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ENTROPY OF AUTOMORPHISMS ON L.C.A. GROUPS

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In this paper we will consider entropy of automorphisms on locally compact abelian groups. Bowen's definition of entropy of a uniformly continuous mapping applies in particular to topological automorphisms of l.c.a. groups. If $h_B(\alpha,G)$ denotes the Bowen entropy of $\alpha \in \mathrm{Aut}(G)$, we investigate the appropriate dual notion $h_\infty(\hat{\alpha}, \hat{G})$ of the adjoint automorphism $\hat{\alpha}$ on the dual group $\hat{G}$, and show $h_B(\alpha, G) = h_\infty(\hat{\alpha}, \hat{G})$. We define the total entropy $h(\alpha, G)$ of α on G to be the sum $h_B(\alpha,G)+h_\infty(\alpha,G)$ and show that with this definition, $h(\alpha, G)$ coincides with Kolmogorov-Sinai entropy if G is compact and furthermore the invariance properties present in the compact case are retained for an arbitrary l.c.a. group G. We also obtain the addition theorem for entropy and a formula for the entropy on projective limits. In conclusion we mention some questions which arise.

0. Let G be a locally compact abelian group (l.c.a. group) and $\mathrm{Aut}(G)$ the group of topological automorphisms of G, i.e., those automorphisms which are also homeomorphisms of G. If Γ is the dual group of G and $\alpha \in \mathrm{Aut}(G)$, the adjoint automorphism $\hat{\alpha}$ is defined by $\hat{\alpha}(\tau(x)) = \tau(\alpha^{-1}(x))$, $\tau \in \Gamma$, $x \in G$. $\hat{\alpha}$ is in $\mathrm{Aut}(\Gamma)$, and in fact $\alpha \to \hat{\alpha}$ describes an antiisomorphism of $\mathrm{Aut}(G)$ to $\mathrm{Aut}(\Gamma)$. Consider the following properties which we would like an entropy function $h(\alpha, G)$ to possess:

(i) if G is compact, $h(\alpha, G)$ is the Kolmogorov-Sinai entropy of α with respect to haar measure; in general $h(\alpha, G) \geqq h_B(\alpha, G)$, the Bowen entropy.

(ii) $h(\alpha^k, G) = k \cdot h(\alpha, G)$, k a positive integer;

(iii) if $\alpha_1, \alpha_2 \in \mathrm{Aut}(G)$ are conjugate (i.e., there exists $\beta \in \mathrm{Aut}(G)$ with $\beta\alpha_1 = \alpha_2\beta$), then $h(\alpha_1, G) = h(\alpha_2, G)$;

(iv) if G_i is a l.c.a. group and $\alpha_i \in \mathrm{Aut}(G_i)$, $i = 1, 2$, then $h(\alpha_1 \times \alpha_2, G_1 \times G_2) = h(\alpha_1, G_1) + h(\alpha_2, G_2)$;

(v) $h(\alpha, G) = h(\alpha^{-1}, G)$;

(vi) $h(\alpha, G) = h(\hat{\alpha}, \Gamma)$;

(vii) $h(\theta, G) = 0$, θ the identity map.

1. Before continuing, let us observe that in the class of compactly generated abelian Lie groups, there is a unique smallest function satisfying (i) through (vii). For any such Lie group G can be written as $\boldsymbol{R}^n \times Z^m \times K$, K a compact Lie group, and so any $\alpha \in \mathrm{Aut}(G)$ has a corresponding decomposition as $\alpha_1 \times \alpha_2 \times \alpha_3$. By (iv),

$h(\alpha, G) = h(\alpha_1, \boldsymbol{R}^n) + (\alpha_2, Z^m) + h(\alpha_3, K)$. Now $h(\alpha_3, K)$ is determined by (i), and $h(\alpha_2, Z^m) = h(\hat{\alpha}_2, T^m)$ is determined by (vi) and (i). α_1 is conjugate in $\mathrm{Aut}(\boldsymbol{R}^n) = \mathrm{GL}(n, \boldsymbol{R})$ to a linear transformation $A = A_1 \times \cdots \times A_l$, where each A_i has at most two eigenvalues, $\lambda_i, \bar{\lambda}_i$ (not counting multiplicities). By (iii) and (iv) $h(\alpha_1, \boldsymbol{R}^n) = h(A, \boldsymbol{R}^n) = \sum_{i=1}^{l} h(A_i, R^{k_i})$, where k_i is the dimension of the subspace corresponding to A_i. Set $\varepsilon_i = \begin{cases} 1 & \text{if } |\lambda_i| \geqq 1 \\ -1 & \text{if } |\lambda_i| < 1 \end{cases}$. By (v), $h(A, \boldsymbol{R}^n) = \sum_{i=1}^{l} h(A^{\varepsilon_i}, \boldsymbol{R}^{k_i})$. Now the Bowen entropy $h_B(A_i^{\varepsilon_i}, \boldsymbol{R}^{k_i}) = k_i |\log|\lambda_i||$ [1; Theorem 15]. Thus by (i), $h(\alpha_1, \boldsymbol{R}^n) \geqq \sum_{i=1}^{l} k_i |\log|\lambda_i||$. In fact if we set $h(\alpha_1, \boldsymbol{R}^n) = \sum_{i=1}^{l} k_i |\log|\lambda_i||$, it is easy to check that (i)-(vii) are satisfied.

