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INVARIANT MANIFOLDS FOR REGULAR POINTS

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In this article we prove, for a differentiable vector field or a diffeomorphism on a smooth manifold, that the set of points such that the semitrajectories issuing from them approach a particular semitrajectory at a given exponential rate, constitute a differentiable submanifold, provided the differential of the flow has a certain similar behavior on that trajectory. (See Theorem 1 below, for a precise statement). In particular, the stable manifold theorem for hyperbolic sets ([3], [6, XI]) follows as a corollary.

Although we only consider the C^1-case, the same methods, which are essentially classical ([2, Ch. XIII]), could be applied to obtain higher differentiability properties.

Since I have not seen in the literature this type of results for points which are neither equillibrium nor periodic points, and on account of [6, XI-8], I thought that their publication might not be entirely devoid of interest.

1. Terminology and notation are standard. If X is a differentiable vector field on a smooth manifold M, ϕ will always denote the corresponding flow, and ϕ_t the diffeomorphism $x \to \phi(x, t)$, $x \in M$, $t \in R$. For brevity, we shall sometimes write $x(t)$ or $y(t)$ instead of $\phi(x, t)$ or $\phi(y, t)$ respectively.

THEOREM 1. *Let M be compact smooth (C^∞) Riemannian manifold and X a C^1-vector field. Assume that for some $x \in M$, there are subspaces E, I; $E \oplus I = T_xM$, such that for some positive mumbers K, λ, μ, $\mu < \lambda$, we have*

$$(1) \qquad \|\phi_s'(x(t))e_t\| < Ke^{-\lambda s}\|e_t\| \quad \textit{for} \quad e_t \in \phi_t'(x)E,\ s,\ t > 0\ ,$$

and

$$(2) \qquad \|\phi_{-s}(x(t))i_t\| < Ke^{\mu s}\|i_t\| \quad \textit{for} \quad i_t \in \phi_t'(x)I,\ 0 < s < t\ .$$

Then, $W_\lambda(x) = \{y \in M / \overline{\lim}\,(1/t) \log \operatorname{dist}(\phi(y, t), \phi(x, t)) < -\lambda\}$ is a C^1-submanifold of M, such that $T_xW_\lambda(x) = E$.

Condition (1) means that ϕ_t' strongly contracts the bundle $\bigcup_{t>0} \phi_t'(x)E$, while (2), which is equivalent to

$$(2') \qquad \|\phi_s'(x(t))i_t\| \geqq He^{-\mu s}\|i_t\|\ , \quad t,\ s > 0$$

for some $H > 0$, only says that ϕ_t' does not contract as strongly on $\bigcup_{t>0} \phi_t'(x)I$.

The following theorem may be proved applying Theorem 1 to the suspension of M. (See [1], Ch. 1.)

THEOREM 2. *Let M be a compact Riemannian smooth manifold and f a C^1-diffeomorphism of M. Assume that there exists a point $x \in M$ and subspaces E_x, I_x, $E_x \oplus I_x = T_x M$ such that for some positive numbers K, p, q, $p < q < 1$, we have*

(1) $\|f^{m\prime}(f^n(x))e_n\| < Kp^n \|e_n\|$, *for* $e_n \in f^{n\prime}(x)E_x$, $m, n > 0$.

(2) $\|f^{-m\prime}(f^n(x))i_n\| < Kq^{-m} \|i_n\|$, *for* $i_n \in f^{n\prime}(x)I_x$, $0 < m < n$.

Then $W_p(x) = \{y \in M / \overline{\lim}_{n\to\infty} (1/n) \log \operatorname{dist}(f^n(y), f^n(x)) < -\log p\}$ is a C^1-submanifold of M, such that $T_x(W_p(x)) = E_x$.

Proof that Theorem 1 *implies Theorem* 2: Consider the suspension $\hat{M}$ of M, equipped with some Riemannian metric, and the corresponding vector field X. (We shall identify M and $\pi(M \times \{0\})$, π being the canonical projection of $M \times R$ onto $\hat{M}$).

