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IMAGINARY VALUES OF MEROMORPHIC FUNCTIONS IN THE DISK

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Let f be a meromorphic function in the unit disk, and let $\phi(r,f)$ be the number of solutions of the equation $\operatorname{Re} f(re^{i\theta}) = 0$ for $0 \leqq \theta \leqq 2\pi$. In this paper we bound $\phi(r,f)$ off an exceptional set of r values, and $\Phi(r,f) = \int_0^r \phi(t,f)(1-t)^{-1}dt$ for all r, in terms of the Nevanlinna characteristic function of f. We then give examples to show that the bounds obtained are the best possible.

The quantity $\phi(r, f)$ was studied for entire functions by A. Gelfond [3] and later by S. Hellerstein and J. Korevaar [5]. The quantities $\phi(r, f)$ and $\Phi(r, f)$ were studied for meromorphic functions in the plane by J. Miles and the author [10].

We will prove the following theorem analogous to Theorem 1 of Miles and Townsend.

THEOREM. *If $c_0(r) = (1-\alpha_0) + \alpha_0 r$ for $0 < \alpha_0 < 1$ and f is a meromorphic function in the unit disk then there is a constant $A = A(\alpha_0)$ and a set $\Delta \subset [0, 1)$ satisfying*

$$\int_\Delta \exp\{T(c_0(r), f) - \log(1-r)\}dr < \infty$$

so that for $r \notin \Delta$ and $r > R$

(i) $\phi(r, f) < A(1-r)^{-1}[T(c_0(r), f) - \log(1-r)]$.

If $\Phi(r, f) = \int_0^r \phi(t, f)(1-t)^{-1}dt$ then there is an α_1 so that $0 < \alpha_1 < 1$, and a constant A' so that for $r > R$ and for $c_1(r) = (1-\alpha_1) + \alpha_1 r$

(ii) $\Phi(r, f) < A'(1-r)^{-1}[T(c_1(r), f) + (1-r)^{-1}]$.

We will then give examples to show that no nontrivial lower bound for $\phi(r, f)$ can be given and that the factor $(1-r)^{-1}$ in (i) and (ii) can not be replaced by any function $b(r)$ satisfying $b(r) = o((1-r)^{-1})$ as $r \to 1$.

It is not known whether the exceptional set for (i) is nonempty, even if f is holomorphic in the unit disk.

We note that the second occurrence of $(1-r)^{-1}$ in (ii) may be replaced by $-\log(1-r)$, using a proof that is much longer and more intricate than the one given in this paper. This alternate proof is a combination of the essential ideas of the proof of Theorem 2 in [12], together with techniques used in this paper to bound $\phi(r, f)$ in terms of the characteristic function of f.

The technique used in [10] to obtain an upper bound for the number of solutions of $\operatorname{Re} g(z) = 0$ on $|z| = r$ for g meromorphic in the plane begins by considering $G_r(\theta) = \operatorname{Re} g(re^{i\theta})$ as a function of a complex variable θ. After showing that $G_r(\theta)$ is a meromorphic function in the θ-plane, Jensen's theorem can be used to bound the number of zeros of G_r in $|\theta| \leqq \pi$, and hence to bound the number of zeros of $\operatorname{Re} g(re^{i\theta})$ for $-\pi \leqq \theta \leqq \pi$. However, if g is meromorphic in $|z| < 1$, then $G_r(\theta)$ is only meromorphic in $|\operatorname{Im} \theta| < A(1 - r)$, where $0 < A < 1$. Thus, to bound the number of zeros of $G_r(\theta)$ on the real θ-axis using the above technique, we would have to apply Jensen's theorem to $G_r(\theta)$ in $O((1 - r)^{-1})$ disks of radius less than $A(1 - r)$, centered on the real θ-axis, and covering the real θ-axis between $-\pi$ and π. This complication alone would introduce an additional factor of $(1 - r)^{-1}$ to the bounds of ϕ and Φ in (i) and (ii) of the theorem. New techniques are used to obtain the correct bounds for ϕ and Φ.

Also, in [10] the bounds on ϕ and Φ involve $T(Ar, f)$ for some constant $A > 1$. Such a bound is impossible for r close to 1 if f is meromorphic in $|z| < 1$. This complication is resolved by denoting a convex linear combination of 1 and r by $c(r) = (1 - b) + br$, $0 < b < 1$, and bounding ϕ and Φ in terms of $T(c(r), f)$.[1]

We assume familiarity with the standard notation of Nevanlinna theory. It is not intended that positive constants such as A and R have the same value with each occurrence. Also, notation such as $A(\alpha_0)$, $A(\alpha, d)$, etc. is used to emphasize the dependence of the constants on α_0, or α and d, etc. Once again it is not intended that these constants have the same value with each occurrence. Throughout the paper, if $c(r) = (1 - b) + br$ for $0 < b < 1$, then we let $c^n(r) = c(c^{n-1}(r))$. It is easy to show that $c^n(r) = (1 - b^n) + b^n r$.

1. Preliminary lemmas.

LEMMA 1.1.[2] *Let $f(z)$ be holomorphic in the circle $|z| < R$ with $|f(0)| = 1$ and let η be an arbitrary positive number not exceeding $(8e)^{-1}$. Inside the circle $|z| \leqq r < R$ but outside of a family of excluded circles, centered at the zeros of f in $|z| < R$, the sum of whose radii is not greater than ηr, we have*

$$\log |f(z)| > A(R - r)^{-2} T(R, f) \log \eta ,$$

provided r and R are sufficiently large.

[1] I wish to thank the referee of this paper for suggesting this very useful notation as well as for making other helpful comments.

[2] This lemma was observed several years ago by A. Baernstein, who in unpublished work used it to obtain a bound for $\phi(r, f)$, off an exceptional set, where f is meromorphic in the plane.

This is an elementary adaptation of Theorem 11 of [7].

LEMMA 1.2. *There are absolute constants $A > 0$, $\gamma \in [0, 1)$ and p, a positive integer, such that if f is meromorphic in $|z| < 1$, then there exist holomorphic functions g and h in $|z| < 1$, such that $f = g/h$ and*

$$\max (T(r, g), T(r, h)) < A(1 - r)^{-p} T((1 - \gamma) + \gamma r, f) .$$

This lemma is contained in [1], which carries a result of J. Miles [9] to the unit disk.

LEMMA 1.3. *If f is a nonconstant meromorphic function in the plane and $0 < \alpha < 1$, then there is an $A = A(\alpha)$ so that for $r > R$*

$$\frac{1}{2\pi} \int_0^{2\pi} |\mathrm{Re}\,(re^{i\theta} f''(re^{i\theta})/f'(re^{i\theta})) + 1| d\theta$$
$$< A(1 - r)^{-1}[T((1 - \alpha) + \alpha r, f) - \log (1 - r)] .$$

This lemma is contained in (3.10) of [8].