2. We now recall Bowen's definition of topological entropy, recast in the slightly more general context of uniform spaces, which has the advantage that we need not restrict ourselves to metric l.c.a. groups, without complicating the proofs. Let $(X, \mathscr{U})$ be a uniform space and $T: X \to X$ uniformly continuous. A set $E \subset X$ is (n, U)-separated $(U \in \mathscr{U})$ if for any distinct $x, y \in E$ there is a j such that $0 \leqq j < n$ and $(T^j(x), T^j(y)) \notin U$. A set F is said to (n, U)-span another set K (with respect to T) provided that for each $x \in K$ there is a $y \in F$ for which $(T^j(x), T^j(y)) \in U$ for all $0 \leqq j < n$.

For a compact set $K \subset X$ let $r_n(U, K)$ (where $U \in \mathscr{U}$) be the smallest cardinality of any (n, U) spanning set F for K (with repect to T) and let $s_n(U, K)$ denote the largest cardinality of any (n, U)-separated set E contained in K. Define

$$\bar{r}_T(U, K) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(U, K)$$

and

$$\bar{s}_T(U, K) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(U, K) .$$

It can be shown that

$$r_n(U, K) \leqq s_n(U, K) \leqq r_n(V, K) < \infty$$

if $V \circ V \subset U$; also if $U_1 \subset U_2$

$$\bar{r}_T(U_1, K) \geqq \bar{r}_T(U_2, K) \quad \text{and} \quad \bar{s}_T(U_1, K) \geqq \bar{s}_T(U_2, K) .$$

Finally set $h_B(T, K, X) = \lim_{U \in \mathscr{U}} \bar{r}_T(U, K)$, and

$$h_B(T, X) = \sup_{K \text{ compact}} h_B(T, K, X) .$$

In our context X will always be a l.c.a. group G and the uniformity $\mathscr{U}$ will be the usual left (right, two-sided) uniformity on G.

3. We wish to introduce a second invariant. In doing so, however, we will restrict our attention to automorphisms of a l.c.a.

group G. Let $\alpha \in \mathrm{Aut}(G)$ and $U \subset G$ a precompact neighborhood of the identity. Set

$$U_{\alpha,n} = U + \alpha^{-1}U + \cdots + \alpha^{-(n-1)}U$$

and

$$h_\infty(\alpha, U, G) = \limsup_n \frac{1}{n} \log \mu(U_{\alpha,n}) ,$$

where μ is a fixed haar measure on G. Finally, set

$$h(\alpha, G) = \lim_U k(\alpha, U, G) ,$$

where the net $\{U\}$ is directed by $U_1 < U_2$ iff $U_1 \subset U_2$.

Note that the value of $h_\infty(\alpha, G)$ is independent of the particular choice of haar measure. $h_\infty(\alpha, G)$ was considered in [5] in the context of discrete groups.

4. THEOREM. *Let G be a locally compact abelian group and $\alpha \in \mathrm{Aut}(G)$. The function $h(\alpha, G)$ defined by $h(\alpha, G) = h_B(\alpha, G) + h_\infty(\alpha, G)$ satisfies properties* (i) *through* (vii) *above.*

5. REMARK. $h_B(\alpha, G)$ can also be computed as follows:

$$h_B(\alpha, G) = \lim_U \limsup_n -\frac{1}{n} \log \mu\Big(\bigcap_{j=0}^{n-1} \alpha^{-j} U\Big) ,$$

where μ is a haar measure on G, and the outer limit is taken over a precompact neighborhood base $\{U\}$ converging to the identity [1; Proposition 7].

6. EXAMPLE. The p-adic shift. Let Ω_p be the p-adic group; $\Omega_p = \{\bar{x} = (\cdots, x_k, x_{k+1}, \cdots): x_k \in \{0, 1, \cdots, p-1\}, -\infty < k < \infty$, and $x_k = 0$ for all $k < n_0$, where n_0 is an integer depending on $\bar{x}\}$. The group operation and haar measure on Ω_p are described, for example, in [2]. Let $\alpha \in \mathrm{Aut}(\Omega_p)$ be the bilateral shift given by $\alpha(\bar{x}) = \bar{y}$, where $y_k = x_{k-1}$, $-\infty < k < \infty$. It is easily seen that for any measurable set $E \subset \Omega_p$, the haar measure $m(\alpha E) = pm(E)$, and so α is a nonunimodular automorphism with modular function $\Delta(\alpha) = p$. Let $U_n = \{\bar{x} \in \Omega_p: x_k = 0 \text{ for } k < n\}$. Then $\{U_n: n = 0, 1, \cdots\}$ forms a neighborhood base at 0 in Ω_p. We normalize the haar measure μ so that $\mu(U_0) = 1$. $h_B(\alpha, U_n) = \lim_{k\to\infty} - 1/k \log \mu(\bigcap_{j=0}^{k-1} \alpha^{-j} U_n) = \lim_{k\to\infty} - 1/k \log \mu(U_{n+k}) = \lim_{k\to\infty} - 1/k \log p^{-(n+k)} = \log p$. Since this is independent of n, $h_B(\alpha, \Omega_p) = \log p$.