Since $X \neq 0$ on M, Theorem 1 may be applied to the semitrajectory $\phi(x, t)$, $t > 0$, taking E_x as E, the subspace spanned by I_x and $X(x)$ as I, and letting $-\log p$, $-\log q$ play, respectively, the roles of λ *and* μ. In this way, we get a C^1-submanifold $W_\lambda(x)$ of M; but if $y = \pi(y, s)$, and s is not an integer, $\operatorname{dist}(\phi(y, t), \phi(x, t))$ is bounded away from zero for $t > 0$. Thus, $W_\lambda(x) \subset M$, and this clearly implies $W_\lambda(x) = W_p(x)$. Since $T_x W_\lambda(x) = E_x$, this completes the proof.

If x lies on a hyperbolic set ([3], [6]), its stable and unstable manifold may be obtained by a direct application of Theorem 2 (Theorem 1, if we were dealing with a vector field) to the diffeomorphisms f and f^{-1}.

2. The results of this section will enable us to replace the manifold M by an open subset of Euclidean space.

Let M be a compact connected smooth submanifold of R^N and let r be the retraction $x \to r(x)$, where $r(x)$ is a point of M with the property

$$\|x - r(x)\| = \operatorname{dist}(x, M).$$

If the domain of r is restricted to a suitable neighborhood Ω of M, then r becomes a well defined smooth function (see [3]), such that $r(x) - x$ is orthogonal to M for each $x \in \Omega$. Since for $x \in \Omega$, $r'(x): R^N \to R^N$ is of maximal rank $n = \dim M$, and $r'(x)v = 0$ if v is orthogonal to $T_{r(x)}M$, we have that for each $u \in T_{r(x)}M$ there is exactly one vector $v \in T_{r(x)}M$ such that $r'(x)v = u$.

If X is a vector field on M we may define a vector field Y on Ω by letting $Y(x)$ be the unique vector of $T_{r(x)}M$ such that $r'(x)Y(x) = X(r(x))$. If $X \in C^r$, $r > 1$, then, clearly, $Y \in C^r$; also $Y/M = X$.

LEMMA 3. *Let a be a real number and Z^a the vector field defined on Ω,*

$$Z^a = a(r(x) - x) + Y .$$

Then, the normal bundle $N(M)$ of M is invariant under the flow ϕ^a determined by Z^a and

$$\| \phi_t^{a\prime}(x)\nu \| = e^{-at} \| v \|$$

for every $x \in M$ and $v \in N_x(M)$.

Proof. The invariance of $N(M)$ follows from the following relation:

$$r'(x)Z^a(x) = r'(x)Y(x) = X(r(x)) = Z^a(r(x)) ,$$

which clearly implies that $r(\phi_t^{a\prime}(x)) = \phi_t^a(r(x))$ for $x \in \Omega$.

The assertion concerning the norm of $\phi_t^{a\prime}$ is a consequence of the following equalities, where we have written (,) for the inner product in R^N:

$$\begin{aligned} Z^a(\| r(x) - x \|^2) &= 2((r(x) - x), (r'(x)Z^a(x) - Z^a(x))) \\ &= 2((r(x) - x), (Z^a(r(x)) - Z^a(x))) \\ &= 2((r(x) - x), X(r(x)) - Y(x) - a(r(x) - x)) . \end{aligned}$$

Since $((r(x)-x), X(r(x)) - Y(x)) = 0$, we have that $Z^a(\| r(x) - x \|^2) = -2a \| r(x) - x \|^2$. Therefore,

$$\| \phi^a(x, t) - \phi^a(r(x), t) \| = e^{-at} \| x - r(x) \| ,$$

which clearly implies the thesis.

Consider now a C^1-vector field X on an open connected subset Ω of R^n, and a semitrajectory $\{\phi(x, t), t > 0\}$ of X, whose closure is compact and contained in Ω. Theorem 1 is a consequence of the following proposition.

