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THE TRANSFER OF INVARIANT PAIRINGS TO LATTICES

T. J. ENRIGHT AND R. PARTHASARATHY

In the article "On the fundamental series of a real semisimple Lie algeba" two covariant functors, the completion functor and the lattice functor, are introduced. In this article, we study the behavior of invariant pairings and invariant Hermitian pairings under the action of these functors.

Let $\mathfrak{m}$ be a finite dimensional reductive Lie algebra over $\boldsymbol{C}$, the field of complex numbers, and let $U(\mathfrak{m})$ denote its universal enveloping algebra. Let $\mathfrak{h}$ be a CSA of $\mathfrak{m}$ with roots Δ, a positive system Q of Δ and Weyl group $\mathscr{W}$. For each $\alpha \in Q$, let $\mathfrak{m}_\alpha$ denote the α root space in $\mathfrak{m}$ and choose vectors $\bar{H}_\alpha$, X_α, $X_{-\alpha}$ such that

$$(1.1) \qquad X_\alpha \in \mathfrak{m}_\alpha\,, \qquad X_{-\alpha} \in \mathfrak{m}_{-\alpha}\,, \qquad \bar{H}_\alpha = [X_\alpha, X_{-\alpha}]\,, \qquad \alpha(\bar{H}_\alpha) = 2\,.$$

Then $\bar{H}_\alpha$, X_α, $X_{-\alpha}$ is called a standard triple and spans a subalgebra $\mathfrak{a}^{(\alpha)}$ of $\mathfrak{m}$ isomorphic to $sl(2)$. Choose and fix once and for all an involutive anti-automorphism σ of $\mathfrak{m}$ such that σ restricted to $\mathfrak{h}$ equals the identity. Fix a real form $\mathfrak{m}_0$ of $\mathfrak{m}$ such that $[\mathfrak{m}_0, \mathfrak{m}_0]$ is a compact real form of $[\mathfrak{m}, \mathfrak{m}]$ and $\mathfrak{m}_0 \cap \mathfrak{h} = \mathfrak{h}_0$ is a real form of $\mathfrak{h}$. Let $\mathfrak{m}_{\boldsymbol{R}}$ be the real Lie algebra underlying $\mathfrak{m}$ and let $\bar{\sigma}$ be the $\boldsymbol{R}$-linear anti-automorphism of $U(\mathfrak{m}_{\boldsymbol{R}})$ uniquely determined by the condition that $-\bar{\sigma}$ restricted to $\mathfrak{m}$ equals the conjugation of $\mathfrak{m}$ with respect to $\mathfrak{m}_0$. For $\mathfrak{m}$-modules A and B, a bilinear (resp. Hermitian) pairing $\langle \cdot, \cdot \rangle$ of A and B is called *invariant* if $\langle x\cdot a, b\rangle = \langle a, x^\sigma \cdot b\rangle$ (resp. $\langle x\cdot a, b\rangle = \langle a, x^{\bar{\sigma}}\cdot b\rangle$), $a \in A$, $b \in B$, $x \in U(\mathfrak{m})$. Denote by $I_\mathfrak{m}(A, B)$ (resp. $IH_\mathfrak{m}(A, B)$) the $\boldsymbol{C}$-linear (resp. $\boldsymbol{R}$-linear) space of invariant (resp. invariant Hermitian) pairings of A and B. If $\mathfrak{g}$ is any finite dimensional Lie algebra over $\boldsymbol{C}$ with $\mathfrak{m} \subset \mathfrak{g}$ and if $\sigma_\mathfrak{g}$ (resp. $\bar{\sigma}_\mathfrak{g}$) is an involutive anti-automorphism of $U(\mathfrak{g})$ (resp. $U(\mathfrak{g}_{\boldsymbol{R}})$) which restricts to σ (resp. $\bar{\sigma}$) on $\mathfrak{m}$, then we define $\mathfrak{g}$-invariant pairings as above with σ replaced by $\sigma_\mathfrak{g}$ (resp. $\bar{\sigma}$ replaced by $\bar{\sigma}_\mathfrak{g}$) and $U(\mathfrak{m})$ replaced by $U(\mathfrak{g})$. For $\mathfrak{g}$-modules A and B, we denote by $I_\mathfrak{g}(A, B)$ (resp. $IH_\mathfrak{g}(A, B)$) the spaces of $\mathfrak{g}$-invariant (resp. $\mathfrak{g}$-invariant Hermitian) pairings of A and B.

For each $\alpha \in Q$, let C_α denote the completion functor determined by the standard triple $\bar{H}_\alpha$, X_α, $X_{-\alpha}$. If $\mathfrak{m} = sl(2)$ then we write H, X, Y in place of $\bar{H}_\alpha$, X_α, $X_{-\alpha}$ and denote the functor C_α by C (there is only one positive root in this case). Also in this case for $n \in \boldsymbol{N}$ (integer $\geqq 0$) and A an $\mathfrak{m}$-module we write $A[n]$ for the subspace of H-eigenvectors with eigenvalue n and $A^X[n]$ for the subspace of $A[n]$

of vectors mapped to zero by X. In the general case, Bouaziz [1] and Deodhar [3] have shown that for any $\mathfrak{g}$-module A in the category $\mathscr{I}_{\mathfrak{g}}(\mathfrak{m})$ (cf. Definition 4.1 [5]) there is a lattice above A. Let $A_s (s \in \mathscr{W})$ denote this lattice above A and define the functor τ by the formula: $\tau(A) = A_1/\sum_{s\neq 1} A_s$. $\tau(A)$ is a $U(\mathfrak{m})$-finite $\mathfrak{g}$-module.