To compute $h_\infty(\alpha, \Omega_p)$, once again it is sufficient to consider the sets U_n, $n \in Z$, since any precompact set E is contained in some U_n. Since $\alpha^{-1}U_n = U_{n+1} \subset U_n$, for any $k \in Z^+$ we have

$$\mu(U_n + \alpha^{-1}U_n + \cdots + \alpha^{-(k-1)}U_n) \leqq \mu(kU_n) .$$

As haar measure on locally compact abelian groups has polynomial growth, we conclude that

$$\begin{aligned} h_\infty(\alpha, \Omega_p, U_n) &= \lim_{k\to\infty} \frac{1}{k} \log \mu(U_n + \alpha^{-1}U_n + \cdots + \alpha^{-(k-1)}U_n) \\ &= 0 , \quad \text{and consequently} \end{aligned}$$
$$h_\infty(\alpha, \Omega_p) = 0 .$$

Consider now α^{-1}. $\alpha^{-1}\bar{x} = \bar{y}$, where $y_k = x_{k+1}$, $k \in Z$. Since $\alpha U_n = U_{n-1} \supset U_n$, $\bigcap_{j=0}^{k-1} \alpha^j U_n = U_n$. It follows that $h_B(\alpha^{-1}, \Omega_p) = 0$. On the other hand, observe that

$$\alpha^{k-1}U_n \subset U_n + \alpha U_n + \cdots + \alpha^{k-1}U_n \subset \alpha^k U_n .$$

(The right hand containment is easily verified by induction.) Thus $1/k \log p^{k-1-n} \leqq 1/k \log \mu(U_n + \alpha U_n + \cdots + \alpha^{k-1}U_n) \leqq 1/k \log p^{k-n}$. Hence $h_\infty(\alpha^{-1}, \Omega_p) = \log p$.

Next, recall that the dual group Ω_p^* of Ω_p is given by $\Omega_p^* = \{\bar{x} = (\cdots, x_k, x_{k+1}, \cdots)\colon x_k \in \{0, 1, \cdots, p-1\}$, and $x_k = 0$ for all $k > n$, where n depends only on $\bar{x}\}$. The group operation and haar measure on Ω_p^* are analogous to those of Ω_p. The adjoint $\hat{\alpha}$ is given by $\hat{\alpha}(\bar{x}) = \bar{y}$, where $y_k = x_{k-1}$. It is easy to compute that $h_B(\hat{\alpha}, \Omega_p^*) = 0$, $h_\infty(\hat{\alpha}, \Omega_p^*) = \log p$, $h_B(\hat{\alpha}^{-1}, \Omega_p^*) = \log p$, and $h_\infty(\hat{\alpha}^{-1}, \Omega_p^*) = 0$. We see that $h(\alpha, \Omega_p) = h(\alpha^{-1}, \Omega_p) = h(\hat{\alpha}, \Omega_p^*) = h(\hat{\alpha}^{-1}, \Omega_p^*) = \log p$.

The situation above contrasts with that of the bilateral shift β on $G = \prod_{j=-\infty}^{\infty} (Z_p)_i$, $((Z_p)_i = Z_p$ for all $i \in Z)$, with dual group $\Gamma = \sum_{j=-\infty}^{\infty} (Z_p)_j$. Here $h_B(\beta, G) = h_B(\beta^{-1}, G) = \log p$ by the Kolmogorov-Sinai theorem. $h_\infty(\beta, G) = h_\infty(\beta^{-1}, G) = 0$. Also $h_B(\hat{\beta}, \Gamma) = h_B(\hat{\beta}^{-1}, \Gamma) = 0$ and $h_\infty(\hat{\beta}, \Gamma) = h_\infty(\hat{\beta}^{-1}, \Gamma) = \log p$.

7. EXAMPLE. We verify directly that if $\alpha \in \mathrm{Aut}(\boldsymbol{R}^n) = \mathrm{GL}(n, \boldsymbol{R})$, then $h(\alpha, \boldsymbol{R}^n) = \sum_{i=1}^{l} k_i |\log|\lambda_i||$, where $\{\lambda_1, \cdots, \lambda_l\}$ are eigenvalues of α and k_i is the multiplicity of λ_i. (Here our notation differs from that of § 1.) Now α gives rise to $\tilde{\alpha} \in \mathrm{GL}(n, \boldsymbol{C})$ by $\tilde{\alpha}(x + iy) = \alpha(x) + i\alpha(y)$, $x, y \in \boldsymbol{R}^n$. Since $\tilde{\alpha}$ can be identified with $\alpha \times \alpha$ on $\boldsymbol{R}^{2n}$, the effect of changing the base field from $\boldsymbol{R}$ to $\boldsymbol{C}$ is to double the entropy. If we can show $h(\tilde{\alpha}, \boldsymbol{C}^n) = \sum_{i=1}^{l} 2k_i |\log|\lambda_i||$, the formula above will follow. There is a basis for $\boldsymbol{C}^n$ such that the matrix A of $\tilde{\alpha}$ is in Jordan canonical form: thus $A = A_1 \times \cdots \times A_l$ and each A_i is a $k_i \times k_i$ Jordan block with eigenvalue λ_i.