PROPOSITION 4. *Assume that there are subspaces E_0, I_0, $E_0 \oplus I_0 = R^n$, such that, writing $E_t(I_t)$ for $\phi_t'(x)E_0$* (resp. $\phi_t'(x)I_0$), *we have*

(1) $\quad \| \phi_s'(x(t))e_t \| < Ke^{-\lambda s} \| e_t \|$, *for* $e_t \in E_t$, $t > 0$, $s > 0$,

(2) $\quad \| \phi_{-s}'(x(t))i_t \| < Ke^{\mu s} \| i_t \|$, *for* $i_t \in I_t$, $0 < s < t$,

for some positive numbers, $K, \lambda, \mu, \mu < \lambda$.

Then $W_\lambda(x) = \{y \in \Omega / \overline{\lim}_{t\to\infty} (1/t) \log\|\phi(y, t) - \phi(x, t)\| < -\lambda\}$ *is a* C^1*-submanifold of* R^n *tangent to* E_0 *at* x.

Proof that Proposition 4 *implies Theorem* 1. We may assume that M is embedded in, say, R^n. Extend the vector field X to a neighborhood Ω of M as in the previous lemma, choosing $a > \lambda$. Let E_0 be the subspace spanned by E and $N_x(M)$ and take $I_0 = I$; we may now apply Proposition 4 to get a C^1-submanifold $W'_\lambda(x)$ of R^n. Then, $W_\lambda(x) = r(W'_\lambda(x))$, is a manifold (see [4], Lemma 3) and since $r'(x)E_0 = E$, the proof is complete.

3. In this section we prove two preliminary results.

Consider, as before, a C^1-vector field X on an open connected subset $\Omega \subset R^n$, and a semitrajectory $\{\phi(x, t), t > 0\}$ whose compact closure is included in Ω. Let $E_t, I_t, t > 0$ be as in Proposition 4, and call $P_t(Q_t)$ the projection of R^n onto $E_t(I_t)$ along I_t(resp. E_t).

LEMMA 5. *There is a positive number* M, *such that* $\|P_t\| < M$, $\|Q_t\| < M, t > 0$.

Proof. Suppose that $\|P_t\|$ is not bounded for $t > 0$. Then we may find a sequence $t_n \to \infty$ and vectors $e_{t_n} \in E_{t_n}$, $i_{t_n} \in I_{t_n}$, $n = 1, 2, \cdots$ such that $\|e_{t_n}\| \to \infty$ and $\|e_{t_n} + i_{t_n}\| = 1$. Moreover, we may assume that $\phi(x, t_n)$ converges to $y \in \Omega$, and that $(e_{t_n}/\|e_{t_n}\|)$ converges to some unit vector $u \in R^n$. Since $(-i_{t_n}/\|i_{t_n}\|)$ must also converge to u, we have that for $t > 0$, $\|\phi'_t(y)u\| < Ke^{-\lambda t}$ and $\|\phi'_t(y)u\| > He^{-\mu t}$ (see 2') in §2) which is absurd. Inasmuch as $P_t + Q_t = Id, t > 0$, this completes the proof.

The following technical lemma will be useful.

LEMMA 6. *Assume that* $\phi(y, t)$ *is defined in* $[0, b)$. *Then, for* $0 < t < b$, *we have*

$$\phi(y, t) - \phi(x, t) = \phi'_t(x)(y - x) + \int_0^t \phi'_{t-s}(x(s))\Delta(x(s), y(s))ds ,$$

where $\Delta(x, y) = X(y) - X(x) - J(x)(y - x)$.