We can now state the main results of this article. First assume $\mathfrak{m} = sl(2)$. For $A, B \in \mathscr{I}(\mathfrak{m})$ and $\varphi \in I_{\mathfrak{m}}(A, B)$ (resp. $IH_{\mathfrak{m}}(A, B)$) there exists a unique pairing $C(\varphi) \in I_{\mathfrak{m}}(C(A), C(B))$ (resp. $IH_{\mathfrak{m}}(C(A), C(B))$) such that

(i) $C(\varphi)$ is zero on $(A \times C(B)) \cup (C(A) \times B)$

(ii) for all $n \in N$, $a \in C(A)^X[n]$, $b \in C(B)^X[n]$

$$C(\varphi)(a, b) = \frac{1}{n!(n+1)!}\varphi(Y^{n+1}a, Y^{n+1}b) .$$

This result is given below as Proposition 3.1 and is a mild variation of a result in [5] which concerns only invariant forms. The main result of § 3 is Theorem 3.4 which asserts that if $A, B \in \mathscr{I}_{\mathfrak{g}}(\mathfrak{m})$ and $\varphi \in I_{\mathfrak{g}}(A, B)$ (resp. $IH_{\mathfrak{g}}(A, B)$) then $C(\varphi) \in I_{\mathfrak{g}}(C(A), C(B))$ (resp. $IH_{\mathfrak{g}}(C(A), C(B))$). For $\varphi \in I_{\mathfrak{g}}(A, A)$, this theorem is precisely Proposition 8.5 [5]. The proof given there is based on an elaborate computation. The proof given in this article is more conceptual and is based on properties of certain vector valued pairings.

In § 4 we apply the results of § 3 to the setting of general reductive Lie algebras $\mathfrak{m}$. Let t_0 be the unique element of $\mathscr{W}$ such that $t_0 Q = -Q$ and let $t_0 = s_{\alpha_1} \circ \cdots \circ s_{\alpha_d}$ be a reduced expression for t_0 ($\alpha_i \in Q$, simple roots). Let $A, B \in \mathscr{I}_{\mathfrak{g}}(\mathfrak{m})$ and $\varphi \in I_{\mathfrak{g}}(A, B)$ (resp. $IH_{\mathfrak{g}}(A, B)$). Then (cf. Proposition 4.2), the pairing $C_{\alpha_1} \circ \cdots \circ C_{\alpha_d}(\varphi)$ on $A_1 \times B_1$ is independent of the reduced expression for t_0 and is zero on $(\sum_{s\neq 1} A_s \times B_1) \cup (A_1 \times \sum_{s\neq 1} B_s)$. This pairing induces a pairing of $\tau(A)$ and $\tau(B)$ which we denote by $\tau(\varphi)$. Clearly, from the results of § 3, $\tau(\varphi) \in I_{\mathfrak{g}}(\tau(A), \tau(B))$ (resp. $IH_{\mathfrak{g}}(\tau(A), \tau(B))$).

Define the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in Q} \mathfrak{m}_\alpha$. For any $\mathfrak{b}$-module M, we denote the induced module from $\mathfrak{b}$ to $\mathfrak{m}$ by $U(M)$; i.e., $U(M) = U(\mathfrak{m}) \otimes_{U(\mathfrak{b})} M$. The main result of this last section concerns an important example where the map $\varphi \mapsto \tau(\varphi)$ is surjective and preserves nondegenerate pairings. This result (Proposition 4.5) asserts the following:

Let M and N be locally finite $\mathfrak{b}$-modules which are semisimple as $\mathfrak{h}$-modules with finite dimensional integral weight spaces. Assume that $U(M)$ and $U(N)$ admit nondegenerate invariant forms. Then the maps:

$$\tau\colon I_{\mathfrak{m}}(U(M), U(N)) \longrightarrow I_{\mathfrak{m}}(\tau(U(M)), \tau(U(N)))$$
$$\tau\colon IH_{\mathfrak{m}}(U(M), U(N)) \longrightarrow IH_{\mathfrak{m}}(\tau(U(M)), \tau(U(N)))$$

are surjections. Moreover, both maps carry nondegenerate pairings to nondegenerate pairings.

This result is an important part of the theory of the functor τ. In particular, it is used in "The representations of complex semisimple Lie groups" [6] to show that every irreducible representation of a complex Lie group is infinitesimally equivalent to the image under τ of an irreducible highest weight module.

2. Notation. We continue with the notation of § 1. Let $\mathfrak{n} = \sum_{\alpha \in Q} \mathfrak{m}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha \in Q} \mathfrak{m}_{-\alpha}$. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{m} = \mathfrak{n}^- \oplus \mathfrak{b}$. For any $\mu \in \mathfrak{h}^*$, let C_μ denote the one dimensional $\mathfrak{b}$-module corresponding to μ and let $V_{\mathfrak{m},Q,\mu} = U(\mathfrak{m}) \otimes_{U(\mathfrak{b})} C_\mu$ denote the Verma module with highest weight μ. For any $\mathfrak{m}$-module A, let A^* be the contragradient module. Let $Z(\mathfrak{m})$ denote the center of $U(\mathfrak{m})$ and $Z(\mathfrak{m})^\wedge$ the set of homomorphisms of $Z(\mathfrak{m})$ into C. It is well known that $Z(\mathfrak{m})^\wedge$ is parametrized by the Weyl group orbits in $\mathfrak{h}^*$. For $\chi \in Z(\mathfrak{m})^\wedge$ and an $\mathfrak{m}$-module A, let A_χ equal the submodule of A where $z - \chi(z)\cdot 1$ is locally nilpotent for all $z \in Z(\mathfrak{m})$. Let $A^\mathfrak{n}$ denote the $\mathfrak{h}$-submodule of A of vectors mapped to zero by $\mathfrak{n}$; and, for $\mu \in \mathfrak{h}^*$, let $A^\mathfrak{n}[\mu]$ be the subspace of $A^\mathfrak{n}$ of vectors of weight μ. Let $U^m(\mathfrak{m})$ denote the usual filtration of $U(\mathfrak{m})$, $m \in N$. For complex vector spaces A, B and F, and F_0 a real form of F, we say that φ is a Hermitian map of $A \times B$ into F if φ is linear in the first variable and conjugate linear in the second variable. If $A = B$, we say φ is a Hermitian form if φ is a Hermitian map and $\varphi(a, b) = \overline{\varphi(b, a)}$, $a, b \in A$ and denoting conjugation of F with respect to F_0.

