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CHOOSING ℓ-ELEMENT SUBSETS OF n-ELEMENT SETS

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We consider axioms that assert the possibility of choosing a subset of an n-element set. We study the interdependence of these axioms and of the more usual axioms of choice for n-element sets.

The discussion takes place within any of the usual systems of set theory without the axiom of choice. Our logical framework is the first-order predicate calculus with identity. Lower case letters stand for natural numbers. Throughout this paper, we let $n \geqq 2$ and $\ell \geqq 1$. At first, we assume $n > \ell$.

Let $[n]$ be the statement: "For every nonempty set X of n-element sets there is a function f with domain X such that for each A in X, $f(A) \in A$." Here, $[n]$ is called the axiom of choice for n-element sets. (See [1].)

Let $S(n, \ell)$ be the statement: "For every nonempty set X of n-element sets there is a function f with domain X such that for each A in X, $f(A)$ is an ℓ-element subset of A."

Let $T(n, \ell)$ be the statement: "For every nonempty set X of n-element sets there is a function f with domain X such that for each A in X, $f(A)$ is a nonempty subset of A with at most ℓ elements."

Finally, let $T^*(n, n-1)$ be the statement: "For every nonempty set X of n-element sets there is a function f with domain X such that for each A in X, $f(A) = \langle A_1, A_2 \rangle$, where A_1 and A_2 are pairwise disjoint nonempty subsets of A whose union is A.

Observe that $T(n, n-1)$ asserts the possibility of choosing a nonempty proper subset of each n-element set, whereas $T^*(n, n-1)$ asserts the possibility of ordering the partition thereby obtained. Clearly, $T(n, n-1) \leftrightarrow T^*(n, n-1)$.

The following relationships are also immediate.

$$S(n, \ell) \longleftrightarrow S(n, n-\ell)$$
$$S(n, 1) \longleftrightarrow S(n, n-1) \longleftrightarrow T(n, 1) \longleftrightarrow [n]$$
$$\left[\binom{n}{\ell}\right] \longrightarrow S(n, \ell)$$
$$[2^n - 2] \longrightarrow T(n, n-1)$$

For convenience, for $n \leqq \ell$, let $S(n, \ell) = S(n, n-1)$ and $T(n, \ell) = T(n, n-1)$.

Now let k, ℓ, m, n be natural numbers such that $k \geqq 0$, $\ell \geqq 1$,

$m \geqq 2$, and $n \geqq 2$.

If $\ell < k$, then clearly,

$$S(n, \ell) \longrightarrow T(n, \ell) \longrightarrow T(n, k) .$$

Theorem 1 generalizes the first of these relationships.

THEOREM 1. *For* $\ell < n$ *and* $n - \ell < m \leqq n$,

$$S(n, \ell) \longrightarrow T(m, \ell) .$$

Proof. Let X be a nonempty set of m-element sets. For each A in X, let A' consist of the first $n - m$ natural numbers that are not in A and let $A'' = A \cup A'$. We use $S(n, \ell)$ to obtain an ℓ-element subset of A''. At least one element of this subset belongs to A.

The next two theorems generalize Tarski's result:

$$[kn] \longrightarrow [n] .$$

(See [2], p. 99.)

THEOREM 2. *For* $\ell < n$ *and for* $k \geqq 0$,

$$(S(n, \ell) \wedge [kn + \ell]) \longrightarrow [n] .$$

Proof. Let X be a nonempty set of n-element sets and let $A \in X$. We use $S(n, \ell)$ to choose an ℓ-element subset B of A.

If $k = 0$, we use $[\ell]$ to pick an element of B.

If $k > 0$, we use $[kn + \ell]$ to pick an element of $(B \times \{0\}) \cup (A \times \{1, 2, \cdots, k\})$. Let $f(A)$ be the first coordinate of this chosen element.

THEOREM 3. *Let* $k \geqq 1$.

(a) *For* $\ell < n$, $T(kn, \ell) \rightarrow T(n, \ell)$.

(b) *For* ℓ *not of the form* jn *for any* $j \leqq k$, $S(kn, \ell) \rightarrow T(n, \ell)$.

Proof. Let X be a nonempty set of n-element sets and let $X' = \{A \times k\colon A \in X\}$.

(a) We choose a subset of at most ℓ elements of each $A \times k$ in X'. For each such chosen subset, let B be the set of first coordinates. Then B is a nonempty subset of A and has at most ℓ elements.

(b) If $\ell \geqq kn$, then

$$S(kn, \ell) \longleftrightarrow S(kn, kn - 1) \longleftrightarrow [kn]$$

and

$$[kn] \longrightarrow [n] \longrightarrow T(n, \ell) .$$

If $\ell < kn$, we choose an ℓ-element subset C of each $A \times k$ in X'. Not every a in A appears the same number of times as the first coordinate of a member of C. Let B be the set of those a that appear the maximal number of times in this role. If $\ell < n$, B is nonempty and has at most ℓ elements. If $\ell > n$, B is a nonempty proper subset of A. In both cases, the axiom $T(n, \ell)$ is realized.