LEMMA 1.4. *Suppose f is a nonconstant meromorphic function in the disk and r is such that $f'(re^{i\theta}) \neq 0, \infty$ for $0 \leqq \theta \leqq 2\pi$. If $\phi(r, f) > 7A(1 - r)^{-1}[T((1 - \alpha) + \alpha r, f) - \log (1 - r)]$, where A and α are as in Lemma 1.3, then*

$$\phi(r, zf''(z)/f'(z) + 1) > \phi(r, f)/6 .$$

Proof. Let $\beta(\theta)$ be a continuous determination of the argument of the vector tangent to the curve $f(re^{i\theta})$, $0 \leqq \theta \leqq 2\pi$. We recall that

$$\beta'(\theta) = \mathrm{Re}\,(re^{i\theta} f''(re^{i\theta})/f'(re^{i\theta}) + 1) . \tag{1.1}$$

Suppose $0 \leqq \alpha_1 < \alpha_2 < \alpha_3 < 2\pi$, $\mathrm{Re}\, f(re^{i\alpha_j}) = 0$ for $j = 1, 2, 3$ and $\mathrm{Re}\, f(re^{i\theta}) \neq 0$ for $\alpha_1 < \theta < \alpha_3$ except for $\theta = \alpha_2$. We distinguish two cases.

Case I. Suppose $|\beta(\phi_1) - \beta(\phi_2)| < \pi$ for all ϕ_1 and ϕ_2 in $[\alpha_1, \alpha_3]$. By Rolle's theorem there exist $\alpha_1' \in (\alpha_1, \alpha_2)$ and $\alpha_2' \in (\alpha_2, \alpha_3)$ and there exist integers n_1 and n_2 such that $\beta(\alpha_j') = n_j\pi + \pi/2$, $j = 1, 2$. Since $|\beta(\alpha_1') - \beta(\alpha_2')| < \pi$, we must have $\beta(\alpha_1') = \beta(\alpha_2')$. By Rolle's theorem we conclude that in Case I there exists γ in $(\alpha_1', \alpha_2') \subset (\alpha_1, \alpha_3)$ such that $\beta'(\gamma) = 0$.

Case II. Suppose there exist ϕ_1 and ϕ_2 in $[\alpha_1, \alpha_2]$ such that $|\beta(\phi_1) - \beta(\phi_2)| \geqq \pi$. Thus, in Case II

$$\frac{1}{2\pi}\int_{\alpha_1}^{\alpha_3}|\beta'(\theta)|d\theta \geqq \frac{1}{2}\,. \tag{1.2}$$

We now let $0 \leqq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ be a complete list of solutions of $\operatorname{Re} f(re^{i\theta}) = 0$ in $[0, 2\pi)$, and consider triples $(\theta_{2k-1}, \theta_{2k}, \theta_{2k+1})$ for $1 \leqq k \leqq [\phi(r, f)/2] - 1$. By Lemma 1.3 and (1.2), no more than $2A(1 - r)^{-1}[T((1 - \alpha) + \alpha r, f) - \log(1 - r)]$ of these triples fall into Case II. Thus at least

$$\begin{aligned}[\phi(r, f)/2] - 1 - [2A(1 - r)^{-1}\{T((1 - \alpha) + \alpha r, f) - \log(1 - r)\}]\\ \geqq [\phi(r, f)/6]\end{aligned}$$

of these triples fall into Case I, and consequently there are at least $\phi(r, f)/6$ zeros of $\beta'(\theta)$ in $[0, 2\pi)$.

LEMMA 1.5. *If f is a nonconstant meromorphic function in the unit disk, $k(r)$ is a function satisfying $k(r) \geqq -\log(1 - r)$ and $c_2(r) = (1 - \alpha_2) + \alpha_2 r$ where $0 < \alpha_2 < 1$, then there is a constant A and a set $\Delta \subset [0, 1)$, both depending on the function k and on α_2, such that*

$$\int_\Delta \exp\{T(c_2(r), f) + k(r)\}dr < \infty$$

and for $r \notin \Delta$ and $r > R$,

$$\int_0^{2\pi}\log|\operatorname{Re}(re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})) + 1|^{-1}d\theta < A[T(c_2(r), f) + k(r)]\,.$$

Proof. We follow closely [6, p. 226–227]. Let $G(z) = zf''(z)/f'(z) + 1$, and

$$\rho(a) = |\operatorname{Re} a|^{-1/2}\Big(\iint_A |\operatorname{Re} a|^{-1/2}dw(a)\Big)^{-1}$$

where $w(a)$ is area measure on the Riemann sphere A. Also, define

$$\lambda(t, G) = \int_0^{2\pi}\rho(G(te^{i\theta}))|G'(te^{i\theta})|^2(1 + |G(te^{i\theta})|^2)^{-2}d\theta\,.$$

From (14.6.18) of [6], we have

$$\int_0^{2\pi}\log\rho(G(re^{i\theta}))d\theta \leqq 8\pi T(r, G) + \log\lambda(r, G) + O(1)\,. \tag{1.3}$$

We set $L(r, G) = \int_0^r \lambda(t, G)t\,dt$ and $K(r, G) = \int_{r_0}^r L(s, G)s^{-1}ds$. Then by (14.6.20) of [6], $T(r, G) \geqq K(r, G) - O(1)$. Denote by Δ_1 the intervals $(\alpha_{1j}, \beta_{1j})$ where

$$\lambda(r, G) > r^{-1}\exp\{k(r) + T(c_2(r), G)\}(L(r, G))^2\,.$$

We have

$$\int_{\Delta_1} \exp\{k(r) + T(c_2(r), G)\}dr < \int_{\Delta_1} r\lambda(r, G)(L(r, G))^{-2}dr$$
$$= \int_{\Delta_1} (L(r, G))^{-2}dL(r, G)$$
$$< (L(\alpha_{11}, G))^{-1} < \infty .$$

Denote by Δ_2 the intervals $(\alpha_{2j}, \beta_{2j})$ where

$$L(r, G) > r\exp\{k(r) + T(c_2(r), G)\}[K(r, G)]^2 .$$

As before, we have

$$\int_{\Delta_2} \exp\{k(r) + T(c_2(r), G)\}dr < \int_{\Delta_2} (K(r, G))^{-2}d(K(r, G))$$
$$< (K(\alpha_{21}, G))^{-1} < \infty .$$

Let $\Delta = \Delta_1 \cup \Delta_2$. If $r \notin \Delta$ and $r > R$, then

$$\lambda(r, G) < r^{-1}\exp\{k(r) + T(c_2(r), G)\}[L(r, G)]^2$$
$$< r\exp\{3k(r) + 3T(c_2(r), G)\}(K(r, G))^4$$
$$< r\exp\{3k(r) + 3T(c_2(r), G)\}(T(r, G) + O(1))^4 .$$

Thus for $r \notin \Delta$ and $r > R$ and for some constant A,

$$\log\lambda(r, G) < A(3k(r) + 7T(c_2(r), G)) . \tag{1.4}$$

From Lemma 1.6 and well known properties of the characteristic function, $T(s, G) < A_2(T(s, f) - \log(1 - s))$ for $s > R$. The lemma follows readily from (1.3) and (1.4).

We state the following elementary lemma without proof.

LEMMA 1.6. *Let f be meromorphic in $|z| < 1$ with $|f(0)| = 1$. If $r < 1$ and $c(r) = (1 - \alpha) + \alpha r$ for some $0 < \alpha < 1$, then*

(i) $n(r, f') < A(\alpha)(1 - r)^{-1}T(c(r), f')$

(ii) $n(r, 1/f') < A(\alpha)(1 - r)^{-1}T(c(r), f')$

(iii) $T(r, f') < A(T(r, f) - \log(1 - r))$ *for* $r > R$

and

(iv) $T(r, 1/f') < A(T(r, f) - \log(1 - r))$ *for* $r > R$.

2. Proof of part (i) of the theorem. Without loss of generality we may assume that $|f(0)| = 1$ since if $f(0) \neq 0, \infty$ we may consider $f(z)/|f(0)|$ and if $f(0) = 0, \infty$ we may consider $f(z) + i$ or $1/f(z) + i$.