Proof. From

$$\begin{aligned}\frac{d}{dt}(\phi(y, t) - \phi(x, t)) &= X(\phi(y, t)) - X(\phi(x, t)) \\ &= J(x(t))(y(t) - x(t)) + \Delta(x(t), y(t)) ,\end{aligned}$$

we get

$$\phi'_{-t}(x(t))\frac{d}{dt}(y(t) - x(t)) - \phi'_{-t}(x(t))J(x(t))(y(t) - x(t))$$
$$= \phi'_{-t}(x(t))\Delta(x(t), y(t)) ,$$

which implies

$$\frac{d}{dt}(\phi'_{-t}(x(t))(y(t) - x(t))) = \phi'_{-t}(x(t))\Delta(x(t), y(t))$$

since $\phi'_{-t}(x(t))\cdot\phi'_t(x) = Id$ and $(d/dt)\phi'_t(x) = J(x(t))\phi'_t(x)$ ([2], Ch. I). By integration we find

$$\phi'_{-t}(x(t))(y(t) - x(t)) = (y - x) + \int_0^t \phi'_{-s}(x(s))\Delta(x(s), y(s))ds$$

and applying $\phi'_t(x)$ on the left we obtain the thesis of the lemma.

4. LEMMA 7. *Assume that $y(t), t > 0$, is a semitrajectory of X such that $\|y(t) - x(t)\| < \alpha e^{-\gamma t}$, where $\alpha > 0$ and $\mu < \gamma < \lambda$. Then $y(t)$ satisfies the integral equation*

$$y(t) = x(t) + \phi'_t(x)P_0(y - x) + \int_0^t \phi'_{t-s}P_s\Delta(x(s), (s))ds$$
$$- \int_t^\infty \phi'_{t-s}(x(s))Q_s\Delta(x(s), y(s))ds .$$

Proof. From Lemma 6 we get

$$y(t) - x(t) = \phi'_t(x)P_0(y - x) + \int_0^t \phi'_{t-s}(x(s))P_s\Delta(x(s), y(s))ds$$
$$+ \phi'_t(x)(Q_0(y - x) + \int_0^t \phi'_{t-s}(x(s))Q_s\Delta(x(s), y(s))ds .$$

Since for large s,

$$X(y(s)) - X(x(s)) = \int_0^1 J((1 - u)x(s) + uy(s))du(y(s) - x(s)) ,$$

we have that $\|\Delta(x(s), y(s))\| < c\|y(s) - x(s)\|$ for some $c > 0$; if c is taken large enough, the same inequality holds for all $s > 0$. Then, from the above formula we obtain, on account of (1), that

$$\left\|\phi'_t(x)(Q_0(y - x)) + \int_0^t \phi'_{-s}(x(s))Q_s\Delta(x(s), y(s))ds\right\|e^{\gamma t}$$
$$< \alpha + KMe^{-(\lambda-\gamma)t}\|y - x\| + KMc\alpha e^{\gamma t}\int_0^t e^{-\lambda(t-s)}e^{-\gamma s}ds ,$$

which is bounded for $t > 0$. By (2′) this implies the boundedness, for $t > 0$, of

$$\left\| Q_0(y - x) + \int_0^t \phi'_{-s}(x(s)) Q_s \Delta(x(s), y(s)) ds \right\| e^{(\gamma-\mu)t} .$$

Thus, $Q_0(y - x) = -\int_0^\infty \phi'_{-s}(x(s)) Q_s \Delta(x(s), y(s)) ds$ as we had to show.

On the other hand it is important to notice that if $y(t)$, $t \geqq 0$ is a continuous function with values in Ω that satisfies the integral equation

$$y(t) = x(t) + \phi'_t(x) e_0 + \int_0^t \phi'_{t-s}(x(s)) P_s \Delta(x(s), y(s)) ds$$
$$- \int_t^\infty \phi'_{t-s}(x(s)) Q_s \Delta(x(s), y(s)) ds ,$$

$e_0 \in E_0$, then $y(t)$ is also a trajectory of X with $P_0(y(0) - x) = e_0$. In fact, since the differentiability of $y(t)$ follows by inspection of the right hand side of the equation, we may differentiate both sides to get

$$\dot{y}(t) = \dot{x}(t) + J(x(t))(y(t) - x(t)) + \Delta(x(t), y(t)) = X(y(t)) .$$