Henceforth, let A be a nonempty finite set of natural numbers greater than 1. If $A = \{a_1, a_2, \cdots, a_m\}$, let $S(A, \ell)$ denote

$$S(a_1, \ell) \wedge S(a_2, \ell) \wedge \cdots \wedge S(a_m, \ell) .$$

For $n > \ell$, we say that n *is an* A_ℓ*-number* if for some $j \geqq 1$ and some k satisfying $0 \leqq k < \ell$, $jn + k \in A$. Furthermore, for all $n \geqq 2$ and $\ell \geqq 1$, we say that A *and* n *satisfy condition* $\mathscr{A}_\ell$ if either

(i) n is an A_ℓ-number, or both

(ii)$_a$ $n = rp$ for some prime p in A, and

(ii)$_b$ whenever $n = n_1 + n_2$ for $n_1 > \ell$ and $n_2 > \ell$, then either A and n_1 or else A and n_2 satisfy condition $\mathscr{A}_\ell$.

LEMMA. *Let p be a prime and let $\ell \geqq 1$ and $r \geqq 1$. Then*

$$S(p, \ell) \longrightarrow T(rp, rp - 1) .$$

(*The lemma is Theorem* 2(*g*) *of* [3]. *See also* [4].)

THEOREM 4. *Let A be as above, let $n \geqq 2$ and $\ell \geqq 1$, and suppose A and n satisfy condition $\mathscr{A}_\ell$. Then $S(A, \ell) \to T(n, \ell)$.*

Proof. Assume A and n satisfy condition $\mathscr{A}_\ell$.

If n is an A_ℓ-number, then for some $j \geqq 1$ and for some k satisfying $0 \leqq k < \ell$, $S(jn + k, \ell)$ is true. By Theorem 1, $T(jn, \ell)$ must be true, and by Theorem 3, $T(n, \ell)$ is true.

If n is not an A_ℓ-number, then $n = rp$ for some prime p in A. By our hypothesis, $S(p, \ell)$ is true. By the lemma, $T(n, n - 1)$ is true.

If $2 \leqq n \leqq \ell$, then $T(n, n - 1) = T(n, \ell)$.

If $\ell < n < 2\ell$, we use $T^*(n, n - 1)$ to obtain $\langle B_1, B_2 \rangle$, where $\{B_1, B_2\}$ forms a partition of an element B of a nonempty set of n-element sets. At least one of these subsets, B_1 or B_2, has at most ℓ elements. We choose the first of these with this property. Thus, $T(n, \ell)$ is true.

Now assume that for $n' < n$, whenever A and n' satisfy condition $\mathscr{A}_\ell$, then $S(A, \ell) \to T(n', \ell)$.

Let $n \geqq 2\ell$ and suppose $S(A, \ell)$ is true. We use $T^*(n, n-1)$ to obtain $\langle B_1, B_2 \rangle$, as in the preceding paragraph. If one of the subsets B_1 or B_2 of B has at most ℓ elements, then $T(n, \ell)$ is realized. Otherwise, one of these subsets has n_1 elements, the other has $n - n_1$ elements, and both $n > \ell$ and $n - n_1 > \ell$. By $(\text{ii})_b$, either A and n_1 satisfy condition $\mathscr{A}_\ell$ or else A and $n - n_1$ satisfy condition $\mathscr{A}_\ell$. By the inductive hypothesis, either $T(n_1, \ell)$ or $T(n - n_1, \ell)$ is true. We can therefore choose a nonempty subset of at most ℓ elements of one of the subsets B_1 or B_2 of B. Thus, $T(n, \ell)$ is true.

Let A be as above and let $n \geqq 2$. Let $P(A, n)$ be the statement: "For every prime partition of n, that is, whenever $n = p_1 + p_2 + \cdots + p_k$, one of these primes is in A."

THEOREM 5. *Assume $P(A, n)$. Then for all ℓ, $S(A, \ell) \to T(n, \ell)$.*

Proof. It suffices to show that if $P(A, n)$, then for all $\ell \geqq 1$, A and n satisfy condition $\mathscr{A}_\ell$.

Assume $P(A, n)$ and let $\ell \geqq 1$.

If n is prime and $n > \ell$, then n is an A_ℓ-number. If n is prime and $n \leqq \ell$, then $(\text{ii})_a$ and $(\text{ii})_b$ of condition $\mathscr{A}_\ell$ are satisfied.

Suppose that n is composite and that for all k, $2 \leqq k < n$, whenever $P(A, k)$, then A and k satisfy condition A_ℓ. By $P(A, n)$, each prime factor of n is in A. Let $n = n_1 + n_2$, where $n_1 > \ell$ and $n_2 > \ell$. Suppose there is a prime partition of n_1 with no summand in A. Then by $P(A, n)$, every prime partition of n_2 has a summand in A. Thus, $P(A, n_1)$ or $P(A, n_2)$. By the inductive hypothesis, either A and n_1 or else A and n_2 satisfy condition $\mathscr{A}_\ell$. Therefore, A and n satisfy condition $\mathscr{A}_\ell$.

Independence results concerning these axioms can be found in [3].

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