With α_0 as in part (i) of the theorem, let

$$\alpha = \alpha_0^{1/2} \quad \text{and} \quad s = c(r) = (1 - \alpha) + \alpha r .$$

Also define

$$F_r(\theta) = \operatorname{Re}\,(re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})) + 1\ , \tag{2.1}$$

and for $x \in [0, 2\pi)$

$$H_r^x(\theta) = F_r(x + \theta)\ . \tag{2.2}$$

We will show that if θ is complex then $H_r^x(\theta)$ is a meromorphic function in a strip containing the real θ-axis. We will apply Jensen's theorem to $H_r^x(\theta)$ in a circle centered on the real θ-axis, and integrate with respect to x to obtain a bound for $\phi(r, (zf'(z)/f(z)) + 1)$, which will yield a bound for $\phi(r, f)$. We first let

$$K(t, a, \theta) = (t^2 - ta\cos\theta)/(t^2 + a^2 - 2at\cos\theta)\ . \tag{2.3}$$

Then, by the differentiated Poisson-Jensen theorem [4, p. 22], we have

$$\begin{aligned} F_r(\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \log|f'(se^{i\mu})|\frac{2rs((r^2+s^2)\cos(\theta-\mu) - 2rs)}{(s^2 + r^2 - 2rs\cos(\theta-\mu))^2}d\mu \\ &\quad - \sum_{0<a_n<s} K(a_n r, s^2, \theta - \alpha_n) + \sum_{0<b_n<s} K(b_n r, s^2, \theta - \beta_n) \\ &\quad + \sum_{a_n<s} K(r, a_n, \theta - \alpha_n) - \sum_{b_n<s} K(r, b_n, \theta - \beta_n) + 1 \\ &= I - II + III + IV - V + 1\ , \end{aligned} \tag{2.4}$$

where $\{a_n e^{i\alpha_n}\}$ and $\{b_n e^{i\beta_n}\}$ are the zeros and poles, respectively, of f', listed in nondecreasing order of magnitude. We let θ be complex and prove

LEMMA 2.1. *The function $F_r(\theta)$ (see (2.1)) is meromorphic in $|\operatorname{Im}\theta| < (1-\alpha)(1-r)$ with poles at values of θ for which $\operatorname{Im}\theta = \pm\log(rd_n^{-1})$ and $\operatorname{Re}\theta = \gamma_n + 2\pi k$, $k = 0, \pm1, \pm2, \cdots$, where $d_n e^{i\gamma_n}$ is a zero or pole of f' and $0 < d_n < s$.*

Proof. If $t = a$ then $K(t, a, \theta) = 1/2$ for all $\theta \neq 2\pi k$, $k = 0, \pm1, \pm2, \cdots$. If $t^2 + a^2 - 2at\cos\theta = 0$ where $a \neq t$ and $\theta = \zeta + i\beta$, then

$$1 < (a^2 + t^2)(2at)^{-1} = \cos\theta = \cos\zeta\cosh\beta - i\sin\zeta\sinh\beta\ . \tag{2.5}$$

Thus, $\zeta = 2\pi k$ and $\cosh\beta = (a^2 + t^2)/2at = (a/t + t/a)/2 = \cosh(\log a/t)$. Hence,

$$\operatorname{Re}\theta = 2\pi k\ ,\quad k \text{ an integer and}\quad \operatorname{Im}\theta = \pm\log at^{-1}\ . \tag{2.6}$$

We have $\log sr^{-1} = \log(1 + (s - r)r^{-1}) > (1-\alpha)(1-r)$ for $r > R$. Thus, term I of (2.2) is a holomorphic function of θ in $|\operatorname{Im}\theta| < (1-\alpha)(1-r)$. Also for $0 < d_n < s$, we have $\log s^2(d_n r)^{-1} > \log sr^{-1}$. Hence terms II and III are also holomorphic in $|\operatorname{Im}\theta| < (1-\alpha)(1-r)$. Finally, from (2.5) and (2.6), terms IV and V are meromorphic in $|\operatorname{Im}\theta| < (1-\alpha)(1-r)$ with poles at values of θ satisfying $\operatorname{Im}\theta =$

$\pm\log rd_n^{-1}$ and $\operatorname{Re}\theta = \gamma_n + 2\pi k$, $k = 0, \pm 1, \pm 2, \cdots$.

We now apply Jensen's theorem to $H_r^x(\theta)$ (see (2.2)) with $h = (1-\alpha)(1-r)/2$, and integrate with respect to x, to obtain

$$\begin{aligned}(2.7)\qquad \int_0^{2\pi} N\left(h, \frac{1}{H_r^x}\right)dx &= -\int_0^{2\pi} \log|H_r^x(0)|dx + \int_0^{2\pi} N(h, H_r^x)dx \\ &\quad + \int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \log|H_r^x(he^{i\mu})|d\mu \\ &= L_1 + L_2 + L_3 .\end{aligned}$$

In the following four lemmas we obtain a lower bound for the left hand side of equation (2.7), and upper bounds for the three terms L_1, L_2 and L_3.

LEMMA 2.2. *For H_r^x defined above we have*

$$\int_0^{2\pi} N\left(h, \frac{1}{H_r^x}\right)dx \geqq 2h\phi(r, zf''(z)/f'(z) + 1) .$$

Proof. By Tonelli's theorem,

$$\int_0^{2\pi} N\left(h, \frac{1}{H_r^x}\right)dx = \int_0^h\int_0^{2\pi} n\left(t, \frac{1}{H_r^x}\right)t^{-1}dxdt .$$

The contribution to the latter integral from a single zero of H_r^x on the real θ-axis at $\theta = a$, where $0 \leqq a - h < a + h < 2\pi$ is $\int_0^h\int_{a-t}^{a+t} t^{-1}dxdt = 2\int_0^h dt = 2h$. Similarly it can be shown that if $a - h < 0$ or $a + h \geqq 2\pi$, then the contribution to the integral is again $2h$. The lemma follows from the fact that the real zeros of H_r^x are just the zeros of $\operatorname{Re}(zf''(z)/f'(z) + 1)$ on $|z| = r$.

LEMMA 2.3. *Let A be the constant and Δ the set in Lemma 1.5 corresponding to $k(r) = -\log(1-r)$ and $\alpha_2 = \alpha^2$. For L_1 as in (2.7) we have for $r \notin \Delta$ and $r > R$,*

$$L_1 < A[T(c^2(r), f) - \log(1-r)] .$$

Proof. If $r \notin \Delta$ and $r > R$, then by Lemma 1.5

$$\begin{aligned}L_1 &= -\int_0^{2\pi} \log|H_r^x(0)|dx = -\int_0^{2\pi} \log|F_r(x)|dx \\ &= -\int_0^{2\pi} \log|\operatorname{Re}(re^{ix}f''(re^{ix})/f'(re^{ix})) + 1|dx \\ &< A[T(c^2(r), f) - \log(1-r)] .\end{aligned}$$

LEMMA 2.4. *For L_2 as in (2.7), we have for $A = A(\alpha)$ and for*

$r > R$

$$L_2 < A[T(c^2(r), f) - \log(1-r)] .$$

Proof. By Tonelli's theorem we have

$$L_2 = \int_0^{2\pi} N(h, H_r^x)dx = \int_0^h \int_0^{2\pi} n(t, H_r^x)t^{-1}dxdt .$$