Let B be a $k \times k$ matrix of the form $B = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & & \lambda \end{bmatrix}$ with

$\lambda \neq 0$ on the diagonal and 1 on the superdiagonal. Then B^m is an upper triangular matrix with λ^m on the diagonal and $1/j!\,(d^{(j)}/d\lambda^j)(\lambda^m)$ on the jth superdiagonal, $j = 1, 2, \cdots, k-1$, $m = 1, 2, \cdots$. A crude estimate shows

$$\|B^m x\| \leqq \begin{cases} km^k\|x\|, & \text{if } |\lambda| \leqq 1 \\ |\lambda|^m km^k \|x\|, & \text{if } |\lambda| > 1 . \end{cases}$$

Thus, if U is a ball in C^k and $|\lambda| \leqq 1$,

$$\begin{aligned} &U + BU + \cdots + B^{N-1}U \\ &\subset U + kU + \cdots + k(N-1)^k U \\ &\subset k^2 N^k U \quad \text{for every positive integer } N . \end{aligned}$$

Since haar measure μ on C^k has polynomial growth, $1/N \log(k^2N^kU) \to 0$ as $N \to \infty$. So if $|\lambda| \leqq 1$, $h_\infty(B^1, C^k) = 0$.

If $|\lambda| > 1$,

$$\begin{aligned} &U + BU + \cdots + B^{N-1}U \\ &\subset U + |\lambda| kU + \cdots + |\lambda|^{N-1}k(N-1)^k U \\ &\subset (1 + |\lambda| + \cdots + |\lambda|^{N-1})kN^kU \\ &\subset \frac{|\lambda|^N - 1}{|\lambda| - 1} kN^kU . \end{aligned}$$

The polynomial growth condition on μ means that for any convex neighborhood V of O in C^k there is a $c > 0$ and an exponent q such that $\mu(tV) \leqq ct^q$, $t > 0$. In fact we can take $q = 2k$. Thus

$$\mu(B^{N-1}U) \leqq \mu(U + BU + \cdots + B^{N-1}U) \leqq \mu\Big(\frac{|\lambda|^N - 1}{|\lambda| - 1} kN^kU\Big) .$$

Since $\mu(B^{N-1}U) = \Delta(B)^{N-1}\mu(U)$, where $\Delta(B) = |\lambda|^{2k}$ is the modular function, we have

$$|\lambda|^{2k(N-1)} \leqq \mu(U + BU + \cdots + B^{N-1}U) \leqq c\Big(\frac{|\lambda|^N - 1}{|\lambda| - 1}\Big)^{2k} k^{2k}N^{2k^2} .$$

Taking the log of both sides and dividing by N and taking the limit as $N \to \infty$ we get that $h_\infty(B^{-1}, C^k) = 2k \log|\lambda|$.

From (8.b) we have

$$h_\infty(B, C^k) = h_\infty(B^{-1}, C^k) - 2k \log|\lambda| .$$

We conclude $h_\infty(B, C^k) = \begin{cases} 0, & |\lambda| \geqq 1. \\ -2k \log|\lambda|, & |\lambda| < 1. \end{cases}$ From [1; Theorem 15] $h_\infty(\tilde{\alpha}, C^n) = \sum_{|\lambda_i|>1} 2k_i \log|\lambda_i|$. $h(\tilde{\alpha}, C^n) = \sum_{i=1}^{l} 2k_i\,|\log|\lambda_i||$ follows.

8. We now turn to the proof of Theorem 4. Property (i) fol-

lows from the definition of $h(\alpha, G)$, while (ii), (iii), (iv) and (vill) need only be verified for h_∞, since they are known to hold for h_B([1]). The verifications are straightforward and are omitted. Actually, in § 9 we will show $h_\infty(\alpha, G) = h_B(\hat{\alpha}, \hat{G})$, and so these properties of h_∞ will in fact follow from those of h_B. Note that (vii) follows since l.c.a. groups have polynomial growth.

Recall that the modular function $\Delta(\alpha)$ ($\alpha \in \mathrm{Aut}(G)$) is defined by

$$\mu(\alpha U) = \Delta(\alpha)\mu(U) ,$$

where U is any measurable subset of G. Equivalently, for $f \in L^1(G)$,

$$\int_G f\circ\alpha d\mu = \Delta(\alpha)^{-1}\int_G f d\mu .$$

To show (v) we will show that

(a) $h_B(\alpha, G) = h_B(\alpha^{-1}, G) + \log \Delta(\alpha)$ and

(b) $h_\infty(\alpha, G) = h_\infty(\alpha^{-1}, G) - \log \Delta(\alpha)$.

If U is a precompact neighborhood of the identity in G and μ is haar measure we have $\mu(\bigcap_{j=0}^{n-1} \alpha^{-j} U) = \mu(\alpha^{-(n-1)}\bigcap_{j=0}^{n-1} \alpha^j U) = \Delta(\alpha)^{-(n-1)}\mu(\bigcap_{j=0}^{n-1} \alpha^j U)(\alpha \in \mathrm{Aut}(G))$, and (a) follows. For (b) notice $\mu(U + \alpha^{-1}U + \cdots + \alpha^{-(n-1)}U) = \mu(\alpha^{-(n-1)}(U + \alpha U + \cdots + \alpha^{(n-1)}U)) = \Delta(\alpha)^{-(n-1)}\mu(U + \mu U + \cdots + \alpha^{(n-1)}U)$.

9. To show $h(\alpha, G) = h(\hat{\alpha}, \hat{G})$ we will prove $h_\infty(\alpha, G) = h_B(\hat{\alpha}, \hat{G})$, as mentioned above. This, together with the Pontryagain-van Kampen duality theorem, will yield the assertion.