3. Invariant and invariant Hermitian pairings. Let notation be as in § 1 except that we shall write $\mathfrak{a}$ in place of $\mathfrak{m}$ and assume $\mathfrak{a} \cong sl(2)$. H, X, Y will be a standard triple for $\mathfrak{a}$ and C will denote the completion functor with respect to H, X, Y defined on the categories $\mathscr{I}_\mathfrak{g}(\mathfrak{a})$, $\mathfrak{g}$ a finite dimensional Lie algebra which contains $\mathfrak{a}$.

PROPOSITION 3.1. *Let $A, B \in \mathscr{I}(\mathfrak{a})$ and let $\xi_n (n \in N)$ be nonzero constants in $\boldsymbol{R}$. For any φ in $I_\mathfrak{a}(A, B)$ (resp. $IH_\mathfrak{a}(A, B)$) there exists a unique pairing $C(\varphi) = {}^c\langle\cdot,\cdot\rangle$ in $I_\mathfrak{a}(C(A), C(B))$ (resp. $IH_\mathfrak{a}(C(A), C(B))$) such that*

(i) *${}^c\langle\cdot,\cdot\rangle$ equals zero on $(C(A) \times B) \cup (A \times C(B))$,*

(ii) *for each $n \in N$, $a \in C(A)^X[n]$, $b \in C(B)^X[n]$,*

$$^c\langle a, b\rangle = \xi_n \langle Y^{n+1}a, Y^{n+1}b\rangle .$$

For φ in $I_\mathfrak{a}(A, A)$, the proposition is precisely Proposition 8.1 [5]. For $\varphi \in IH_\mathfrak{a}(A, A)$, the proof of the proposition is entirely analogous. The case $B \neq A$ is proved exactly as the case $A = B$ and we omit the details.

By Proposition 3.1, for each set of nonzero constants $\xi_n (n \in N)$ we have the transfer maps for $A, B \in \mathscr{F}(\mathfrak{a})$,

$$\begin{aligned} &\varphi \longmapsto C(\varphi) \\ &I_{\mathfrak{a}}(A, B) \longrightarrow I_{\mathfrak{a}}(C(A), C(B)) \quad \text{and} \\ &IH_{\mathfrak{a}}(A, B) \longrightarrow IH_{\mathfrak{a}}(C(A), C(B)) . \end{aligned} \tag{3.1}$$

DEFINITION 3.2. We say the map $\varphi_{A,B} \mapsto C(\varphi_{A,B})$ for A, B in $\mathscr{F}(\mathfrak{a})$ and $\varphi_{A,B} \in I_{\mathfrak{a}}(A, B)$ (resp. $IH_{\mathfrak{a}}(A, B)$) is *compatible with tensoring* if for any finite dimensional $\mathfrak{a}$-module F and any $\varphi_F \in I_{\mathfrak{a}}(F, F)$ (resp. $IH_{\mathfrak{a}}(F, F)$)

$$C(\varphi_F \otimes \varphi_{A,B}) = \varphi_F \otimes C(\varphi_{A,B}) .$$

(Here we identify canonically $C(F \otimes A)$ with $F \otimes C(A)$ and $C(F \otimes B)$ with $F \otimes C(B)$.)

PROPOSITION 3.3. *The maps $\varphi \to C(\varphi)$ given in* (3.4) *are compatible with tensoring if and only if there is a nonzero constant ξ such that*

$$\xi_n = \xi / n!(n+1)! \quad (n \in N;\ 0! = 1) .$$

In the case $\varphi \in I_{\mathfrak{a}}(A, A)$, the proposition is precisely Proposition 8.4 [5] and for $\varphi \in IH_{\mathfrak{a}}(A, A)$ the proof of the proposition is entirely analogous. The proof in the case $A \neq B$ is exactly the same as the case $A = B$ and we omit the details.

Now we consider the relative situation $\mathfrak{a} \subseteq \mathfrak{g}$. *Fix the constants $\xi_n = 1/n!(n+1)!$ $(n \in N)$ and let $\varphi \mapsto C(\varphi)$ be the map given by* (3.1) *for this choice of constants.*

THEOREM 3.4. *Let $A, B \in \mathscr{F}_{\mathfrak{g}}(\mathfrak{a})$ and let $\varphi \in I_{\mathfrak{g}}(A, B)$* (resp. $IH_{\mathfrak{g}}(A, B)$). *Then $C(\varphi) \in I_{\mathfrak{g}}(C(A), C(B))$* (resp. $IH_{\mathfrak{g}}(C(A), C(B))$).

For $\varphi \in I_{\mathfrak{g}}(A, A)$ this theorem is precisely Proposition 8.5 [5]. The proof given here is different from the proof in [5], is more conceptual and is based on properties of certain vector valued pairings. Before giving the proof, we establish a few properties of vector valued pairings.

For $\mathfrak{g}$-modules A, B and F, we call ψ an invariant pairing (resp. invariant Hermitian pairing) of A and B with values in F if ψ is a bilinear (resp. Hermitian) map $\psi: A \times B \to F$ and, for $x \in U(\mathfrak{g})$, $a \in A$, $b \in B$, $\psi(x \cdot a, b) - \psi(a, x^{\sigma_{\mathfrak{g}}} \cdot b) = x \cdot \psi(a, b)$ (resp. $\psi(x \cdot a, b) - \psi(a, x^{\bar{\sigma}_{\mathfrak{g}}} \cdot b) = x \cdot \psi(a, b)$). Denote the set of such pairings by $I_{\mathfrak{g}}(A, B, F)$ (resp.