5. For each $\alpha > 0$, and γ, $\mu < \gamma < \lambda$, let $y_\alpha(\gamma)$ be the space of continuous functions $t \to y(t)$, $y(t) \in R^n$, $t \geqq 0$, such that $\| y(t) - x(t) \| < \alpha e^{-\gamma t}$. If $y, z \in y_\alpha(\gamma)$, let

$$d(y, z) = \sup_{t>0} \| y(t) - z(t) \| e^{\gamma t} ;$$

it is not difficult to check, that with d as the distance, $y_\alpha(\gamma)$ becomes a complete metric space.

Now for $e_0 \in E_0$, consider the operator $T_{e_0}: y \to z$, where $y \in y_\alpha(\gamma)$ and $z: [0, \infty) \to R^n$ is given by

$$z(t) = x(t) + \phi'_t(x) e_0 + \int_0^t \phi'_{t-s}(x(s)) P_s \Delta(x(s), y(s)) ds$$
$$- \int_t^\infty \phi'_{t-s}(x(s)) Q_s \Delta(x(s), y(s)) ds ;$$

the fact that $\gamma > \mu$ ensures the convergence of the improper integral.

Since for y close to x

$$\Delta(x, y) = \left(\int_0^1 (J(1 - u)x + uy) - J(x) du \right) (y - x) ,$$

the continuity of J implies that for each $\varepsilon > 0$, it is possible to choose $\alpha = \alpha(\varepsilon) > 0$, such that if $\| y - x \| < \alpha$,

$$\| \Delta(x, y) \| < \varepsilon \| y - x \| .$$

For a given γ, $\mu < \gamma < \lambda$, choose $\varepsilon = \varepsilon(\gamma)$ such that $\varepsilon KM((\lambda - \gamma)^{-1} + (\gamma - \mu)^{-1}) = 1/2$, and let $\alpha(\gamma)$ or simply α, be the corresponding $\alpha(\varepsilon(\gamma))$.

LEMMA 8. *For each $e_0 \in E_0$ with $\|e_0\| < \alpha/(2K)$, T_{e_0} is a contraction of $y_\alpha(\gamma)$.*

Proof. We first show that for those e_0, $T_{e_0}: y_\alpha(\gamma) \to y_\alpha(\gamma)$.

Let $t \to y(t)$ belong to $y_\alpha(\gamma)$, and let $z = T_{e_0}(y)$; then, by (1) and (2), we have, for $t > 0$,

$$\begin{aligned}\|z(t) - x(t)\| e^{\gamma t} &\leqq Ke^{-(\lambda-\gamma)t}\|e_0\| \\ &\quad + KM \in \alpha e^{-(\lambda-\gamma)t}\int_0^t e^{(\lambda-\gamma)s}ds + KM\varepsilon\alpha e^{(\gamma-\mu)t}\int_t^\infty e^{(\mu-\gamma)s}ds \\ &< K\|e_0\| + \alpha\varepsilon KM\Big(\frac{1}{\lambda-\gamma} + \frac{1}{\mu-\gamma}\Big) \leqq \alpha \,.\end{aligned}$$

On the other hand, if $y, \bar{y} \in y_\alpha(\gamma)$ and $z = T_{e_0}(y)$, $\bar{z} = T_{e_0}(\bar{y})$, we have that

$$\begin{aligned}\|\bar{z}(t) - z(t)\| e^{\gamma t} &\leqq KM\varepsilon e^{-(\lambda-\gamma)t}\int_0^t d(y, \bar{y})e^{(\lambda-\gamma)s}ds \\ &\quad + KM\varepsilon e^{(\gamma-\mu)t}\int_t^\infty d(y, \bar{y})e^{(\mu-\gamma)s}ds \,,\end{aligned}$$

for $t \geqq 0$, and consequently, $d(z, \bar{z}) < (1/2)d(y, \bar{y})$. This completes the proof.