The contribution to L_2 from a pole of $F_r(\theta)$ at b, where $|\operatorname{Im} b| < h$, is no more than

$$\int_{|\operatorname{Im} b|}^h \left(\int_{\operatorname{Re} b - \sqrt{t^2-(\operatorname{Im} b)^2}}^{\operatorname{Re} b + \sqrt{t^2-(\operatorname{Im} b)^2}} dx\right)t^{-1}dt = \int_{|\operatorname{Im} b|}^h 2\sqrt{t^2 - (\operatorname{Im} b)^2}\, t^{-1}dt$$
$$\leqq 2\int^h dt = 2h .$$

The poles of $F_r(\theta)$ (see (2.1)) in $\{\theta\colon 0 \leqq \operatorname{Re}\theta < 2\pi \text{ and } |\operatorname{Im}\theta| < h\}$ arise from zeros or poles of $f'(z)$ in $|z| < s$. Thus, by Lemma 1.6, $F_r(\theta)$ has no more than $2(n(s, f') + n(s, 1/f')) < A(\alpha)(1-r)^{-1}[T(c^2(r), f) - \log(1-r)]$ poles in the above region for $r > R$. Hence

$$L_2 < 2hA(\alpha)(1-r)^{-1}[T(c^2(r), f) - \log(1-r)] ,$$

and the lemma follows since $h(1-r)^{-1} = (1-\alpha)/2$.

LEMMA 2.5. *For L_3 as in (2.7) we have for some constant $A = A(\alpha)$ and for $r > R$*

$$L_3 < A[T(c^2(r), f) - \log(1-r)] .$$

Proof. We have from (2.4) that

$$(2.8)\quad L_3 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \log|F_r(x + he^{i\mu})|dxd\mu$$
$$\leqq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \log^+\left|\frac{1}{2\pi}\int_0^{2\pi} \log|f'(se^{it})|\right.$$
$$\left.\times \frac{2rs((s^2+r^2)\cos(x + he^{i\mu} - t) - 2rs)}{(r^2 + s^2 - 2rs\cos(x + he^{i\mu} - t))^2}\,dt\right|dxd\mu$$
$$+ \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \log^+\left|\sum_{d_n<s} K(r, d_n, x + he^{i\mu} - \gamma_n)\right|dxd\mu$$
$$+ \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \log^+\left|\sum_{0<d_n<s} K(d_n r, s^2, x + he^{i\mu} - \gamma_n)\right|dxd\mu + \log 5$$
$$= E_1 + E_2 + E_3 + \log 5 ,$$

where $d_n e^{i\gamma_n}$ is a zero or pole of f'.

We analyze terms E_1, E_2 and E_3 separately.

Term E_1. Since $h = (1-\alpha)(1-r)/2$ and $\log sr^{-1} > (1-\alpha)(1-r)$

for $r > R$, we have for some $w \in ((1-\alpha)(1-r)/2, (1-\alpha)(1-r))$, for $\mu \in [0, 2\pi)$ and for $r > R$,

$$
\begin{aligned}
(2.9)\quad & |(r^2+s^2)(2rs)^{-1} - \cos(x + he^{i\mu} - t)| \\
& \geqq |\cosh(\log sr^{-1}) - \cosh(h \sin \mu)| \\
& \geqq \left|\cosh((1-\alpha)(1-r)) - \cosh\left(\frac{1}{2}(1-\alpha)(1-r)\right)\right| \\
& = \sinh \omega\left((1-\alpha)(1-r) - \frac{1}{2}(1-\alpha)(1-r)\right) \\
& \geqq \frac{1}{2}(1-\alpha)(1-r)\sinh\left(\frac{1}{2}(1-\alpha)(1-r)\right) \\
& \geqq \frac{1}{2}(1-\alpha)(1-r)\frac{1}{4}(1-\alpha)(1-r) = \frac{1}{8}(1-\alpha)^2(1-r)^2 .
\end{aligned}
$$

Also, since $r < s < 1$ and $\cosh(h) + \sinh(h) = e^h < 4$, we have from (2.5) that

$$|(s^2+r^2)\cos(x + he_i^{\mu} - t) - 2rs| \leqq 2(\cosh(h) + \sinh(h)) + 2 < 10 .$$

Thus, for constants $A_j = A_j(\alpha)$, $j = 1, 2$, and for $r > R$, from (2.7) and Lemma 1.6,

$$
\begin{aligned}
(2.10)\quad E_1 & < 2\pi\left(-A_1 \log(1-r) + \log^+\left|\frac{1}{2\pi}\int_0^{2\pi} |\log|f'(se^{it})||dt\right|\right) \\
& = 2\pi\left(-A_1\log(1-r) + \log^+\left(T(s, f') + T\left(s, \frac{1}{f'}\right)\right)\right) \\
& < A_2(\log T(c(s), f) - \log(1-r)) .
\end{aligned}
$$

Term E_3. Since $0 < d_n < s$ we have $(s^4 + d_n^2 r^2)(2d_n rs^2)^{-1} \geqq (s^2 + r^2)(2rs)^{-1}$. As in (2.9) we have for $r > R$ that the denominator of $|K(d_n r, s^2, x + he^{i\mu} - \gamma_n)|$ (see (2.3)) divided by $|2d_n rs^2|$ is

$$(2.11)\quad |(s^4 + d_n^2 r^2)(2d_n rs^2)^{-1} - \cos(x + he^{i\mu} - \gamma_n)| > \frac{1}{8}(1-\alpha)^2(1-r)^2 .$$

Also as above we have for $r > R$ and $d_n \neq 0$ that the numerator of $|K(d_n r, s^2, x + he^{i\mu} - \gamma_n)|$ divided by $|2d_n rs^2|$ is

$$
\begin{aligned}
(2.12)\quad & |(2d_n rs^2)^{-1}(d_n rs^2 \cos(x + he^{i\mu} - \gamma_n) - d_n^2 r^2)| \\
& = \frac{1}{2}|\cos(x + he^{i\mu} - \gamma_n) - d_n rs^{-2}| \\
& \leqq \frac{1}{2}(\cosh(h) + \sinh(h)) + \frac{1}{2} \\
& = \frac{1}{2}(e^h + 1) < 3 .
\end{aligned}
$$

We conclude from (2.11) and (2.12) that for $r > R$

$$|K(d_n r, s^2, x + he^{i\mu} - \gamma_n)| < A(\alpha)(1 - r)^{-2},$$

and therefore from (2.8) and Lemma 1.6, for $r > R$

$$\begin{aligned} (2.13)\qquad E_3 &< 2\pi(\log(n(s, f') + n(s, 1/f')) + \log(A(\alpha)(1 - r)^{-2})) \\ &< A(\alpha)[\log T(c^2(r), f) - \log(1 - r)] . \end{aligned}$$

Term E_2. We change the variables of integration in E_2 to $u = x + h\cos\mu - \gamma_n$ and $v = h\sin\mu$. Since this transformation takes $\{(x, \mu): 0 \leqq x < 2\pi, 0 \leqq \mu < 2\pi\}$ onto $\{(u, v): 0 \leqq u \leqq 2\pi, -h \leqq v \leqq h\}$ exactly twice, it follows that

$$(2.14)\qquad E_2 = \frac{2}{\pi}\int_0^h\int_0^{2\pi}\left(\log^+\left|\sum_{d_n<s} K(r, d_n, u + iv)\right|\right)(h^2 - v^2)^{-1/2}du dv .$$