$IH_{\mathfrak{g}}(A, B, F)$). If F is the trivial module then $I_{\mathfrak{g}}(A, B, F) = I_{\mathfrak{g}}(A, B)$ and $IH_{\mathfrak{g}}(A, B, F) = IH_{\mathfrak{g}}(A, B)$.

Next we shall associate to any pairing ψ two scalar valued pairings. For any vector space V, let V^* denote the algebraic dual of V. Let $\bar{V}$ be the set V^* with multiplication by elements of $\boldsymbol{C}$ denoted by $*$ and given by $\alpha * \lambda = \bar{\alpha}\lambda$, $\alpha \in \boldsymbol{C}$, $\lambda \in V^*$. $\bar{V}$ is a vector space over $\boldsymbol{C}$ called the conjugate dual of V. If V is a $\mathfrak{g}$-module, then $\bar{V}$ becomes a $\mathfrak{g}$-module by composing the $\boldsymbol{R}$-linear automorphism $-\bar{\sigma}$ and the contragradient representation of $\mathfrak{g}$ on V^*. We shall call this the *conjugate dual module* to V and denote it by $\bar{V}$. For $\psi \in I_{\mathfrak{g}}(A, B, F)$ define $\psi^R \in I_{\mathfrak{g}}(A, F^* \otimes B)$ and ${}^L\psi \in I_{\mathfrak{g}}(F^* \otimes A, B)$ by the formulae: for $\lambda \in F^*$, $a \in A$, $b \in B$,

$$\psi^R(a, \lambda \otimes b) = \lambda(\psi(a, b)) \, , \tag{3.2}$$

$${}^L\psi(\lambda \otimes a, b) = \lambda(\psi(a, b)) \, . \tag{3.3}$$

If $\psi \in IH_{\mathfrak{g}}(A, B, F)$ then define $\psi^R \in IH_{\mathfrak{g}}(A, \bar{F} \otimes B)$ and ${}^L\psi \in IH_{\mathfrak{g}}(F^* \otimes A, B)$ by the formulae (3.2) and (3.3) respectively. We note that if F is finite dimensional then ψ is determined by ψ^R and by ${}^L\psi$.

Assume F is finite dimensional and $A, B \in \mathscr{I}(\mathfrak{a})$. Using ψ^R and ${}^L\psi$ we define two transfers of ψ as follows. For $\psi \in I_{\mathfrak{a}}(A, B, F)$ (resp. $IH_{\mathfrak{a}}(A, B, F)$), define $RC(\psi)$ and $LC(\psi)$ to be the unique elements of $I_{\mathfrak{a}}(C(A), C(B), F)$ (resp. $IH_{\mathfrak{a}}(C(A), C(B), F)$) such that $(RC(\psi))^R = C(\psi^R)$ and ${}^L(LC(\psi)) = C({}^L\psi)$.

PROPOSITION 3.5. *For $\psi \in I(A, B, F)$ or $IH(A, B, F)$, $RC(\psi) = LC(\psi)$.*

Proof. First assume $\psi \in I_{\mathfrak{g}}(A, B, F)$. Let φ be the canonical pairing of F and F^*. Tensoring we obtain a pairing

$$\varphi \otimes {}^L\psi \colon (F \otimes F^* \otimes A) \times (F^* \otimes B) \longrightarrow \boldsymbol{C} \, . \tag{3.4}$$

Since $F \otimes F^*$ is canonically isomorphic to Hom (F^*, F^*), the identity element of Hom (F^*, F^*) induces a canonical inclusion:

$$i \colon A \longrightarrow F \otimes F^* \otimes A \, . \tag{3.5}$$

Via this inclusion, we restrict $\varphi \otimes {}^L\psi$ to obtain the pairing:

$$\operatorname{Res}(\varphi \otimes {}^L\psi) \colon A \times (F^* \otimes B) \longrightarrow \boldsymbol{C} \, . \tag{3.6}$$

With an easy calculation, one verifies that ${}^L\psi$ and hence ψ is determined by Res $(\varphi \otimes {}^L\psi)$; and moreover, if $\psi_i \in I_{\mathfrak{g}}(A, B, F)$, $(i = 1, 2)$, then

$$\psi_1 = \psi_2 \Longleftrightarrow \operatorname{Res}(\varphi \otimes {}^L\psi_1) = \psi_2^R \, . \tag{3.7}$$

Assertion (3.7) shows that to prove $LC(\psi) = RC(\psi)$ we need only prove $\operatorname{Res}(\varphi \otimes {}^{L}(LC(\psi))) = (RC(\psi))^{R}$. Thus we need only prove the identity:

$$\operatorname{Res}(\varphi \otimes C({}^{L}\psi)) = C(\psi^{R}) . \tag{3.8}$$

By Proposition 3.3, $\varphi \otimes C({}^{L}\psi) = C(\varphi \otimes {}^{L}\psi)$ and by functoriality of $C(\cdot)$ applied to (3.5), $C(i)$ is the canonical inclusion $C(A) \to F \otimes F^{*} \otimes C(A)$. This implies: $\operatorname{Res}(\varphi \otimes C({}^{L}\psi)) = C(\operatorname{Res}(\varphi \otimes {}^{L}\psi))$. However, by (3.7), $\operatorname{Res}(\varphi \otimes {}^{L}\psi) = \psi^{R}$; and so, the identity (3.8) is established. This completes the proof for $\psi \in I_{\mathfrak{g}}(A, B, F)$. If $\psi \in IH_{\mathfrak{g}}(A, B, F)$ then since the map φ gives an invariant Hermitian pairing of F and $\bar{F}$ the same argument applies and the proof is complete.

