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COVERINGS OF A PROJECTIVE ALGEBRAIC MANIFOLD

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Let M be a projective algebraic manifold. Suppose $\pi: D \to M$ is a covering of M. If D satisfies $H^1(D, O^*)=0$, then D is a Stein manifold with $H^2(D, Z)=0$, where O^* is the sheaf of germs of nowhere-vanishing holomorphic functions and Z is the additive group of integers.

Let D be a domain in C^n and Γ be a discrete subgroup of $\mathrm{Aut}(D)$. It is well-known that if the quotient manifold D/Γ is compact, then D is a domain of holomorphy. Recently, Carlson-Harvey [1] showed that if D is a domain in a Stein manifold and $D \to M$ is a covering of a compact Moisheson manifold M, then D is a Stein manifold. On the other hand, we showed in [4] that if a pseudoconvex domain D in a projective algebraic manifold satisfies $H^1(D, O^*) = 0$, then D is a Stein manifold with $H^2(D, Z) = 0$.

In this paper, we study the case where a covering of a manifold is not contained in a larger manifold. We shall prove the following:

THEOREM. *Let M be a projective algebraic manifold. Suppose $\pi: D \to M$ is a covering of M. If D satisfies $H^1(D, O^*) = 0$, then D is a Stein manifold with $H^2(D, Z) = 0$.*

We remark that the condition $H^1(D, O^*) = 0$ cannot be replaced by $H^1(D, O) = 0$, where O is the sheaf of germs of holomorphic functions. To see this it is enough to consider the case $D = M = P_2(C)$ and π is the identity mapping.

Proof of theorem. Let $\{V_i\}$ be an open covering of M such that each V_i is a local coordinate neighborhood and is biholomorphic to a connected component $\pi^{-1}(V_i)$. Since M is a projective algebraic manifold, there is a positive line bundle F over M. Choosing a suitable refinement $\{U_j\}$ of $\{V_i\}$, we can represent F by a system of transition functions $\{f_{jk}\}$ and find a Harmitian metric $\{a_j\}$ along the fibers of F which satisfies the following conditions:

(i) Each a_j is a C^∞, real-valued and positive function on U_j,

(ii) If $U_j \cap U_k \neq \phi$, then we have $a_k = |f_{jk}|^2 a_j$,

(iii) For every point P in M, the Hessian of $-\log a_j$ relative to a local coordinate system $(z_1, \cdots, z_n)$ at P

$$L(-\log a_j; P) = \left(-\frac{\partial^2 \log a_j}{\partial z_\alpha \partial \bar{z}_\beta}(P)\right)$$
$$(\alpha, \beta = 1, \cdots, n)$$

is positive definite. By the compactness of M, M has a finite open covering $\{U_j: j = 1, \cdots, m\}$.

Since U_j is biholomorphic to each of the connected components of $\pi^{-1}(U_j)$, we have the functions $\{a_j \circ \pi\}$ which satisfies the following conditions:

(i) Each $a_j \circ \pi$ is a C^∞, real-valued and positive function on $\pi^{-1}(U_j)$,

(ii) If $\pi^{-1}(U_j) \cap \pi^{-1}(U_k) \neq \phi$, then we have $a_j \circ \pi = |f_{jk} \circ \pi|^2 a_k \circ \pi$,

(iii) $W(-\log a_j \circ \pi; P)$ is positive at every point P in D, where

$$W(\phi; P) := \min\left\{\sum_{\alpha,\beta} \frac{\partial^2 \phi}{\partial w_\alpha \partial \bar{w}_\beta}(P)\lambda_\alpha \bar{\lambda}_\beta : \sum_\alpha |\lambda_\alpha|^2 = 1\ , \quad \alpha, \beta = 1, \cdots, n\right\}$$

and $(w_1, \cdots, w_n)$ is a local coordinate at P.