The strategy of the proof is to rewrite the defining expressions for h_B and h_∞ as limits of convolutions of characteristic functions of sets, and then replace characteristic functions with nonnegative positive definite L^1-functions, which is self-dual under the Fourier transform. It will be convenient to introduce some notations. If ϕ is a function on G and $\alpha \in \mathrm{Aut}(G)$, $\alpha\phi$ will denote the function $(\alpha\phi)(x) = \phi(\alpha^{-1}x)$, $x \in G$. Also, we will let

$$C_n(\phi, \alpha) = \phi*(\alpha^{-1}\phi)*\cdots*(\alpha^{-(n-1)}\phi),\ n > 0 .$$

LEMMA 9.1. *Let $n \geqq 1$ and $U_i = -U_i$ be symmetric neighborhoods of* 0, $1 \leqq i \leqq n$. *Then*

$$\frac{\mu(U_1)^2\mu(U_2)^2\cdots\mu(U_n)^2}{\chi_{U_1}*\chi_{U_1}*\chi_{U_2}*\chi_{U_2}*\cdots*\chi_{U_n}*\chi_{U_n}(0)} \leqq \mu(2U_1 + 2U_2 + \cdots + 2U_n) .$$

Proof. The convolution of characteristic functions $\chi_{U_i}*\chi_{U_i}$ is positive definite, $1 \leqq i \leqq n$, since $\chi_{U_i}^*(x) = \chi_{U_i}(-x)^- = \chi_{U_i}(x)$. Thus

$\chi_{U_1}*\chi_{U_1}*\chi_{U_2}*\chi_{U_2}*\cdots*\chi_{U_n}*\chi_{U_n}$ is positive definite, and achieves its maximum at 0. Now

$$\mu(U_i)^2\mu(U_2)^2\cdots\mu(U_n)^2 = \int_G \chi_{U_1}*\chi_{U_1}*\chi_{U_2}*\chi_{U_2}*\cdots*\chi_{U_n}*\chi_{U_n}(x)d\mu(x)$$
$$\leqq \chi_{U_1}*\chi_{U_1}*\chi_{U_2}*\chi_{U_2}*\cdots*\chi_{U_n}*\chi_{U_n}(0)\mu(2U_1+2U_2+\cdots+2U_n)\ ,$$

since the integrand is supported in $2U_1+2U_2+\cdots+2U_1$.

LEMMA 9.2. *Let U_i, $1 \leqq i \leqq n$ be as in Lemma 9.1 and r a positive integer. Then*

$$\mu(2U_1+2U_2+\cdots+2U_n) \leqq \frac{\mu((r+1)U_2)^2((r+1)U_2)^2\cdots((r+1)U_n)^2}{\chi_{rU_1}*\chi_{rU_1}*\cdots*\chi_{rU_n}*\chi_{rU_n}(0)}\ .$$

Proof. Let λ_x be the point mass at x, so that $\lambda_x*f(z) = f(z-x)$. Let $u_i, u_i' \in U_i$, $1 \leqq i \leqq n$, and $x = u_1+u_1'+\cdots+u_n+u_n' \in 2U_1+\cdots+2U_n$. We have

$$\begin{aligned}
&\chi_{(r+1)U_1}*\chi_{(r+1)U_1}*\cdots*\chi_{(r+1)U_n}*\chi_{(r+1)U_n}(x)\\
&\quad = \lambda_{-x}*\chi_{(r+1)U_1}*\chi_{(r+1)U_1}*\cdots*\chi_{(r+1)U_n}*\chi_{(r+1)U_n}(0)\\
&\quad = (\lambda_{-u_1}*\chi_{(r+1)U_1})*(\lambda_{-u_1'}*\chi_{(r+1)U_1})*\cdots*(\lambda_{-u_n}*\chi_{(r+1)U_n})*(\lambda_{-u_n'}*\chi_{(r+1)U_n})(0)\\
&\quad \geqq \chi_{rU_1}*\chi_{rU_1}*\cdots*\chi_{rU_n}*\chi_{rU_n}(0)\ .
\end{aligned}$$

Integrating over $2U_1+2U_2+\cdots+2U_n$ gives the result.

LEMMA 9.3. *Any precompact neighborhood V of 0 in G is contained in a precompact neighborhood U satisfying*

$$\lim_{r\to\infty}\frac{\mu((r+1)U)}{\mu(rU)} = 1\ .$$

Proof. Let H be the compactly generated open subgroup generated by V. Then H is a projective limit of abelian lie groups [2; Theorem 9.6]; hence it is enough to observe that the assertion holds for $G = \boldsymbol{R}^n$ or $G = Z^m$, the proofs of which are straightforward and omitted.

Using the notation introduced at the beginning of this section we define

$$h_\infty^1(\alpha, G) = \sup_{\substack{U\in G \text{ open}\\ \text{precompact}}} \limsup_n -\frac{1}{n}\log\left\{\frac{\Delta(\alpha)^{n(n-1)}}{\mu(U)^{2n}}C_n(\chi_U*\chi_U, \alpha)(0)\right\}\ .$$

LEMMA 9.4. $h_\infty(\alpha, G) = h_\infty^1(\alpha, G)$.