We define

$$(2.15)\qquad \varepsilon = \varepsilon(r) = \min\{\exp(-T(c^2(r), f)), (1 - r)^5\} ,$$

and

$$\begin{aligned} (2.16)\qquad D = D(\varepsilon) = \bigcup_{d_n<s} \{&(\log(d_n r^{-1}) - \varepsilon, \log(d_n r^{-1}) + \varepsilon) \\ &\cup (-\log(d_n r^{-1}) - \varepsilon, -\log(d_n r^{-1}) + \varepsilon)\} . \end{aligned}$$

We will evaluate the integral in (2.14) over values in $[0, h] - D$ and then over v values in $D \cap [0, h]$. We begin by obtaining a lower bound for the denominator of $|K(r, d_n, v + iv)|$ (see (2.3)). If $r^2 + d_n^2 - 2rd_n\cos(u_0 + iv_0) = 0$ for $|v_0| \leqq h$, then

$$\begin{aligned} &r^2 + d_n^2 - 2rd_n\cos(u + iv) \\ &\quad = r^2 + d_n^2 - 2rd_n\cos(u + iv) - (r^2 + d_n^2 - 2rd_n\cos(u_0 + iv_0)) \\ &\quad = -2rd_n(\cos(u + iv) - \cos(u_0 + iv_0)) \\ &\quad = 4rd_n\sin\left(\frac{1}{2}(u - u_0) + \frac{i}{2}(v - v_0)\right)\sin\left(\frac{1}{2}(u + u_0) + \frac{i}{2}(v + v_0)\right) . \end{aligned}$$

There is an absolute constant B so that $|\sin z|/|\operatorname{Im} z| > B$. If $v \notin D$, then $|v \pm v_0| > \varepsilon$ and $|\sin((u \pm u_0)/2 + i(v \pm v_0)/2)| > B|v \pm v_0| > B\varepsilon$. Hence, for $v \notin D$, $d_n \neq 0$ and $r > R$, the denominator of $|K(r, d_n, u + iv)|$ is

$$|r^2 + d_n^2 - 2rd_n\cos(u + iv)| > 4rd_n B^2\varepsilon^2 .$$

Also, since $|v| \leqq h$ and $\cos(u + iv) = \cos u\cosh v - i\sin u\sinh v$, we have that the numerator of $|K(r, d_n, u + iv)|$ is

$$(2.17)\qquad |r^2 - rd_n\cos(u + iv)| \leqq 1 + \cosh v + \sinh|v| < 4 .$$

Thus, since $K(r, 0, u+iv) = 1$ and $\int_0^h (h^2 - v^2)^{-1/2} dv = \pi/2$, we have for $d_0 = \min\{d_k \neq 0 : k = 1, 2, 3, \cdots\}$, and for $r > R$

$$\text{(2.18)} \quad \int_{[0,h]-D} \frac{1}{2\pi} \int_0^{2\pi} \left(\log^+ \left| \sum_{d_n < s} K(r, d_n, u+iv) \right| \right) (h^2 - v^2)^{-1/2} du dv$$

$$\leqq \int_0^h \left(\log\left(n(s, f') + n\left(s, \frac{1}{f'}\right)\right) - A(d_0) \log \varepsilon \right) (h^2 - v^2)^{-1/2} dv$$

$$< A(\alpha, d_0)(T(c(s), f) - \log(1-r)) .$$

Furthermore, since $\int_0^{2\pi} |\log |c - \cos t|^{-1}| dt < A$ for all real c, (2.17) and a straight forward calculation yield that for all $d_n \neq 0$,

$$\int_0^{2\pi} \log^+ |K(r, d_n, u+iv)| du$$

$$= \int_0^{2\pi} \log^+ \left| \frac{r^2 - r d_n \cos(u+iv)}{r^2 + d_n^2 - 2 r d_n \cos(u+iv)} \right| du$$

$$< 8\pi + |\log(2rd_0)| + \int_0^{2\pi} \log^+ |(r^2 + d_n^2)(2rd_n)^{-1} - \cos u|^{-1} du$$

$$< 8\pi + |\log(2rd_0)| + A = A(d_0) .$$

Hence, using Lemma 1.6, for $r > R$

$$\text{(2.19)} \quad \int_0^{2\pi} \log^+ \left| \sum_{d_n < s} K(r, d_n, u+iv) \right| du$$

$$\leqq 2\pi \log\left(n(s, f') + n\left(s, \frac{1}{f'}\right)\right)$$

$$+ \sum_{d_n < s} \int_0^{2\pi} \log^+ |K(r, d_n, u+iv)| du$$

$$\leqq 2\pi \log\left(n(s, f') + n\left(s, \frac{1}{f'}\right)\right) + A(d_0)\left(n(s, f') + n\left(s, \frac{1}{f'}\right)\right)$$

$$< A(\alpha, d_0)(1-r)^{-1}(T(c(s), f) - \log(1-r)) .$$

The measure of D is no more than $\delta = \delta(\varepsilon) = 2(n(s, f') + n(s, 1/f'))\varepsilon$. Also,

$$\int_{D \cap [0,h]} (h^2 - v^2)^{-1/2} dv \leqq \int_{h-\delta}^h (h^2 - v^2)^{-1/2} dv = \sin^{-1}(1) - \sin^{-1}(1 - \delta h^{-1})$$

$$= \frac{\pi}{2} - y$$

where $y = \sin^{-1}(1 - \delta h^{-1})$. Since $\lim_{w \to \pi/2} (\sin \pi/2 - \sin w)/(\pi/2 - w)^2 = 1/2$, we have for $r > R$

$$\frac{\pi}{2} - y \leqq 2\left(\sin\frac{\pi}{2} - \sin y\right)^{1/2} = 2(1 - (1 - \delta h^{-1}))^{1/2} = (4\delta h^{-1})^{1/2} .$$

Therefore,

$$\int_{D\cap[0,h]}(h^2-v^2)^{-1/2}dv \leqq (4\delta h^{-1})^{1/2} = \left(8h^{-1}\left(n(s, f') + n\left(s, \frac{1}{f'}\right)\right)\varepsilon\right)^{1/2}$$

and from (2.19) and Lemma 1.6,

$$\begin{aligned}(2.20)\quad \int_{D\cap[0,h]}\int_0^{2\pi}&\left(\log^+\left|\sum_{d_n<s}K(r, d_n, u+iv)\right|\right)(h^2-v^2)^{-1/2}dudv\\ &\leqq A(\alpha, d_0)(1-r)^{-1}(T(c(s), f) - \log(1-r))\\ &\quad\times\left(8h^{-1}\left(n(s, f') + n\left(s, \frac{1}{f'}\right)\right)\varepsilon\right)^{1/2}\\ &\leqq A(\alpha, d_0)(1-r)^{-2}(T(c(s), f) - \log(1-r))^{3/2}\varepsilon^{1/2} = o(1)\end{aligned}$$

by the definition of ε (see (2.15)). From (2.14), (2.18) and (2.20) we conclude that for $r > R$

$$(2.21)\qquad E_2 < A(\alpha, f)(T(c(s), f) - \log(1-r))\ .$$

Since $s = c(r)$ it follows from (2.10), (2.13) and (2.21) that for $r > R$ and for some constant $A = A(\alpha, f)$

$$L_3 < A(\alpha, f)(T(c^2(r), f) - \log(1-r))\ .$$

Finally, we conclude from (2.7) and Lemmas 2.2, 2.3, 2.4 and 2.5 for $r \notin \Delta$, $r > R$ and for some constant $A = A(\alpha, f)$

$$2h\phi(r, zf''(z)/f'(z) + 1) < A(T(c^2(r), f) - \log(1-r))\ .$$

Part (i) of the theorem now follows from Lemma 1.4 since $h = (1-\alpha)(1-r)/2$, and $c^2(r) = c_0(r)$.