Proof of Theorem 3.4. For any $\varphi \in IH_{\mathfrak{a}}(A, B)$ with $A, B \in \mathscr{I}_{\mathfrak{g}}(\mathfrak{a})$, define $\mathfrak{g}^{*}$ valued pairings ${}_{L}\varphi$ and φ_{R} by the formulae: for $X \in \mathfrak{g}$, $a \in A$, $b \in B$,

$$\begin{aligned} {}_{L}\varphi(a, b)(X) &= \varphi(X \cdot a, b) \\ \varphi_{R}(a, b)(X) &= \varphi(a, X^{\sigma_{\mathfrak{g}}} \cdot b) . \end{aligned} \tag{3.9}$$

One checks that ${}_{L}\varphi$ and φ_{R} are elements of $IH_{\mathfrak{a}}(A, B, \mathfrak{g}^{*})$. We now claim the following identities hold:

$${}^{L}({}_{L}C(\varphi)) = C({}^{L}({}_{L}\varphi)) \tag{3.10}$$

$$(C(\varphi)_{R})^{R} = C((\varphi_{R})^{R}) . \tag{3.11}$$

Let π be the $\mathfrak{a}$-module map: $\pi\colon \mathfrak{g} \otimes A \to A$, $X \otimes a \mapsto X \cdot a$, $(X \in \mathfrak{g},\ a \in A)$ and $\pi \times 1\colon (\mathfrak{g} \otimes A) \times B \to A \times B$. Then ${}^{L}({}_{L}\varphi) = \varphi \circ (\pi \times 1)$; and so, $C({}^{L}({}_{L}\varphi)) = C(\varphi) \circ (C(\pi) \times 1)$. By uniqueness of $C(\pi)$, $C(\pi)(X \otimes a) = X \cdot a$ $(X \in \mathfrak{g},\ a \in C(A))$; and thus, identity (3.10) is true. Identity (3.11) is proved in essentially the same way. Combining (3.3), (3.10) and (3.2), (3.11), we obtain:

$$LC({}_{L}\varphi) = {}_{L}C(\varphi) \tag{3.12}$$

$$RC(\varphi_{R}) = C(\varphi)_{R} . \tag{3.13}$$

Now assume $\varphi \in IH_{\mathfrak{g}}(A, B)$. Invariance is equivalent to saying that ${}_{L}\varphi$ equals φ_{R}. By Proposition 3.5 and identities (3.12) and (3.13) we have: ${}_{L}C(\varphi) = C(\varphi)_{R}$. This says that $C(\varphi)$ is invariant.

In the case where $\varphi \in I_{\mathfrak{g}}(A, B)$ the argument is essentially the same. We need only replace $\bar{\sigma}_{\mathfrak{g}}$ by $\sigma_{\mathfrak{g}}$ in definition (3.9). We omit the details.

4. Pairings and lattices. Let notation be as in § 1. In particular, $\mathfrak{m}$ is a reductive Lie algebra over C which is contained in a finite dimensional Lie algebra $\mathfrak{g}$. Let $t_0 \in \mathscr{W}$ be the unique element such that $t_0 Q = -Q$.

LEMMA 4.1. *Let $t_0 = s_{\alpha_1} \cdots s_{\alpha_d} = s_{\beta_1} \cdots s_{\beta_d}$ be two reduced expressions for t_0 (α_i, β_j simple). For any Q-dominant integral weight μ, set*

$$m_i = (s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\mu + \delta))(\bar{H}_{\alpha_i}) , \qquad n_i = (s_{\beta_{i-1}} \cdots s_{\beta_1}(\mu + \delta))(\bar{H}_{\beta_i}) \quad (1 \leqq i \leqq d) .$$

Then, in $U(\mathfrak{n}^-)$ we have the identity:

$$X_{-\alpha_d}^{m_d} \cdots X_{-\alpha_1}^{m_1} = X_{-\beta_d}^{n_d} \cdots X_{-\beta_1}^{n_1} . \tag{4.1}$$

Proof. Let x (resp. y) denote the left (resp. right) side of (4.1). $x \otimes 1$ and $y \otimes 1$ both span the space of $\mathfrak{n}$-invariants of weight $t_0(\mu + \delta) - \delta$ in $V_{\mathfrak{m},Q,\mu}$; and so, x is a nonzero multiple of y. However, since the simple roots are linearly independent and $\sum m_i \alpha_i = \mu - t_0(\mu + \delta) + \delta = \sum n_i \beta_i$, identity (4.1) is true in the symmetric algebra. This implies $x - y$ is an element of $U^{r-1}(\mathfrak{m})$, $r = \sum m_i = \sum n_i$. But x is a nonzero multiple of y and $x, y \notin U^{r-1}(\mathfrak{m})$ so $x = y$.

PROPOSITION 4.2. *Let $t_0 = s_{\alpha_1} \cdots s_{\alpha_d} = s_{\beta_1} \cdots s_{\beta_d}$ be two reduced expressions for t_0 (α_i, β_i simple). Let $A, B \in \mathscr{J}_{\mathfrak{g}}(\mathfrak{m})$ and $\varphi \in I_{\mathfrak{g}}(A, B)$ or $IH_{\mathfrak{g}}(A, B)$. Then the pairings $C_{\alpha_1} \circ \cdots \circ C_{\alpha_d}(\varphi)$ and $C_{\beta_1} \circ \cdots \circ C_{\beta_d}(\varphi)$ are equal and this pairings is zero on $(\sum_{s \neq 1} A_s \times B_1) \cup (A_1 \times \sum_{s \neq 1} B_s)$.*

Proof. Assume $\varphi \in I_{\mathfrak{g}}(A, B)$. Let $\varphi_{\gamma,i} = C_{\gamma_i} \circ \cdots \circ C_{\gamma_d}(\varphi)$, ($\gamma = \alpha$ or β). Clearly $\varphi_{\gamma,1} \in I_{\mathfrak{g}}(A_1, B_1)$ and, directly from the definition, $\varphi_{\alpha,1}$ is zero on $(A_{s_{\alpha_1}} \times B) \cup (A \times B_{s_{\alpha_1}})$. For any simple root ξ, we have (Theorem 1 [2]):

$$s_\xi t_0 = s_{\alpha_1} \cdots \hat{s}_{\alpha_j} \cdots s_{\alpha_d} \quad \text{for some } j , \quad 1 \leqq j \leqq d , \tag{4.2}$$

(^ denotes omission) .