Proof. Let U be a precompact open neighborhood of O and apply Lemma 9.1 with $U_j = \alpha^{-(j-1)}U$. Take the logarithm of both sides of the inequality and divide by n; since $\chi_{\alpha^{-j}U} = \alpha^{-j}\chi_U$, this yields

$$-\frac{1}{n}\log\frac{C_n(\chi_U*\chi_U,\alpha)(0)}{\mu(U)^2\mu(\alpha^{-1}U)^2\cdots\mu(\alpha^{-(n-1)}U)^2} \leqq \frac{1}{n}\log\mu(2U+\alpha^{-1}(2U)+\cdots+\alpha^{-(n-1)}(2U)) .$$

From $\mu(\alpha^{-j}U) = \Delta(\alpha)^{-j}\mu(U)$, we conclude that $h^1_\infty(\alpha, G) \leqq h_\infty(\alpha, G)$.

By Lemma 9.3 we may suppose U satisfies

$$\lim_{r\to\infty}\frac{\mu((r+1)U)}{\mu(rU)} = 1 .$$

From Lemma 9.2 we obtain

$$\frac{1}{n}\log\mu(2U+\alpha^{-1}(2U)+\cdots+\alpha^{-(n-1)}(2U)) \leqq -\frac{1}{n}\log\left\{\frac{\Delta(\alpha)^{n(n-1)}}{\mu(rU)^{2n}}C_n(\chi_{rU}*\chi_{rU},\alpha)\right\} + 2\log\frac{\mu((r+1)U)}{\mu(rU)} .$$

It follows from Lemma 9.2 that $h_\infty(\alpha, G) \leqq h^1_\infty(\alpha, G)$ and thus $h_\infty(\alpha, G) = h^1_\infty(\alpha, G)$.

Let $P(G)$ denote the continuous positive definite functions on G with compact support and $C_{00}(G)^+$ the nonnegative continuous functions on G with compact support. Set

$$h^2_\infty(\alpha, G) = \sup_{\phi\in P(G)\cap C_{00}(G)^+}\limsup_n -\frac{1}{n}\log\left\{\frac{\Delta(\alpha)^{n(n-1)/2}}{\left(\int_G\phi d\mu\right)^n}C_n(\phi,\alpha)(0)\right\} .$$

Clearly, $h^2_\infty(\alpha, G) \geqq h^1_\infty(\alpha, G)$. On the other hand, by an estimate very similar to the first part of Lemma 9.2, it follows that $h^2_\infty(\alpha, G) \leqq h_\infty(\alpha, G)$. Thus $h^2_\infty(\alpha, G) = h^1_\infty(\alpha, G)$.

Let $L^1(G)^+$ denote the positive cone in $L^1(G) = L^1(G, \mu)$. Define

$$h^3_\infty(\alpha, G) = \sup_{\phi\in P(G)\cap L^1(G)^+}\limsup_n -\frac{1}{n}\log\left\{\frac{\Delta(\alpha)^{n(n-1)/2}}{\left(\int_G\phi d\mu\right)^n}C_n(\phi,\alpha)(0)\right\} .$$

Clearly $h^3_\infty(\alpha, G) \geqq h^2_\infty(\alpha, G)$.

LEMMA 9.5. $h^2_\infty(\alpha, G) = h^3_\infty(\alpha, G)$.

Proof. We sketch the proof briefly, since the details are essentially the same as in [5; Lemma 11] except for the presence of the modular function. First, a routine estimate shows that in the definition of h^3_∞ each ϕ which appears inside the brackets { } may be replaced by $\phi*\phi$. Then given $\phi\in P(G)\cap L^1(G)^+$, $\|\phi\|_1=1$, there is a compact symmetric neighborhood U of O such that if $f=\phi\chi_U$, $\|\phi-f\|_1<\varepsilon/2$. If $\psi=f*f$, $\psi\in P(G)\cap C_{00}(G)^+$ and

$$\left|-\frac{1}{n}\log\left\{\frac{\Delta(\alpha)^{n(n-1)/2}}{\left(\int_G\psi d\mu\right)^n}C_n(\psi,\alpha)\right\}\right.$$
$$\left.+\frac{1}{n}\log\{\Delta(\alpha)^{n(n-1)/2}C_n(\phi*\phi,\alpha)\}\right|\leqq\log(1-\varepsilon),\ n=1,2,\cdots.$$

Hence $h^3_\infty(\alpha,G)=h^2_\infty(\alpha,G)$.

Suppose now Γ is another locally compact abelian group and $\beta\in\mathrm{Aut}(\Gamma)$. As observed in Remark 5, we have

$$h_B(\beta,\Gamma)=\sup_U\limsup_n\left\{-\frac{1}{n}\log\omega\Big(\bigcap_{j=1}^n\beta^{-j}U\Big)\right\}$$

where ω is a haar measure on Γ and U is a precompact open neighborhood of the neutral element 1 in Γ. Define

$$h^1_B(\beta,\Gamma)=\sup_{\phi\in P(\Gamma)\cap L^1(\Gamma)^+}-\frac{1}{n}\left\{\log\frac{1}{\phi(e)^n}\int_\Gamma\phi(\beta^{-1}\phi)\cdots(\beta^{-(n-1)}\phi)(y)d\omega(y)\right\}.$$

LEMMA 9.6. $h^1_B(\beta,\Gamma)=h_B(\beta,\Gamma)$.

Proof. Formally the proof is identical to [5; Lemma 12]; the compactness of Γ, which is assumed there, is not needed.

9.7. Let G be a locally compact abelian group with fixed haar measure μ, and let Γ be the dual group with haar measure ω, which is appropriately normalized so that the inversion theorem holds.

