PACIFIC JOURNAL OF MATHEMATICS
Vol. 96, No. 1, 1981

INTERSECTIONS OF TERMS OF POLYCENTRAL SERIES OF FREE GROUPS AND FREE LIE ALGEBRAS, II

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This paper investigates intersections of second terms of polycentral series of free groups and free Lie algebras and derives bases for the lower central factors of the resulting factors.

Let G_m denote the nth term of the lower central series a group G and L_m the ideal in a Lie Algebra L generated by of products of m elements. Define $G_{m,n} = (G_m)_n$ and $L_{m,n} = (L_m)_n$. Let F denote a free group and $\boldsymbol{L}$ a free Lie Algebra. This paper investigates $F_{m,n} \cap F_{p,q}$ and $\boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q}$ and the factors $F/(F_{m,n} \cap F_{p,q})$ and $\boldsymbol{L}/(\boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q})$. Bases for the lower central factors of $F/(F_{m,n} \cap F_{p,q})$ and for the additive group of $\boldsymbol{L}/(\boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q})$ are derived. This enables us to describe $\boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q}$ as a product of certain ideals in $\boldsymbol{L}$ and also to describe $F_{m,n} \cap F_{p,q}$ as a product of commutator subgroups of F. Some special cases of the bases have been communicated to me by M. Boral.

If $m = p$ then

$$\begin{aligned} F_{m,n} \cap F_{p,q} &= F_{m,n} \text{ for } n \geqq q \\ &= F_{p,q} \text{ for } q \geqq n . \end{aligned}$$

Assume without loss of generality that $m \geqq p$. Then $F_m \subseteqq F_p$ and if further $n \geqq q$ then $F_{m,n} \subseteqq F_{p,q}$ and hence $F_{m,n} \cap F_{p,q} = F_{m,n}$. Similar remarks of course apply for the Lie Algebra. So we shall assume in what follows that $m > p$ and $n < q$.

In [1] it is defined what is meant by saying that a basic group commutator is structurally contained in $F_{m,n}$; and what is meant by saying that a basic element is structually contained in $\boldsymbol{L}_{m,n}$ is similarly defined.

It follows from [1] Lemma 4 that the basic commutators structurally contained in both $F_{m,n}$ and in $F_{p,q}$ are contained in $F_{m,n} \cap F_{p,q}$. Let S_r denote the set of basic commutators of weight r structurally contained in both $F_{m,n}$ and in $F_{p,q}$ and let $T_r = B_r \backslash S_r$ where B_r denotes the totality of basic commutators of weight r. Define $T = \bigcup_1^\infty T_r$. We shall also use S_r, T_r, B_r and T for the corresponding Lie elements in $\boldsymbol{L}$.

THEOREM A.

(i) *The* rth *lower central factor of* $F/(F_{m,n} \cap F_{p,q})$ *is free abelian on the set* T_r.

(ii) T *is an additive basis for* $\boldsymbol{L}/(\boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q})$.

Proof. (i) It follows from [1] Lemma 4 that T_r generates $F_r(F_{m,n} \cap F_{p,q})/F_{r+1}(F_{m,n} \cap F_{p,q})$ and so it is only necessary to show that T_r is linearly independent modulo $F_{r+1}(F_{m,n} \cap F_{p,q})$. If $r+1 \leqq mn$ or $r+1 \leqq pq$ then this is clear. Hence suppose $r+1 = mn + s = pq + t$, for $s, t \geqq 1$.

Suppose a product Π of elements from T_r is contained in $F_{r+1}(F_{m,n} \cap F_{p,q})$. Then $\Pi = ab$, where $a \in F_{r+1}$ and $b \in F_{m,n} \cap F_{p,q}$. Then by Theorem C of [1] b is a product modulo $F(m, n; s)$ of basic commutators structurally contained in $F_{m,n}$ and also a product modulo $F(p, q; t)$ of basic commutators structurally contained in $F_{p,q}$. Of course $b \in F_r$. Now from the uniqueness modulo F_{r+1} of the expression for an element as a product of basic commutators, it follows that b is a product of basic commutators each of which is structurally contained in both $F_{m,n}$ and in $F_{p,q}$, i.e., b is a product of elements from S_r. However, $T_r \cap S_r = \Phi$ and thus Π induces the identity in $F_r(F_{m,n} \cap F_{p,q})/F_{r+1}(F_{m,n} \cap F_{p,q})$.

(ii) The proof for the Lie Algebra case is similar and is omitted. (It also follows from part (i) by setting up a homomorphism (which is consequently an isomorphism) from $\boldsymbol{L}/(\boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q})$ to the Lie Algebra formed from the direct sum of the lower central factors of $F/(F_{m,n} \cap F_{p,q})$.) □

Now define (for $m \geqq p$, $q > n$),

$$F(m, 1; p, q) = F_m \cap F_{p,q} ,$$
$$F(m, n; p, q) = \Pi[F_m \cap F_{p,i_1}, F_m \cap F_{p,i_2}, \cdots, F_m \cap F_{p,i_n}], \text{ for } n > 1 ,$$

where the product is over all positive integers $i_1, i_2, \cdots, i_n$ such that $i_1 + i_2 + \cdots + i_n = q$.

