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A GENERALIZATION OF A CLASSICAL NECESSARY CONDITION FOR CONVERGENCE OF CONTINUED FRACTIONS[1]

ROBERT HELLER AND F. A. ROACH

One of the most frequently cited necessary conditions for convergence of continued fractions is the divergence of a particular series. In this paper, we show that convergence of a continued fraction implies divergence of each member of an infinite collection of series.

We will be concerned with continued fractions which are of, or can be put into (cf. Wall [4], pp. 19-26), the form

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots . \tag{1}$$

If we let

$$\begin{aligned} &A_0 = b_0 , \quad A_1 = b_1 A_0 + 1 , \quad B_0 = 1 , \quad B_1 = b_1 , \\ &A_p = b_p A_{p-1} + A_{p-2} , \quad \text{and} \\ &B_p = b_p B_{p-1} + B_{p-2} , \quad p = 2, 3, 4, \cdots , \end{aligned} \tag{2}$$

then the nth approximant of (1) is given by A_n/B_n. As is customary, we say that (1) converges provided that not infinitely many of the denominators B_p are zero and $\{A_p/B_p\}$ converges to a finite limit.

The principal result given in this paper is the following theorem.

THEOREM. *Suppose that u is a complex number, v is a complex number such that $-4 < uv \leqq 0$, and $u = 0$ if $v = 0$. If both $\sum |b_{2p-1} - u|$ and $\sum |b_{2p} - v|$ converge, then* (1) *diverges.*

Considering the case where $u = v$, we immediately have the following result.

COROLLARY. *In order for* (1) *to converge, it is necessary that for each real number k between -2 and 2, $\sum |b_p - ki|$ diverge.*

If $k = 0$, this becomes what is often referred to as von Koch's theorem. According to Perron [2] p. 235, it was first proved by Stern [3] in 1860; additional information concerning the numerators and denominators of the approximants was obtained by von Koch [1] in 1895 (cf. Wall [4], pp. 27-29).

[1] This paper is dedicated to the memory of Keith Heller.

The proof is accomplished by establishing that the sequence $\{B_p\}$ is bounded. This, together with the identity

$$|(A_n/B_n) - (A_{n+1}/B_{n+1})| = 1/|B_nB_{n+1}|$$

implies that (1) is divergent. We will establish the boundedness of this sequence by comparing it with the sequence $\{D_p\}$ of denominators of the approximants of the periodic continued fraction

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{u} + \frac{1}{v} + \cdots \tag{3}$$

which is divergent if and only if u and v satisfy the hypothesis. The sequence $\{D_p\}$ is bounded if $u = 0$. If $uv \neq 0$, then (3) is equivalent to

$$\frac{z}{u}\left[\frac{1}{z} + \frac{1}{z} + \frac{1}{z} + \cdots\right]$$

where $z = \sqrt{(|uv|)}i$. Let r denote $[z + \sqrt{(z^2+4)}]/2$ and let s denote $[z - \sqrt{(z^2+4)}]/2$. Since $r + s = z$ and $-rs = 1$, from (2) we have that for $p = 2, 3, 4, \cdots$,

$$D_p = (r+s)D_{p-1} - rsD_{p-2} .$$

From this it follows that $D_p - rD_{p-1} = s^p$ and $D_p - sD_{p-1} = r^p$ and hence,

$$(r-s)D_{p-1} = r^p - s^p .$$

Since $-4 < z^2 < 0$, we have that $z^2 + 4$ is a positive number and therefore the complex conjugate of r is $[-z + \sqrt{(z^2+4)}]/2$. Thus, $|r^p| = |s^p| = 1$ and $2/\sqrt{(z^2+4)}$ is a bound for $|D_p|$.

Let x_p denote $B_p - D_p$, c_{2p-1} denote $b_{2p-1} - u$, and c_{2p} denote $b_{2p} - v$. We will now show that for each positive integer n,

$$\begin{aligned} x_{2n-1} &= \sum_{p=1}^{2n-1} c_pD_{2n-1-p}B_{p-1} \qquad \text{and} \\ x_{2n} &= \sum_{p=1}^{2n} c_pD'_{2n-p}B_{p-1} , \end{aligned} \tag{4}$$

where $D'_p = (u/v)D_p$ if p is odd and $D'_p = D_p$ if p is even.