3. Proof of part (ii) of the theorem. We have obtained an upper bound for $\phi(r, f)$ off an exceptional set of r values, but the techniques used in § 2 do not yield any upper bound for $\phi(r, f)$ on the exceptional set. In this section we obtain an upper bound for $\phi(r, f)$ on the exceptional set by bounding $\phi(r, zf''/f' + 1)$. This upper bound for $\phi(r, f)$ will yield, upon integration, the appropriate bound for $\Phi(r, f)$.

We let $c(r) = (1-\gamma) + \gamma r$ with γ as in Lemma 1.2. By Lemma 1.2 we can write $zf''(z)/f'(z) + 1 = g_1(z)/g_2(z)$ where g_1 and g_2 are holomorphic in the unit disk and for $r > R$

$$\begin{aligned}(3.1)\quad \max(T(r, g_1), T(r, g_2)) &< A(1-r)^{-p}T(c(r), zf''z/f'(z) + 1)\\ &< A(1-r)^{-p}(T(c(r), f) - \log(1-r))\end{aligned}$$

where p is a positive integer and we have used Lemma 1.6 and well known properties of the characteristic function.

We have $\operatorname{Re}(zf''(z)/f'(z)+1) = \operatorname{Re}(g_1(z)\overline{g_2(z)})/|g_2(z)|^2$. We let $u_{j,r}(\theta) = \operatorname{Re} g_j(re^{i\theta})$ and $v_{j,r}(\theta) = \operatorname{Im} g_j(re^{i\theta})$ for $j = 1, 2$ and define J_r by

$$(3.2)\quad J_r(\theta) = \operatorname{Re}(g_1(re^{i\theta})\overline{g_2(re^{i\theta})}) = |g_2(re^{i\theta})|^2 \operatorname{Re}((re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})) + 1) = u_{1,r}(\theta)u_{2,r}(\theta) + v_{1,r}(\theta)v_{2,r}(\theta) .$$

Now choose $r_0 > 0$ so that (3.1), Lemma 3.3, (3.8) and (3.12) of this section hold for $r > r_0$. For γ as in Lemma 1.2 let

$$(3.3)\qquad c_0(r) = (1 - \gamma^{1/4}) + \gamma^{1/4}r \quad \text{and} \quad s_n = c_0^n(r) .$$

We note that if we let $s_0 = r_0$ then $c_0^4(r) = c(r)$ and $\bigcup_{n=0}^{\infty}[s_n, s_{n+1}) = [r_0, 1)$.

LEMMA 3.1. *If $r \in [s_n, s_{n+1})$, $f(re^{i\theta}) \neq 0$ for $0 \leqq \theta \leqq 2\pi$, and the distance from $|z| = r$ to the nearest zero of $g_2(z)$ is no less than ηr, where $\eta < \eta_0 < 1$, then there is a $\theta_0 \in [0, 2\pi)$ such that*

$$\log|J_r(\theta_0)| > A(s_{n+2} - s_{n+1})^{-2}(T(c(s_{n+2}), f) - \log(1 - s_{n+2}))\log\eta .$$

Proof. Applying Lemma 1.1 to $g_2(z)/|g_2(0)|$ or $g_2(z)/c_kz^k$ for appropriate k and c_k in $|z| \leqq s_{n+2}$, we obtain a union of disks $C(s_n, \eta)$, centered at the zeros of g_2 in $0 < |z| \leqq s_{n+2}$, the sum of whose radii does not exceed ηs_{n+1}, such that in $\{r_0 \leqq |z| \leqq s_{n+1}\} - C(s_n, \eta)$

$$(3.4)\quad \log|g_2(z)| > A(s_{n+2} - s_{n+1})^{-2}T(s_{n+2}, g)\log\eta > A(s_{n+2} - s_{n+1})^{-2}(T(c(s_{n+2}), f) - \log(1 - s_{n+2}))\log\eta .$$

We let $B(s_n, \eta) = \{r: f(re^{i\theta}) \in C(s_n, \eta)$ for some $0 \leqq \theta < 2\pi\}$, and

$$(3.5)\quad E(s_n, \eta) = [s_n, s_{n+1}) \cap \{B(s_n, \eta) \cup \{r: f \text{ has a zero of modulus } r\}\} .$$

If $r \in [s_n, s_{n+1}) - E(s_n, \eta)$, then $g_1(z)/g_2(z)$ has no poles (and hence f has no zeros or poles) on $|z| = r$. Thus $\omega = f(re^{i\theta})$, $0 \leqq \theta \leqq 2\pi$ is a closed path in the plane and by (1.1)

$$\frac{1}{2\pi}\int_0^{2\pi}|\operatorname{Re}(re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})) + 1|d\theta \geqq 1 .$$

Consequently, there is a $\theta_0 \in [0, 2\pi)$ such that

$$|\operatorname{Re}(re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})) + 1| \geqq 1 ,$$

which together with (3.2) and (3.4) yields the lemma.

LEMMA 3.2. *If $r \in [s_n, s_{n+1})$ and θ is complex, then $H_r(\theta)$ is holomorphic in $|\operatorname{Im}\theta| < -\log r$ and for $|\operatorname{Im}\theta| \leqq \log(c(s_{n+1})/s_{n+1})$ we have for some positive integer p,*

$$|J_r(\theta)| < (s_{n+3} - s_{n+2})^{-1/2} \exp\{A(s_{n+4} - s_{n+3})^{-(p+1)} \times [T(c(s_{n+4}), f) - \log(1 - s_{n+4})]\} .$$

Proof. If $g_1(z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = \alpha_n + i\beta_n$, α_n, β_n real, then let $g_1^*(z) = \sum_{n=0}^{\infty} |a_n| z^n$. We note that by Lemma 4 of [10]

$$M(r, g_1^*) < (R - r)M(R, g)$$

for $0 < r < R < 1$. Also, for real θ

$$u_{1,r}(\theta) = \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n . \tag{3.6}$$

If we let θ be complex, (3.6) implies that $u_{1,r}(\theta)$ is holomorphic in $|\operatorname{Im} \theta| < -\log r$. If $|\operatorname{Im} \theta| < \log(s_{n+2}/s_{n+1}) < -\log r$, then

$$\begin{aligned} |u_{1,r}(\theta)| &\leqq 2 \sum_{n=0}^{\infty} |a_n| (r \exp(\log(s_{n+2}/s_{n+1})))^n \leqq 2g_1^*(s_{n+2}) \\ &\leqq 2M(s_{n+2}, g_1^*) < 2(s_{n+3} - s_{n+2})^{-1/2} M(s_{n+3}, g_1) \\ &< 2(s_{n+3} - s_{n+2})^{-1/2} \exp\{2(s_{n+4} - s_{n+3})^{-1} T(s_{n+4}, g_1)\} \\ &< 2(s_{n+3} - s_{n+2})^{-1/2} \exp\{A(s_{n+4} - s_{n+3})^{-(p+1)} \\ &\quad \times [T(c(s_{n+4}), f) - \log(1 - s_{n+4})]\} , \end{aligned}$$

where p is a positive integer and we have used Lemma 1.2 and a well known relationship between $\log^+ M(r, f)$ and $T(r, f)$, see [4, p. 18]. Identical statements can be made for $v_{1,r}(\theta)$, $u_{2,r}(\theta)$ and $v_{2,r}(\theta)$ and the lemma follows.