Set $A_i = C_{\alpha_i} \circ \cdots \circ C_{\alpha_d}(A)$, $B_i = C_{\alpha_i} \circ \cdots \circ C_{\alpha_d}(B)$, $(1 \leqq i \leqq d)$. By (4.2), the restriction of $\varphi_{\alpha,1}$ to $(A_{s_\xi} \times B_1) \cup (A_1 \times B_{s_\xi})$ equals $C_{\alpha_1} \circ \cdots \circ C_{\alpha_{j-1}}$ of $\phi_{\alpha,j}$ restricted to $(A_{j+1} \times B_j) \cup (A_j \times B_{j+1})$. But $\varphi_{\alpha,j}$ is zero on $(A_{j+1} \times B_j) \cup (A_j \times B_{j+1})$; and so, $\varphi_{\alpha,1}$ is zero on $(\sum_{s \neq 1} A_s \times B_1) \cup (A_1 \times \sum_{s \neq 1} B_s)$. The same argument applies to $\varphi_{\beta,1}$. Therefore both $\varphi_{\alpha,1}$ and $\varphi_{\beta,1}$ induce invariant pairings on $\tau(A) \times \tau(B)$. These modules are $U(\mathfrak{m})$-finite; and so, we need only check that for Q-dominant integral μ and $\mathfrak{n}$-invariant vectors $z \in A_1$, $w \in B_1$ of weight μ

$$\varphi_{\alpha,1}(z, w) = \varphi_{\beta,1}(z, w) . \tag{4.3}$$

Using the definitions directly we obtain:

$$(4.4)\qquad \begin{aligned}\varphi_{\alpha,1}(z, w) &= \prod_i \frac{1}{m_i!\,(m_i+1)!}\varphi(X^{m_d}_{-\alpha_d}\cdots X^{m_1}_{-\alpha_1}z,\ X^{m_d}_{-\alpha_d}\cdots X^{m_1}_{-\alpha_1}w)\\ \varphi_{\beta,1}(z, w) &= \prod_i \frac{1}{n_i!\,(n_i+1)!}\varphi(X^{n_d}_{-\beta_d}\cdots X^{n_1}_{-\beta_1}z,\ X^{n_d}_{-\beta_d}\cdots X^{n_1}_{-\beta_1}w)\ .\end{aligned}$$

Now, since $\{m_i: 1 \leqq i \leqq d\} = \{n_i: 1 \leqq i \leqq d\} = \{\mu + \delta(\bar{H}_\gamma): \gamma \in Q\}$, identities (4.1) and (4.4) imply (4.3) and the proof is complete for $\varphi \in I_{\mathfrak{g}}(A, B)$. The case $\varphi \in IH_{\mathfrak{g}}(A, B)$ is handled by the same argument.

DEFINITION 4.3. With notation as in Proposition 4.2, the invariant pairing $C_{\alpha_1} \circ \cdots \circ C_{\alpha_d}(\varphi)$ on $A_1 \times B_1$ is independent of the reduced expression and induces a pairing on $\tau(A) \times \tau(B)$, which we denote by $\tau(\varphi)$. We have:

$$(4.5)\qquad \begin{aligned}\tau &: I_{\mathfrak{g}}(A, B) \longrightarrow I_{\mathfrak{g}}(\tau(A), \tau(B))\ ,\\ \tau &: IH_{\mathfrak{g}}(A, B) \longrightarrow IH_{\mathfrak{g}}(\tau(A), \tau(B))\ .\end{aligned}$$

Next we consider an important example where the maps τ in (4.5) are surjective. Let $d_{t_0,1}(\mu)$ denote the element of $U(\mathfrak{n}^-)$ given in the identity (4.1).

LEMMA 4.4. *Let E and F be finite dimensional $\mathfrak{m}$-modules and let ν and ξ be $-Q$-dominant integral elements of $\mathfrak{h}^*$. Let $A = E \otimes V_{\mathfrak{m},Q,\nu-\delta}$ and $B = F \otimes V_{\mathfrak{m},Q,\xi-\delta}$. Then the maps τ in* (4.5) *are surjective.*

Proof. Let $A_s(s \in \mathscr{W})$ and $B_s(s \in \mathscr{W})$ be lattices above A and B respectively. Replacing A and B by summands we assume that for some $\chi, \chi' \in Z(\mathfrak{m})^\wedge$, $A = A_\chi$ and $B = B_{\chi'}$. Since generalized $Z(\mathfrak{m})$ eigenspaces for distinct characters are orthogonal it is sufficient to prove the lemma in the setting where $\chi = \chi'$. Choose $\mu \in \mathfrak{h}^*$ such that $\mu + \delta$ is Q-dominant integral and χ is parameterized by the orbit $\mathscr{W} \cdot (\mu + \delta)$.