Notice that $x_1 = c_1D_0B_0$ and $x_2 = c_1D'_1B_0 + c_2D'_0B_1$. Suppose that for some n, (4) holds true. From (2) we see that

$$x_{2n+1} = (u + c_{2n+1})B_{2n} + B_{2n-1} - (uD_{2n} + D_{2n-1})$$

which is $ux_{2n} + x_{2n-1} + c_{2n+1}B_{2n}$. Replacing x_{2n} and x_{2n-1} with the appropriate sums, we have that x_{2n+1} is

$$\sum_{p=1}^{2n-1} c_p(uD'_{2n-p} + D_{2p-1-p})B_{p-1} + c_{2n}D_1B_{2p-1} + c_{2p+1}D_0B_{2p} .$$

This expression can be written as $\sum_{p=1}^{2n+1} c_p D_{2n+1-p} B_{p-1}$. In a similar manner, we find that

$$x_{2n+2} = \sum_{p=1}^{2n+2} c_p D'_{2n+2-p} B_{p-1} .$$

So, for each n, the equations (4) hold true.

We will now show that there exists a nonnegative number K and a positive number M such that

$$(5) \qquad |x_n| \leqq K \prod_{p=1}^{n-1} (1 + M|c_{p+1}|) , \quad n = 2, 3, 4, \cdots .$$

Let M denote a number such that for each n, $|D_n| \leqq M$ and $|D'_n| \leqq M$. From (4),

$$|x_n| \leqq M \sum_{p=1}^{n} |c_p B_{p-1}| \leqq M \sum_{p=1}^{n} |c_p|(|D_{p-1}| + |x_{p-1}|) .$$

By hypothesis, $\sum |c_p|$ is convergent and hence this sum does not exceed

$$K + \sum_{p=1}^{n} M|c_p x_{p-1}|$$

where $K = M^2 \sum_{p=1}^{\infty} |c_p|$. Since $x_0 = 0$, we have that

$$(6) \qquad |x_n| \leqq K + \sum_{p=1}^{n-1} M|c_{p+1} x_p| , \quad n = 1, 2, 3, \cdots ,$$

where $\sum_{p=1}^{0} M|c_{p+1} x_p| = 0$. Suppose that j is a positive integer such that for $n = 1, 2, \cdots, j$, (5) holds true. Then, combining (5) and (6), we have

$$|x_{j+1}| \leqq K + \sum_{p=1}^{j} \left[M|c_{p+1}| K \prod_{q=1}^{p-1} (1 + M|c_{q+1}|) \right] .$$

The right-hand member of this inequality can be reduced to

$$K \prod_{p=1}^{j} (1 + M|c_{p+1}|) .$$

Thus by mathematical induction, (5) is established.

Since the series $\sum M|c_{p+1}|$ is convergent, so is the product $\prod (1 + M|c_{p+1}|)$. So, the sequence $\{x_p\}$ is bounded. Consequently, the sequence $\{B_p\}$ is bounded and (1) is divergent.

This theorem yields a new necessary condition for convergence of (1) which is considerably stronger than the classical condition. The new condition is not, however, sufficient for the convergence of (1). In fact, even the divergence of both of the series $\sum |b_{2p-1} - u|$ and $\sum |b_{2p} - v|$ for every u and v satisfying the conditions of the theorem is not sufficient for convergence of (1). This can be seen by considering the following example. For $p = 1, 2, 3, \cdots$, let

$$\begin{aligned} b_{3p-2} &= 1\,, \\ b_{3p-1} &= -1 \qquad \text{and} \\ b_{3p} &= 1\,. \end{aligned}$$

In this case, $B_{3p-1} = 0$, $p = 1, 2, 3, \cdots$, but both of the series above are divergent regardless of the values of u and v.

References

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Mississippi State University
Mississippi State, MS 39762
and
University of Houston
Houston, TX 77004