We claim that $\widehat{\alpha\phi}=\Delta(\alpha)\hat{\alpha}\hat{\phi}$, $\phi\in L^1(G)$:

$$\begin{aligned}\widehat{\alpha\phi}(\tau)&=\int_G\alpha\phi(x)\tau(x)^-d\mu(x)\\&=\int_G\phi(\alpha^{-1}x)\tau(x)^-d\mu(x)\\&=\Delta(\alpha)\int_G\phi(x)\tau(\alpha x)^-d\mu(x)\\&=\Delta(\alpha)\int_G\phi(x)(\hat{\alpha}^{-1}\tau)(x)^-d\mu(x)\\&=\Delta(\alpha)\hat{\alpha}\hat{\phi}(\tau).\end{aligned}$$

For $\phi\in P(G)\cap L^1(G)^+$,

$$\frac{\Delta(\alpha)^{n(n-1)/2}}{\left(\int_G \phi d\mu\right)^n} C_n(\phi, \alpha)(0) = \frac{\Delta(\alpha)^{n(n-1)/2}}{(\hat{\phi}(e))^n} \int_\Gamma \widehat{C_n(\phi, \alpha)}(y) d\omega(y)$$

$$= \frac{\Delta(\alpha)^{n(n-1)/2}}{(\hat{\phi}(e))^n} \int_\Gamma \hat{\phi}(\widehat{\alpha^{-1}\phi}) \cdots (\widehat{\alpha^{-(n-1)}\phi})(y) d\omega(y)$$

$$= \frac{1}{(\hat{\phi}(e))^n} \int_\Gamma \hat{\phi}(\hat{\alpha}^{-1}\hat{\phi}) \cdots (\hat{\alpha}^{-(n-1)}\hat{\phi})(y) d\omega(y) .$$

But by the Fourier Invension Theorem,

$$\{\hat{\phi} : \phi \in P(G) \cap L^1(G)^+\} = P(\Gamma) \cap L^1(\Gamma)^+ .$$

The above calculation shows that $h^3_\infty(\alpha, G) = h^1_B(\hat{\alpha}, \Gamma)$. But $h^3_\infty(\alpha, G) = h_\infty(\alpha, G)$ and $h^1_B(\hat{\alpha}, \Gamma) = h_B(\hat{\alpha}, \Gamma)$. Thus $h_\infty(\alpha, G) = h_B(\hat{\alpha}, \Gamma)$, as desired.

10. *Addition Theorem.* Let G be a l.c.a. group, $\alpha \in \mathrm{Aut}(G)$ and $H \subset G$ a closed subgroup such that $\alpha(H) = H$. If α_1 denotes the restriction of α to H and α_2 the induced automorphism on the quotient G/H, then

$$h(\alpha, G) = h(\alpha_1, H) + h(\alpha_2, G/H) .$$

Note. Juzvinskii proves the addition theorem using structure theory in case G, H are compact but not necessarily abelian. The compact case also follows from [1; Theorem 19].

Proof. Let $\phi: G \to G/H$ be the canonical map and write $\dot{x}$ for $\phi(x)$. Let $K \subset G$, $C \subset H$ be compact, and U a neighborhood of O in G. Suppose $\dot{E}_2$ is an $(n, U + H)$-separated set in $\dot{K}$ with respect to α_2 and E_1 is an $(n, U \cap H)$-separted set in C with respect to α_1. For each $\dot{x} \in \dot{E}_2 \subset \dot{K}$ choose a representative $x \in K$ and denote the set of representatives by E_2. The set $E = E_1 + E_2 = \{h + x : h \in E_1, x \in E_2\}$ is (n, U) separated in $K + C$. For suppose $h + x, h' + x' \in E$, $h + x \neq h' + x'$. If $x = x'$, since h, h' are $(n, U \cap H)$-separated there is a j, $0 \leqq j < n$, with $\alpha_1^{-j}h - \alpha_1^{-j}h' \notin U \cap H$. But then $\alpha^{-j}(x + h) - \alpha^{-j}(x' + h') \notin U$. If $x \neq x'$, then $\dot{x} \neq \dot{x}'$ and there is a j, $0 \leqq j < n$, with $\alpha_2^{-j}\dot{x} - \alpha_2^{-j}\dot{x}' \notin U + H$. Hence $\alpha^{-j}(x + h) - \alpha^{-j}(x' + h') \notin U$. In either case, $h + x$ and $h' + x'$ are (n, U)-separated. Thus

$$s_n(U, K + C) \geqq s_n(U \cap H, C) s_n(U + H, \dot{K}) ,$$

and $h_B(\alpha, G) \geqq h_B(\alpha_1, H) + h_B(\alpha_2, G/H)$.

To get the reverse inequality, let $K \subset G$ be compact and U a compact neighborhood of O in G. Let $C = (K - K - 2U) \cap H$ and $\dot{F}_2$ and $(n, \dot{U})$-spanning set for $\dot{K}$ with respect to α_2, which we assume is minimal. Let F_1 $(n, U\cap H)$-span C. For any $y\in K$ there is an $x\in\phi^{-1}(\dot{F}_2)$ with $\alpha^{-j}y-\alpha^{-j}x\in U+H$, $0\leqq j<n$. Equivalently, $y-x\in\bigcap_{j=0}^{n-1}\alpha^jU+H$. The minimality assumption on $\dot{F}_2$ implies $\dot{F}_2\subset\dot{K}+\bigcap_{j=0}^{n-1}\alpha^j\dot{U}$. So for each $\dot{x}\in\dot{F}_2$ we can choose a representative $x\in K+\bigcap_{j=0}^{n-1}\alpha^jU$ and denote the resulting set F_2. We claim that $F=F_1+F_2$ $(n, 2U)$-spans K with respect to α. Given $y\in K$ there is an $x\in F_2$ and a $u\in\bigcap_{j=0}^{n-1}\alpha^jU$ so that $y-x-u\in H$. But $y-x-u\in(K-(K+U)-U)\cap H=(K-K-2U)\cap H=C$.