Thus, if e_0 is small enough, there is one and only one fixed point $y(t, e_0)$ of T_{e_0} in $y_\alpha(\gamma)$, and on account of previous remarks, this fixed point is the unique semitrajectory of the vector field X, satisfying $P_0(y(0, e_0) - x) = e_0$ that belongs to $y_\alpha(\gamma)$.

Since the continuity in e_0 of $y(t, e_0)$ is an easy consequence of uniqueness, and $y(0, e_0) = y(0, e_0')$ implies readily $e_0 = e_0'$, we may state, letting $f = y(0, e_0)$:

COROLLARY 9. *Let $B_\alpha = \{e_0 \in E_0/\|e_0\| < \alpha/2K\}$. There is a continuous injective function $f: B_\alpha \to R^n$ with the following property: a semitrajectory of X, $\phi(y, t)$, $t \geqq 0$, satisfies*

$$\|\phi(y, t) - x(t)\| < \alpha e^{-\gamma t},\ t \geqq 0 \,, \quad \textit{and} \quad P_0(y - x) = e_0 \in B_\alpha \,,$$

if and only if, $y = f(e_0)$.

6. Now we study the differentiability properties of $f(e_0)$ or

$y(t, e_0)$. If the derivative of $y(t, e_0)$ in the direction of the unit vector $u \in E_0$ exists at e_0, and if we could differentiate under the integral sign, we would have that this derivative, $z_u(t, e_0)$, $\|e_0\| < \alpha/(2K)$, satisfies:

$$z_u(t, e_0) = \phi_t'(x)u + \int_0^t \phi_{t-s}'(x(s))P_s(J(y(s, e_0)) - J(x(s)))z_u(s, e_0)ds$$
$$- \int_t^\infty \phi_{t-s}'(x(s))Q_s(J(y(s, e_0)) - J(x(s)))z_u(s, e_0)ds \, .$$

Let V be the space of continuous functions$(t, e_0) \to z(t, e_0)$, $t > 0$, $\|e_0\| < \alpha/2K$, $z(t, e_0) \in R^n$, such that $\|z(t, e_0)\| < 2Ke^{-\gamma t}$. With the distance d,

$$d(z, \bar{z}) = \sup_{\substack{t>0 \\ \|e_0\|<\alpha/2K}} \|z(t, e_0) - \bar{z}(t, e_0)\| e^{\gamma t} \, ,$$

V is a complete metric space.

LEMMA 10. *For $z \in V$, define $T_u(z) = w$ by*

$$w(t, e_0) = \phi_t'(x)u + \int_0^t \phi_{t-s}'(x(s))P_s(J(y(s, e_0)) - J(x(s)))z(s, e_0)ds$$
$$- \int_t^\infty \phi_{t-s}'(x(s))Q_s(J(y(s, e_0)) - J(x(s)))z(s, e_0)ds \, .$$

Then, for each $u \in E_0$, $\|u\| = 1$, T_u is a contraction of V.

Proof. Since

$$\|w(t, e_0)\| \leqq Ke^{-\lambda t} + 2K^2M\varepsilon e^{-\lambda t}\int_0^t e^{(\lambda-\gamma)s}ds$$
$$+2K^2M\varepsilon e^{-\mu t}\int_t^\infty e^{(\mu-\gamma)s}ds$$
$$\leqq 2Ke^{-\gamma t} \, ,$$

T_u maps V into V. The fact that T_u is a contraction follows at once from the inequality

$$\|w(t, e_0) - \bar{w}(t, e_0)\| < KM\varepsilon e^{-\lambda t}\int_0^t e^{(\lambda-\gamma)s}d(z, \bar{z})ds$$
$$+ KM\varepsilon e^{-\mu t}\int_t^\infty e^{(\mu-\gamma)s}d(z, \bar{z})ds$$

and the choice of ε.