Note that it is easily verified that $[F_m \cap F_{p,j_1}, F_m \cap F_{p,j_2}, \cdots, F_m \cap F_{p,j_n}] \leqq F(m, n; p, q)$ for $j_1 + j_2 + \cdots + j_n \geqq q$ and hence in the definition we could have taken $i_1, i_2, \cdots, i_n$ such that $i_1 + i_2 + \cdots + i_n \geqq q$. In [1] Theorem A, $F_m \cap F_{p,s}$ is identified as a product of certain commutator subgroups of F. If $ps \geqq m$ then $F_m \cap F_{p,s} = F_{p,s}$; for $m = ps + r$ the notation $F(p, s; r)$ is used for the product of commutator subgroups which is identified with $F_m \cap F_{p,s}$. In [1] "structurally contained in $F_m \cap F_{p,s}$" has been defined and we wish to extend this definition to defining what is meant by saying that a basic commutator is structurally contained in $F(m, n; p, q)$. If $n = 1$, then $F(m, 1; p, q) = F_m \cap F_{p,q}$ and structurally contained in $F_m \cap F_{p,q}$ has been defined. Assume $n > 1$ and suppose it has been defined what is meant by saying that a basic commutator is structurally contained in $F(m, k; p, q)$ for all k, $1 \leqq k < n$, and for all m, p, q with $m \geqq p$. If $q \leqq n$ say a basic commutator a is structurally contained in $F(m, n; p, q)$ $(=F_{m,n})$ iff a is structurally contained in $F_{m,n}$. If $q > n$ (this forces $q \geqq 2$) say the basic commutator a is structurally contained in $F(m, n; p, q)$ iff $a = [b, c]$ for basic commutators b, c with b structurally contained in $F(m, n_1, p, q_1)$, c structurally contained in $F(m, n_2; p, q_2)$, for positive integers n_1, n_2, q_1, q_2 satisfying $n_1 + n_2 = n$ and $q_1 + q_2 = q$.

For a basic commutator a, use $a \,\overline{\in}\, F(m, n; p, q)$ to mean that a is structurally contained in $F(m, n; p, q)$.

LEMMA 1.
(i) $F(m, n; p, q) \leqq F_{m,n} \cap F_{p,q}$.
(ii) *If* $a \,\overline{\in}\, F(m, n; p, q)$ *then* $a \in F(m, n; p, q)$.

Proof. This follows easily from e.g., [1], Lemma 1.

PROPOSITION B. $a \,\overline{\in}\, F_{m,n}$ *and* $a \,\overline{\in}\, F_{p,q}$ *if and only if* $a \,\overline{\in}\, F(m, n; p, q)$.

Proof. If $a \,\overline{\in}\, F(m, n; p, q)$ then it follows easily from the definitions that $a \,\overline{\in}\, F_{m,n}$ and $a \,\overline{\in}\, F_{p,q}$.

Suppose on the other hand, $a \,\overline{\in}\, F_{m,n}$ and $a \,\overline{\in}\, F_{p,q}$. We can assume that $m > p$ and $n \leqq q$. If $p = 1$ or $n = 1$ there is nothing to be shown. Hence we can assume $p > 1$ and $n > 1$. Therefore $a = [b, c]$ with b, c basic commutators and satisfying

$$(1) \qquad b \,\overline{\in}\, F_{m,k_1},\ c \,\overline{\in}\, F_{m,k_2} \text{ with } k_1 + k_2 = n$$

$$(2) \qquad b \,\overline{\in}\, F_{p,j_1},\ c \,\overline{\in}\, F_{p,j_2} \text{ with } j_1 + j_2 = q$$

((1) follows since $n > 1$ and $a \,\overline{\in}\, F_{m,n}$. (2) follows since $q > 1$ and $a \,\overline{\in}\, F_{p,q}$.)

Hence, by induction, $b \,\overline{\in}\, F(m, k_1; p, j_1)$, $c \,\overline{\in}\, F(m, k_2; p, j_2)$ giving that $a \,\overline{\in}\, F(m, n; p, q)$. □

Let $a \in F_{m,n} \cap F_{p,q}$. Then from Theorem A (i), for any r, a is a product modulo F_{r+1} of basic commutators of weight $\leqq r$ each of which is structurally contained in both $F_{m,n}$ and in $F_{p,q}$. Hence for any r, $a = b_r c_r$ for $b_r \in F(m, n; p, q)$ and $c_r \in F_{r+1}$. Thus up to a residual part $a \in F(m, n; p, q)$. More specifically let $R = \bigcap_{i=1}^{\infty} F(m, n; p, q)F_i$ and then $a \in F(m, n; p, q)R$. Also $R \subseteqq F_{m,n} \cap F_{p,q}$, since both $F/F_{m,n}$ and $F/F_{p,q}$ are residually nilpotent. I have proved

PROPOSITION C.

$$F_{m,n} \cap F_{p,q} = F(m, n; p, q)R . \qquad \square$$

If it could be shown that $F/F(m, n; p, q)$ is residually nilpotent then it would follow that $\bigcap_i^{\infty} F(m, n; p, q)F_i = F(m, n; p, q)$. Then $F_{m,n} \cap F_{p,q}$ would be identified with $F(m, n; p, q)$.

This is no residual problem for the Lie Algebra case, and so I get the following proposition. ($L(m, n; p, q)$ is defined by analogy to the group case.)

PROPOSITION D.

$$\boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q} = \boldsymbol{L}(m, n; p, q) .$$

Proof. Suppose $a \in \boldsymbol{L}_{m,n} \cap \boldsymbol{L}_{p,q}$. Then as above we get that for any r, $a = b_r + c_r$ with $b_r \in \boldsymbol{L}(m, n; p, q)$ and $c_r \in \boldsymbol{L}_r$. If a_j denotes the jth homogeneous part of a then

$$a = a_0 + a_1 + \cdots + a_s \quad (s < \infty) .$$

Therefore $c_{s+1} = 0$ giving that $a \in \boldsymbol{L}(m, n; p, q)$. $\square$

REFERENCES

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Received November 15, 1978.

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