Now choose a positive integer q so that

$$\frac{1}{2} \log(s_{n+2}/s_{n+1}) \leqq \pi(2q)^{-1} < \log(s_{n+2}/s_{n+1}) ,$$

which can always be done provided r_0 is sufficiently large. If $U_1 = \{\theta: |\operatorname{Im} \theta| < \pi(2q)^{-1}\}$, then $f_1(z) = e^z$ is a one-to-one transformation of U_1 onto $U_2 = \{\theta \neq 0: |\arg \theta| < \pi(2q)^{-1}\}$, and $f_2(z) = z^q$ is a one-to-one transformation of U_2 onto $U_3 = \{\theta \neq 0: |\arg \theta| < \pi/2\}$. Also, $f_3(z) = (z - e^{\theta_0 q})/(z + e^{\theta_0 q})$ is a one-to-one transformation of U_3 onto the unit disk, satisfying $f_3(e^{\theta_0 q}) = 0$, where θ_0 is as in Lemma 3.1. If we let $L^{-1}(z) = f_3(f_2(f_1(z)))$, then $L(z)$ is a one-to-one transformation of the unit disk onto U_1, satisfying $L(0) = \theta_0$. We let $p(q) = (e^{\pi q} - 1)/(e^{\pi q} + 1)$. Elementary calculations show that L maps $\{|w| < p(q)\}$ onto a region in U_1 containing the interval $[\theta_0 - \pi, \theta_0 + \pi]$ on the real θ-axis. We will use L to prove

LEMMA 3.3. *If* $r \in [s_n, s_{n+1}) - E(s_n, \eta)$, *then*

$$\phi(r, f) < \exp\{A(s_{n+2} - s_{n+1})^{-1}\}[T(c(s_{n+4}), f) - \log(1 - s_{n+4})] \log \frac{1}{\eta}$$

provided $r > R$.

Proof. We let $n_r(t)$ be the number of zeros of $J_r(L(\omega))$ in $|\omega| \leqq t$. Since $J_r(L(\omega))$ is holomorphic in $|\omega| < 1$, we apply Jensen's theorem to $J_r \circ L$ to obtain

$$(3.7) \qquad \int_0^t n_r(x)x^{-1}dx = -\log|J_r(L(0))| + \frac{1}{2\pi}\int_0^{2\pi} \log|J_r(L(te^{i\zeta}))|d\zeta .$$

For $t > p(q)$ we have

$$(3.8) \qquad \int_0^t n_r(x)x^{-1}dx > n_r(p(q)) \log(t(p(q))^{-1}) .$$

We note that $-\log p(q) > \exp(-\pi q)$ for sufficiently large q, and q will be large enough if s_n (or, equivalently, r_0) is large enough. Also, from the definition of q, we have $\exp(\pi q) < \exp(A(s_{n+2} - s_{n+1})^{-1})$. This observation together with (3.7), (3.8), Lemma 3.1 and Lemma 3.2 yield, upon letting t approach 1,

$$\begin{aligned} n_r(p(q)) &< [\log(t(p(q))^{-1})]^{-1} \int_0^t n_r(x)x^{-1}dx \\ &= [\log(t(p(q))^{-1})]^{-1} \left\{-\log|J_r(\theta_0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|J_r(L(te^{i\zeta}))|d\zeta\right\} \\ &< \exp(A(s_{n+2} - s_{n+1})^{-1})\Big\{A(s_{n+2} - s_{n+1})^{-2}[T(c(s_{n+2}), f) \\ &\quad - \log(1 - s_{n+2})] \log\frac{1}{\eta} - \frac{1}{2}\log(s_{n+3} - s_{n+4}) \\ &\quad + A(s_{n+4} - s_{n+3})^{-(p+1)}[T(c(s_{n+4}), f) - \log(1 - s_{n+4})]\Big\} \\ &< \exp(A(s_{n+2} - s_{n+1})^{-1})[T(c(s_{n+4}), f) - \log(1 - s_{n+4})] \log\frac{1}{\eta} . \end{aligned}$$

Since the zeros of $J_r(L(\omega))$ in $|\omega| < p(q)$ include the zeros of $\operatorname{Re}(re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta}) + 1)$ in the interval $[\theta_0 - \pi, \theta_0 + \pi]$, the lemma follows from Lemma 1.4.

Let A_0 be the constant in Lemma 3.3, and let $\delta_n = \exp(-3T(c(s_{n+4}), f) - 4A_0(s_{n+2} - s_{n+1})^{-1})$. Define $E = \bigcup_{n=0}^{\infty} E(s_n, \delta_n)$, where s_n and $E(s_n, \delta_n)$ are defined by (3.3) and (3.5), respectively. Let Δ' be the set in Lemma 1.5 corresponding to $\alpha_2 = \gamma^2$ and $k(r) = B(1-r)^{-1}$ with B a sufficiently large constant to be specified in (3.12) below. Finally, let $P_1 = [0, r_0]$, $P_2 = \Delta' \cap E$, and $P_3 = (\Delta' - E) \cap [r_0, 1)$. We will bound

$$\int_{P_j} \phi(t, f)(1-t)^{-1}dt \quad \text{for} \quad j = 1, 2, 3 .$$

If $D(n) = \{r < s_{n+2}\colon g_2 \text{ has a zero of modulus } r\}$, and if $r_1 \in D(n)$ then by Lemma 3.3, for $s_n > R$

$$(3.9)\quad \int_{\max(r_1-\delta_n, s_n)}^{\min(r_1+\delta_n, s_{n+1})} \phi(t, f)(1-t)^{-1}dt$$
$$< \exp\{A_0(s_{n+2}-s_{n+1})^{-1}\}(T(c(s_{n+4}), f) - \log(1-s_{n+4}))$$
$$\times \int_{r_1-\delta_\eta}^{r_1+\delta_\eta} (-\log|t-r_1|)dt$$
$$< 2\exp\{A_0(s_{n+2}-s_{n+1})^{-1}\}(T(c(s_{n+4}), f) - \log(1-s_{n+4}))(\delta_n - \delta_n \log \delta_n)$$
$$< \exp\{-2T(c(s_{n+4}), f) - 2A_0(s_{n+2}-s_{n+1})^{-1}\} .$$

Since $E(s_n, \delta_n) \subset \bigcup_{r \in D(n)} (r - \delta_\eta, r + \delta_\eta) \cup \{r: f$ has a zero of modulus $r\}$, and g_2 has no more than $n(s_{n+2}, g_2)$ zeros in $|z| < s_{n+2}$, we have from Lemma 1.6 and (3.9) for $r > R$

$$\int_{E(s_n, \delta_n)} \phi(t, f)(1-t)^{-1}dt$$
$$< \exp\{-2T(c(s_{n+4}), f) - 2A_0(s_{n+2}-s_{n+1})^{-1}\} n(s_{n+2}, g)$$
$$< 1 - s_n = \gamma^{n/4}(1-r_0) .$$

