n - 1$. These actions are distinguished by their local representation at fixed points.*

Proof. The verification splits into different cases according to the value of n modulo 4.

Case (1). $n = -1$ or $0 \pmod{4}$

This follows from Theorems 2.1 and 4.1. These actions are not semi-free.

Case (2). $n = 1$ or $2 \pmod{4}$

Theorem 4.2 gives semi-free smooth effective actions on Σ^{8k+2} with S^{4t-2} as fixed point sets. Theorem 3.2 gives smooth semi-free actions on Σ^{8k+1} with S^{4t+1} as fixed point sets provided $2k - 2$ is nonzero mod 32. We are not able to remove this provision keeping the actions semi-free. However, there is no difficulty in obtaining the spheres in Case (2) as fixed point sets of nonsemi-free smooth effective actions on Σ^{8k+1} , Σ^{8k+2} not bounding spin manifolds. Computations identical to those in § 2 show that it is enough to consider actions with local representation $[\text{trivial} \oplus \psi^{8n}C \oplus (2q - 1)C]$.

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