By Lemma 7 [4], the maps $z \to \bar{z}$ of $A_1^{\mathfrak{n}}$ to $\tau(A)^{\mathfrak{n}}$, $B_1^{\mathfrak{n}}$ to $\tau(B)^{\mathfrak{n}}$ are surjective. Choose subspaces $M_1 \subseteqq A^{\mathfrak{n}}[\mu]$, $N_1 \subseteqq B_1^{\mathfrak{n}}[\mu]$ such that the induced maps give bijections $M_1 \simeq \tau(A)^{\mathfrak{n}}$, $N_1 \simeq \tau(B)^{\mathfrak{n}}$. Define linear subspaces $M = d_{t_0,1}(\mu) \cdot M_1$, $N = d_{t_0,1}(\mu) \cdot N_1$. Then M and M_1 (resp. N and N_1) are linearly isomorphic and $M \subseteqq A^{\mathfrak{n}}[t_0'\mu]$, $N \subseteqq B^{\mathfrak{n}}[t_0'\mu]$. Let $\bar{\varphi} \in I_{\mathfrak{g}}(\tau(A), \tau(B))$. Clearly, $\tau(A)$ and $\tau(B)$ being semisimple, $\bar{\varphi}$ is determined by its restriction to $\tau(A)^{\mathfrak{n}} \times \tau(B)^{\mathfrak{n}}$. Let ψ_1 denote the pull back of this restriction to $M_1 \times N_1$ and define ψ on $M \times N$ by the formula: $\psi(d_{t_0,1}(\mu)a,\ d_{t_0,1}(\mu)b) = \psi_1(a, b)$, $a \in M_1$, $b \in N_1$. We claim that there exist submodules $A' \subseteqq A$, $B' \subseteqq B$ such that:

$$(4.6)\qquad A = A' \oplus U(\mathfrak{m})M\ , \qquad B = B' \oplus U(\mathfrak{m})N\ .$$

Assume, for the moment, A' and B' exist and (4.6) holds. $U(\mathfrak{m})M$ and $U(\mathfrak{m})N$ are the direct sums of irreducible Verma modules all of highest weight $t'_0 \cdot \mu$. Now Proposition 6.12 [5] implies that there exists an invariant pairing φ of $U(\mathfrak{m})M$ and $U(\mathfrak{m})N$ which restricts to ψ on $M \times N$. We extend φ to $A \times B$ by setting it equal to zero on $(A' \times B) \cup (A \times B')$. But then, for some nonzero constant Γ, $\bar{\varphi} = \tau(\Gamma \cdot \varphi)$. Thus to complete the proof of surjectivity of $I_{\mathfrak{g}}(A, B)$ onto $I_{\mathfrak{g}}(\tau(A), \tau(B))$ we need only prove (4.6) holds for some submodules A' and B'.

Let φ_E be a nondegenerate invariant form on E and let φ_ν be a nondegenerate invariant form on $V_{\mathfrak{m},Q,\nu-\delta}$ (cf. Proposition 6.8 [5]). $\tau(\varphi_\nu)$ is a nonzero invariant form on $\tau(V_{\mathfrak{m},Q,\nu-\delta})$. But this module is the irreducible $\mathfrak{m}$-module with highest weight $t_0(\nu) - \delta$; and so, $\tau(\varphi_\nu)$ is nondegenerate. Then $\tau(\varphi_E \otimes \varphi_\nu) = \varphi_E \otimes \tau(\varphi_\nu)$ is nondegenerate. Now $\tau(\varphi_E \otimes \varphi_\nu)$ restricted to $\tau(A)^n \times \tau(A)^n$ is nondegenerate; and therefore, $\varphi_E \otimes \varphi_\nu$ restricted to $M \times M$ is nondegenerate. So since $U(\mathfrak{m})M$ is the direct sum of irreducible Verma modules $\varphi_E \otimes \varphi_\nu$ restricted to $U(\mathfrak{m})M \times U(\mathfrak{m})M$ is nondegenerate. We put A' equal to the orthogonal complement to $U(\mathfrak{m})M$ in A with respect to $\varphi_E \otimes \varphi_\nu$. The argument for B is identical; and so, the proof of (4.6) is complete.

For the case of invariant Hermitian pairings we note that if $\lambda \in \mathfrak{h}^*$ is real valued on $\bar{H}_\alpha (\alpha \in Q)$, and $V_{\mathfrak{m},Q,\lambda}$ is irreducible, then $V_{\mathfrak{m},Q,\lambda}$ and its conjugate dual module (w.r.t. $\bar{\sigma}$) are isomorphic. Here we are using the fact that $\bar{\sigma}$ was determined by a compact real form of $\mathfrak{m}$. With this fact in mind, essentially the same argument as above applies to show $\tau: IH_{\mathfrak{g}}(A, B) \to IH_{\mathfrak{g}}(\tau(A), \tau(B))$ is surjective.

For any $\mathfrak{b}$-module M, we denote the induced module from $\mathfrak{b}$ to $\mathfrak{m}$ by $U(M)$; i.e., $U(M) = U(\mathfrak{m}) \otimes_{U(\mathfrak{b})} M$.

PROPOSITION 4.5. *Let M and N be locally finite $\mathfrak{b}$-modules which are semisimple as $\mathfrak{h}$-modules and have finite dimensional weight spaces. Assume that $U(M)$ and $U(N)$ admit nondegenerate invariant forms. Then the following maps are surjections:*

$$\tau: I_{\mathfrak{m}}(U(M), U(N)) \longrightarrow I_{\mathfrak{m}}(\tau(U(M)), \tau(U(N)))$$
$$\tau: IH_{\mathfrak{m}}(U(M), U(N)) \longrightarrow IH_{\mathfrak{m}}(\tau(U(M), \tau(U(N)))\ .$$

Moreover, both maps carry nondegenerate pairings to nondegenerate pairings.