Since $E = \bigcup_{n=0}^{\infty} E(s_n, \delta_n)$, an elementary calculation shows

$$(3.10)\qquad \int_{P_2} \phi(t, f)(1-t)^{-1}dt < \infty .$$

It follows from [10, paragraph after (2.16)] that

$$(3.11)\qquad \int_{P_1} \phi(t, f)(1-t)^{-1}dt < \infty .$$

If $r \in (\varDelta' - E) \cap [s_n, s_{n+1})$, then from Lemma 3.3, for $r_0 > R$

$$(3.12)\quad \phi(r, f)(1-r)^{-1}$$
$$< (1-r)^{-1}\exp\{A_0(s_{n+2}-s_{n+1})^{-1}\}[T(c(s_{n+4}), f) - \log(1-s_{n+4})]$$
$$\times [3T(c(s_{n+4}), f) + 4A_0(s_{n+2}-s_{n+1})^{-1}]$$
$$< \exp\{2A_0(s_{n+2}-s_{n+1})^{-1}\}T^2(c(s_{n+4}), f)$$
$$< \exp\{B(1-r)^{-1}\}T^2(c(c_0^4(r)), f)$$
$$= \exp\{B(1-r)^{-1}\}T^2(c^2(r), f)$$
$$< \exp\{T(c^2(r), f) + B(1-r)^{-1}\} ,$$

where B is a constant and we have used the fact that $c_0^4(r) = c(r)$. Thus, by Lemma 1.5 we have

$$(3.13)\qquad \int_{P_3} \phi(t, f)(1-t)^{-1}dt < \infty .$$

Finally, we note that the proof of part (i) of the theorem may be altered using Lemma 1.5 with $\varDelta'$ corresponding to $k(r) = B(1-r)^{-1}$ (B as in (3.12)) and $\alpha_2 = \gamma^2$ to yield that for $r \notin \varDelta'$ and $r > R$

$$(3.14)\qquad \phi(r, f) < A(1-r)^{-1}[T(c^2(r), f) + (1-r)^{-1}] .$$

From (3.10), (3.11), (3.13) and (3.14) we conclude for $r > r_0$,

$$\begin{aligned}\int_0^r \phi(t, f)(1-t)^{-1}dt &< \int_0^r A(1-t)^{-2}[T(c^2(t), f) + (1-t)^{-1}]dt + O(1) \\ &< A[T(c^2(r), f) + (1-r)^{-1}]((1-r)^{-1} - 1) + O(1) \\ &< A(1-r)^{-1}[T(c^2(r), f) + (1-r)^{-1}] .\end{aligned}$$

The proof of part (ii) of the theorem follows by letting $\alpha_1 = \gamma^2$.

4. **Examples.** *We first give an example to show that* $\phi(r, f)$ *may equal* $O(1)$, *and that* $\Phi(r, f)$ *may equal* $O(-\log(1-r))$, *for functions of arbitrarily large order.* For $\lambda > 0$, let

$$f(z) = \exp\{((1+z)/(1-z))^\lambda\} ,$$

where the branch is chosen so that $f(0) = e$. Note that $|f(z)| = 1$ implies $\operatorname{Re}\{((1+z)/(1-z))^\lambda\} = 0$. Since $(1+z)/(1-z)$ takes $|z| = r$ onto a circle in the right half plane, $|\arg((1+z)/(1-z))^\lambda| < \pi\lambda/2$. Also, for $k = 0, \pm 1, \pm 2, \cdots, \pm[\lambda/2], -[\lambda/2]-1$, $\arg((1+z)/(1-z))^\lambda = (k+1/2)\pi$ if and only if $\arg((1+z)/(1-z)) = 1/\lambda(k+1/2)\pi$. For each such k, the latter equality holds at most twice on $|z| = r$. Thus, $|f(z)| = 1$ at no more than $4([\lambda/2]+1) \leqq 2\lambda + 4$ points on $|z| = r$. If $L(z)$ is a linear fractional transformation taking $|z| = 1$ onto the imaginary axis, and if $g(z) = L(f(z))$, then $\phi(r, g) \leqq 2\lambda + 4$ and $\Phi(r, g) \leqq (2\lambda + 4)\log(1-r)^{-1}$. The order of g can be made arbitrarily large by choosing λ sufficiently large.

Now we give an example to show that the factor $(1-r)^{-1}$ *in* (i) *and* (ii) *of the theorem cannot be replaced by any function* $b(r)$ *satisfying* $b(r) = o((1-r)^{-1})$. We use the Lindelöf functions. If q is a positive integer and $q \leqq \lambda \leqq q+1$, then we let

$$f(z, \lambda) = \prod_{k=1}^{\infty}(1 - za_n^{-1})\exp\left\{(za_n^{-1}) + \frac{1}{2}(za_n^{-1})^2 + \cdots + \frac{1}{q}(za_n^{-1})^q\right\} ,$$

where $a_n = n^{1/\lambda}$. It is known [11, p. 18] that $f(z, \lambda)$ has order λ and mean type 1. Thus, for $\varepsilon > 0$ and $|z| > R(\varepsilon)$, we have

$$\log|f(z, \lambda)| < (1+\varepsilon)|z|^\lambda . \tag{4.1}$$

We let $g(z, \lambda) = f((1+z)/(1-z), \lambda)$. Thus, for $|(1+z)/(1-z)| > R(\varepsilon)$, (4.1) implies

$$\log|g(z, \lambda)| < (1+\varepsilon)|(1+z)/(1-z)|^\lambda . \tag{4.2}$$

Also, there is a constant $K(\varepsilon)$ so that, if $|(1+z)/(1-z)| \leqq R(\varepsilon)$, then

$$\log|g(z, \lambda)| < K(\varepsilon) . \tag{4.3}$$

Since $(1+\varepsilon)(|1+re^{i\theta}|/|1-re^{i\theta}|)^\lambda = (1+\varepsilon)|1+re^{i\theta}|^\lambda(|1-re^{i\theta}|^2)^{-\lambda/2} \leqq$

$(1+\varepsilon)2^{\lambda}(1-2r\cos\theta+r^2)^{-\lambda/2}$, we have from (4.2) and (4.3)

$$(4.4)\qquad m(r,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}|g(re^{i\theta})|d\theta \leqq \frac{2^{\lambda}(1+\varepsilon)}{2\pi}\int_{-\pi}^{\pi}(1-2r\cos\theta+r^2)^{-(\lambda/2)}d\theta+K(\varepsilon)\ .$$

By [2, p. 65], the latter integral in (4.4) equals $O((1-r)^{-(\lambda-1)})$. Thus

$$(4.5)\qquad T(r,g)=m(r,g)=O((1-r)^{-(\lambda-1)})\ .$$

Since the image of $|z|\leqq r$ under $(1+z)/(1-z)$ contains the interval $[(1-r)/(1+r),(1+r)/(1-r)]$ on the real θ-axis, we have $n(r,1/g)\geqq(1-r)^{-\lambda}$, for $r>R$. By the argument principle, if $f(z)\neq 0$ on $|z|=r$ and $r<R$, then

$$(4.6)\qquad \phi(r,g)\geqq 2(1-r)^{-\lambda}\ .$$

From (4.5) and (4.6), it follows that if $f(z)\neq 0$ on $|z|=r$ and if $r>R$,

$$(1-r)^{-1}T((1-\beta)+\beta r,g)=O[(1-r)^{-1}(1-((1-\beta)+\beta r))^{-(\lambda-1)}] = O[\beta^{-(\lambda-1)}(1-r)^{-\lambda}]\leqq A\phi(r,g)\ .$$

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