Now, for $h \neq 0$, consider the quotient

$$q_u(h, t, e_0) = \frac{1}{h}(y(t, e_0 + hu) - y(t, e_0))$$
$$= \phi_x'(t)u$$

$$+ \int_0^t \phi'_{t-s}(x(s))P_s \frac{1}{h}(X(y(s, e_0 + hu)) - X(y(s, e_0))$$
$$- J(x(s))q_u(h, s, e_0))ds$$
$$- \int_t^\infty \phi'_{t-s}(x(s))Q_s \frac{1}{h}(X(y(s, e_0 + hu)) - X(y(s, e_0))$$
$$- J(x(s))q_u(h, s, e_0))ds ,$$

and the difference

$$\begin{aligned}\delta_u(h, t, e_0) &= q_u(h, t, e_0) - z_u(t, e_0) \\ &= \int_0^t \phi'_{t-s}(x(s))P_s(J(y(s, e_0)) - J(x(s)))\delta_u(h, s, e_0)ds \\ &\quad + \int_0^t \phi'_{t-s}(x(s))P_s D_u(h, s, e_0)ds \\ &\quad - \int_t^\infty \phi'_{t-s}(x(s))Q_s(J(y(s, e_0)) - J(x(s)))\delta_u(h, s, e_0)ds \\ &\quad - \int_t^\infty \phi'_{t-s}(x(s))Q_s D_u(h, s, e_0)ds ,\end{aligned}$$

where

$$D_u(h, s, e_0) = \frac{1}{h}(X(y(s, e_0 + hu)) - X(y(s, e_0))) - J(y(s, e_0))q_u(h, s, e_0) .$$

Let $m(h) = \sup_{t>0} \| \delta_u(h, t, e_0) \| e^{\gamma t}$, $h \neq 0$; then, since

$$\| q(h) \| \leqq (m(h) + 2K)e^{-\gamma t} ,$$

from the last equation we get, on account of

$$\begin{aligned}&\| D_u(h, t, e_0) \| \\ &\leqq \left\| \int_0^1 J((1 - r)y(t, e_0) + ry(t, e_0 + hu))dr - J(y(t, e_0)) \right\| \| q_u(h, t, e) \| ,\end{aligned}$$

that

$$\begin{aligned}\| \delta_u(h, t, e_0) \| e^{\gamma t} \leqq \frac{KM\varepsilon m(h)}{\lambda - \gamma} &+ \frac{KM\rho(h)}{\lambda - \gamma}(m(h) + 2K) \\ &+ \frac{KM\varepsilon m(h)}{\gamma - \mu} + \frac{KM\rho(h)}{\gamma - \mu}(m(h) + 2K) ,\end{aligned}$$

where

$$\rho(h) = \sup_{t \geqq 0} \left\| \int_0^1 dr J((1 - r)y(t, e_0)) + r(y(t, e_0 + hu)) - J(y(t, e_0)) \right\| .$$

Because of the choice of ε, we may write the last inequality, as

$$\left(\frac{1}{2} - KM\left(\frac{1}{\lambda - \gamma} + \frac{1}{\gamma - \mu}\right)\rho(h)\right)m(h) \leqq 2K^2M\left(\frac{1}{\lambda - \gamma} + \frac{1}{\gamma - \mu}\right)\rho(h) .$$

Since $\lim_{h\to 0} \rho(h) = 0$, we get that $\lim_{h\to 0} m(h) = 0$.

This shows that *the derivative of* $y(t, e_0)$ *in the* u *direction is the continuous function* $z_u(t, e_0)$. In particular, it follows that f (see Corollary 9) is a C^1-function.