Proof. Since M and N are locally finite, $U(M)$ and $U(N)$ are each the direct sums of their generalized $Z(\mathfrak{m})$ eigenspaces $U(M)_\chi$ and $U(N)_\chi$, $\chi \in Z(\mathfrak{m})^\wedge$. By assumption the weight spaces of M and N are finite dimensional; and thus, for each $\chi \in Z(\mathfrak{m})^\wedge$ there exist finite dimensional sub $\mathfrak{b}$-modules $M' \subseteq M$, $N' \subseteq N$ such that:

$$(4.7) \qquad U(M)_\chi \subseteq U(M') , \qquad U(N)_\chi \subseteq U(N') .$$

For any pairing $\varphi \in I_{\mathfrak{m}}(U(M), U(N))$ or $IH_{\mathfrak{m}}(U(M), U(N))$ and $\chi, \chi' \in Z(\mathfrak{m})^\wedge$, $\chi \neq \chi'$, we have:

$$(4.8) \qquad U(M)_\chi \subseteq (U(N)_{\chi'})^\perp , \qquad U(N)_{\chi'} \subseteq (U(M)_\chi)^\perp .$$

The inclusions (4.8) imply that we need only prove the proposition for $U(M)$ replaced by $U(M)_\chi$ and $U(N)$ replaced by $U(N)_\chi$. For convenience we set $A = U(M)_\chi$ and $B = U(N)_\chi$.

Using Lemma 4.7 [5], choose a finite dimensional $\mathfrak{m}$-module F and integral weights $\mu, \nu \in \mathfrak{h}^*$ with $\mu(\bar{H}_\alpha) \ll 0$, $\nu(\bar{H}_\alpha) \ll 0$ $(\alpha \in Q)$, such that we have embeddings:

$$(4.9) \qquad M' \hookrightarrow F \otimes C_\mu , \qquad N' \hookrightarrow F \otimes C_\nu .$$

Extending scalars to $U(\mathfrak{m})$ and setting $V_\mu = V_{\mathfrak{m},Q,\mu}$, $V_\nu = V_{\mathfrak{m},Q,\nu}$, we obtain embeddings:

$$(4.10) \qquad U(M') \hookrightarrow F \otimes V_\mu , \qquad U(N') \hookrightarrow F \otimes V_\nu .$$

By assumption $U(M)$ and hence A admits a nondegenerate invariant form, say ζ. Then using Propositions 6.13 and 6.7 [5] there exists an invariant form $\bar{\zeta}$ on $F \otimes V_\mu$ which is nondegenerate and restricts to ζ on A. This implies that A is a direct summand of $F \otimes V_\mu$; and so, $\tau(A)$ is a direct summand of $\tau(F \otimes V_\mu)$. The same argument implies that $\tau(B)$ is a summand of $\tau(F \otimes V_\nu)$. But then if Res denotes the restriction map for pairings, the map

$$(4.11) \qquad \text{Res}\colon I_{\mathfrak{m}}(\tau(F \otimes V_\mu), \tau(F \otimes V_\nu)) \longrightarrow I_{\mathfrak{m}}(\tau(A), \tau(B))$$

is a surjection. Let $\bar{\varphi} \in I_{\mathfrak{m}}(\tau(A), \tau(B))$. Then by (4.11), choose an invariant pairing $\bar{\psi}$ such that $\text{Res}(\bar{\psi}) = \bar{\varphi}$. By Lemma 4.4, choose $\psi \in I(F \otimes V_\mu, F \otimes V_\nu)$ such that $\tau(\psi) = \bar{\psi}$. If φ denotes the restriction of ψ to $A \times B$, then $\tau(\varphi) = \bar{\varphi}$. This proves surjectivity.

Let $\varphi \in I_{\mathfrak{m}}(A, B)$ and assume φ is nondegenerate. Using Propositions 6.13 and 6.7 [5], there exists a nondegenerate invariant pairing ψ of $(F \otimes V_\mu) \times (F \otimes V_\nu)$ which restricts to φ. We use ψ to obtain the orthogonal decomposition:

$$(4.12) \qquad F \otimes V_\mu = A \oplus B^\perp , \qquad F \otimes V_\nu = B \oplus A^\perp .$$

Let $\xi \in \mathfrak{h}^*$ be Q-dominant integral and let z_1 be an $\mathfrak{n}$-invariant of weight ξ in A_1. Assume $z_1 \notin \sum_{s \neq 1} A_s$. From (4.12) we consider $(F \otimes V_\mu)_s = A_s \oplus B^\perp{}_s$ $(s \in \mathscr{W})$; and so, $z_1 \notin \sum_{s \neq 1} (F \otimes V_\mu)_s$. Then, by Proposition 9.8 [5], $z = d_{t_{0,1}}(\xi) \cdot z_1$ is a split invariant of $F \otimes V_\mu$. By Lemma 9.3 [5] and (4.12), there exists an $\mathfrak{n}$-invariant w of weight

$t'_0 \cdot \xi$ in B such that $\varphi(z, w) = \psi(z, w) \neq 0$. Let w_1 be the unique $\mathfrak{n}$-invariant in B_1 such that $w = d_{t_{0,1}}(\xi) \cdot w_1$ and let $\bar{z}$ and $\bar{w}$ denote the images of z_1 and w_1 in $\tau(A)$ and $\tau(B)$. Then $\tau(\varphi)(\bar{z}, \bar{w}) \neq 0$. This implies that $\bar{z}$ is not contained in $\tau(B)^\perp$. But $\bar{z}$ denotes any $\mathfrak{n}$-invariant of $\tau(A)$; and so, $\tau(A) \cap \tau(B)^\perp = 0$. Likewise $\tau(B) \cap \tau(A)^\perp = 0$. So $\tau(\varphi)$ is nondegenerate on $A \times B$ and hence on $U(M) \times U(N)$. The argument for $\varphi \in IH_{\mathfrak{m}}(U(M), U(N))$ is identical and we omit it.

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA 92093
AND
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
BOMBAY 400 005, INDIA