Since $U = \{\pi^{-1}(U_j)\}$ is an open covering of D, $\{f_{jk} \circ \pi\}$ defines an element of $H^1(U, O^*)$. By the assumption of $H^1(D, O^*) = 0$, there is a cochain $\{f_j\}$ of $C^0(U, O^*)$ such that $f_{jk} \circ \pi = f_k/f_j$. We can define a C^∞ function ϕ on D in the following way:

$$\phi(P) := -\log(a_j \circ \pi(P)|f_j(P)|^2)$$

for P in $\pi^{-1}(U_j)$. Since M is paracompact, M has a finite open covering $\{W_j: j = 1, \cdots, m\}$ with $\bar{W}_j \subset U_j$. By the property (iii) there is a positive constant C_j such that $W(\phi; P) > C_j$ for P in $\pi^{-1}(W_j)(j = 1, \cdots, m)$. Hence we have

$$(1) \qquad W(\phi; P) > C: = \min\{C_j: j = 1, \cdots, m\}$$

for P in D. We remark that D is not finitely sheeted, because D has the strongly plurisubharmonic function ϕ.

On the other hand, M is a projective algebraic manifold, so D has a real-analytic Kähler metric. Let $d(P, Q)$ be the distance between P and Q measured by the Kähler metric. Let us fix a point P_0 in D and define a continuous function ψ on D in the following way:

$$\psi(P) := d(P_0, P)$$

for P in D. We see that for every $c > 0$, the set $\{P \in D: \psi(P) < c\}$ is relatively compact in D. Denotes by $\Gamma(P, \varepsilon)$ the set $\{Q \in D: d(P, Q) < \varepsilon\}$, where a positive constant ε is chosen so that $\pi(\Gamma(P, \varepsilon))$

is contained in some U_j and $\Gamma(P, \varepsilon)$ is homeomorphic to a hypersphere. We define the following operator A_ε mapping continuous function f on D into C^1 function on D:

$$A_\varepsilon f(P) := \frac{1}{V}\int_{\Gamma(P,\varepsilon)} f(Q)dv ,$$

where dv is the volume element determined by the Kähler metric and V is the volume of $\Gamma(P, \varepsilon)$. We see that the set $\{P \in D: A_\varepsilon\psi(P) < c\}$ is relatively compact in D. Let define

$$\psi_1 = A_\varepsilon\psi \text{ and } \psi_2 = A_\varepsilon\psi_1$$

on D, then ψ_2 is C^2 and the set $\{P \in D: \psi_2(P) < c\}$ is also relatively compact in D. Let compute the Hessian of ψ_2. Since D has a real-analytic Kähler metric, there are a local coordinate $(w_1, \cdots, w_n)$ of $\Gamma(P, \varepsilon)$ and a positive constant K_1 such that

$$|\psi(Q) - \psi(Q')|^2 \leqq K_1\{|w_1 - w_1'|^2 + \cdots + |w_n - w_n'|^2\}$$

for two points $Q = (w_1, \cdots, w_n)$ and $Q' = (w_1', \cdots, w_n')$ in $\Gamma(P, \varepsilon)$ (see [3] Lemma 1). By the compactness of M, K_1 can be chosen independent of P. Choosing K_1 large enough if necessary, we have

$$\left|\frac{\partial\psi_1}{\partial w_j}(P)\right| \leqq K_1 \quad (j = 1, \cdots, n)$$

and consequently

$$\left|\frac{\partial^2\psi_2}{\partial w_j\partial \bar{w}_k}(P)\right| \leqq K_1 \quad (j, k = 1, \cdots, n)$$

for P in D. Therefore a positive constant K can be chosen so that

$$(2) \qquad W(\psi_2; P) > -K$$

for P in D. Now we define a C^2 function Φ on D in the following way:

$$\Phi(P) := K\cdot\phi(P) + C\cdot\psi_2(P)$$

for P in D. Then (1) and (2) induce

$$W(\Phi; P) \geqq K\cdot W(\phi; P) + C\cdot W(\psi_2; P) > 0$$

for P in D. Hence Φ is a strongly plurisubharmonic function on D and the set $\{P \in D: \Phi(P) < c\}$ is relatively compact in D for every $c > 0$. Therefore D is a Stein manifold by Narasimhan [2]. Moreover from the exact sequence $0 \to Z \to O \to O^* \to 0$ we obtain the exact cohomology sequence

$$\cdots \longrightarrow H^1(D, O) \longrightarrow H^1(D, O^*) \longrightarrow H^2(D, Z) \longrightarrow H^2(D, O) \longrightarrow \cdots .$$

Since $H^2(D, O) = 0$ by the Cartan's Theorem B and $H^1(D, O^*) = 0$ by the assumption, we have $H^2(D, Z) = 0$. This completes the proof.

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