COROLLARY 11. *Let* $B_{\alpha,t_0} = \{e_{t_0} \in E_{t_0} / \|e_{t_0}\| \leqq \alpha/(2K)\}$. *For each* $t_0 \geqq 0$ *there is a continuously differentiable injective function* $f_{t_0}: B_{\alpha,t_0} \to R^n$ *with the following property: a semitrajectory of* X, $\phi(y, t)$, $t > 0$, *satisfies* $\|\phi(y, t) - x(t_0 + t)\| < \alpha e^{-\gamma t}$ *for* $t > 0$, *and* $P_{t_0}(y - x(t_0)) = e_{t_0} \in B_{\alpha,t_0}$, *if and only if*, $y = f_{t_0}(e_{t_0})$. *Furthermore*, $f'_{t_0}(0)u = u$, $u \in E_{t_0}$.

Proof. It is clear that we would have obtained the same results if we had started from any semitrajectory $\phi(x(t_0), t)$, $t \geqq 0$, $t_0 \geqq 0$. Moreover, it is easy to check that, for a fixed γ, the constants $\varepsilon(\gamma)$ and $\alpha(\gamma)$ that we have chosen for the semitrajectory $x(t)$, $t \geqq 0$, are also adequate for the semitrajectories $\phi(x(t_0), t)$, $t \geqq 0$, $t_0 \geqq 0$. So, with the exception of the last one, all the assertions of the corollary are a consequence of previous arguments. The last statement follows by inspection of the integral equation satisfied by $z_u(t, e_{t_0})$ in the case $e_{t_0} = 0$.

7. LEMMA 12. *Assume that for some* $L > 0$ *and some* γ, $\mu < \gamma < \lambda$, $\|\phi(y, t) - x(t)\| \leqq Le^{-\gamma t}$, $t \geqq 0$. *Then* $y \in W_\lambda(x)$.

Proof. Let γ' be a number greater than γ and less than, but close enough to λ. We may assume that $\alpha(\gamma') < \alpha(\gamma)$; take $t_0 > 0$ such that

$$Le^{-\gamma t_0} < \alpha(\gamma') ; \quad Le^{-\gamma t_0} < \frac{M\alpha(\gamma')}{2K}$$

and observe that as a consequence of the last inequality, there is a point $z \in R^n$, such that

$$\|\phi(z, t) - x(t_0 + t)\| < \alpha(\gamma')e^{-\gamma' t} ,$$

for $t \geqq 0$ and $P_{t_0}(z - x(t_0)) = P_{t_0}(\phi(y, t_0) - x(t_0))$.

As both, $\|\phi(z, t) - x(t_0 + t)\|$ and $\|\phi(y, t_0 + t) - x(t_0 + t)\|$ are less than $\alpha(\gamma)e^{-\gamma t}$ we must have $\phi(z, t) = \phi(\phi(y, t_0), t)$ for $t \geqq 0$, which implies $\|\phi(y, t) - x(t)\| \leqq Ne^{-\gamma' t}$, $t \geqq 0$, for some $N > 0$.

Since γ' may be chosen arbitrarily close to λ, this completes the proof.

Proof of Proposition 4. Let $y \in W_\lambda(x)$; we have that for some $L > 0$, and some γ, $\mu < \gamma < \lambda$, $\|\phi(y, t) - x(t)\| \leqq Le^{-\gamma t}$, if $t \geqq 0$. Take a $t_0 > 0$ such that $Le^{-\gamma t_0} < \alpha(\gamma)$, $Le^{-\gamma t_0} < M(2K)^{-1}\alpha(\gamma)$. Then $\phi_{-t_0} \circ f_{t_0}$: $B_{\alpha, t_0} \to R^n$ is an injective C^1-function such that its range contains y and, by the previous lemma, it lies on $W_\lambda(x)$. Define the topology of $W_\lambda(x)$ making $\phi_{-t_0} \circ f_{t_0}$ to be a homeomorphism onto a neighborhood of y in $W_\lambda(x)$. The C^1-compatibility of the atlas constructed in this way is a consequence of Corollary 11 and the differentiability properties of the flow. The assertion concerning the tangent space to $W_\lambda(x)$ at x also follows from the corollary.

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