Thus there exists $h\in F_1$ so that

$$\alpha^{-j}(y-x-u)-\alpha^{-j}h\in U\cap H,\ 0\leqq j<n\ ;$$

i.e.,

$$y-x-u-h\in\Big(\bigcap_{j=0}^{n-1}\alpha^jU\Big)\cap H\ .$$

Hence

$$\begin{aligned}y-(x+h)\in u+\Big(\bigcap_{j=0}^{n-1}\alpha^jU\Big)\cap H\ ,\\ \subset\bigcap_{j=0}^{n-1}\alpha^j(2U)\ ,\end{aligned}$$

which shows $F=F_1+F_2(n, 2U)$-spans K, as claimed. It follows

$$r_n(2U, K)\leqq r_n(\dot{U}, \dot{K})r_n(U\cap H, C)\ ,$$

and consequently $h_B(\alpha, G)\leqq h_B(\alpha_1, H)+h_B(\alpha_2, G/H)$.

We prove the corresponding equality for h_∞ by passing to the dual and utilizing the result above. If K is the annihilator of H in $\hat{G}$, then $K=\widehat{G/H}$ and $\hat{G}/K=\hat{H}$. By the proof of part (vi) of Theorem 4,

$$\begin{aligned}h_\infty(\alpha, G)&=h_B(\hat{\alpha}, \hat{G})\\&=h_B(\hat{\alpha}_1, \hat{G}/K)+h_B(\hat{\alpha}_2, K)\\&=h_B(\hat{\alpha}_1, H)+h_B(\hat{\alpha}_2, G/H)\\&=h_\infty(\alpha_1, H)+h_\infty(\alpha_2, G/H)\ .\end{aligned}$$

COROLLARY 11. *Suppose G is a projective limit of Lie groups, $G=\text{proj}\lim G_\nu$, $G_\nu=G/H_\nu$, H_ν compact and α-invariant, and α_ν is the induced automorphism on G_ν. Then*

$$h(\alpha, G)=\lim_\nu h(\alpha_\nu, G_\nu)\ .$$

Proof. Set α^ν equal the restriction of α to H_ν. First suppose G is compact.

$$h_B(\alpha, G)=h_\infty(\hat{\alpha}, \hat{G})=h_\infty(\hat{\alpha}_\nu, \hat{G}_\nu)+h_\infty(\hat{\alpha}_\nu, \hat{H}_\nu)\ .$$

Now $\hat{G}_\nu = K_\nu$, the annihilator in $\hat{G}$ of H_ν. Furthermore, by properties of projective limits, for any $\tau \in \hat{G}$ there is a ν with $H_\nu \subset$ kernel τ. Thus $\tau \in K_\nu$, and we can write $\hat{G} = \bigcup_\nu K_\nu$. Hence, given any finite set $E \subset \hat{G}$ there is a ν_0 so that $E \subset K_\nu$ for $\nu \geqq \nu_0$. By definition, $h(\hat{\alpha}, \hat{G})$ can be approximated arbitrarily closed by taking a sufficiently large finite set $E \subset \hat{G}$ and forming the limit, $\lim_n (1/n)\log \operatorname{card}(E_{\hat{\alpha},n})$, where $E_{\hat{\alpha},n} = E + \alpha^{-1}E + \cdots + \hat{\alpha}^{-(n-1)}E$. Thus $h_\infty(\hat{\alpha}, \hat{G}) = \lim_\nu h_\infty(\hat{\alpha}_\nu, K_\nu)$ i.e., $h_B(\alpha, G) = \lim_\nu h_B(\alpha_\nu, G_\nu)$.

Now drop the compactness assumption on G and write

$$\begin{aligned} h_B(\alpha, G) &= h_B(\alpha_\nu, G_\nu) + h_B(\alpha^\nu, H_\nu) \\ &= h_B(\alpha_\nu, G_\nu) + h_B(\alpha^{\nu'}, H_{\nu'}) + h_B(\alpha_{\nu\nu'}, H_\nu/H_{\nu'}) , \end{aligned}$$

for $\nu' \geqq \nu$ and $\alpha_{\nu\nu'}$ the induced automorphism on $H_\nu/H_{\nu'}$. Suppose $h_B(\alpha, G) < \infty$ and $\lim_\nu h_B(\alpha^\nu, H_\nu) = \varepsilon > 0$. (The limit exists since the net is nonincreasing.) Choose ν so that $h_B(\alpha^\nu, H_\nu) < 3/2\varepsilon$. By the first part of the proof, we can choose ν' sufficiently large so that $h_B(\alpha_{\nu\nu'}, H_\nu/H_{\nu'})$ is close to $h_B(\alpha_\nu, H_\nu)$, and in particular greater than ε. Since $h_B(\alpha^{\nu'}, H_{\nu'})$ is at least ε, we have that

$$\begin{aligned} h_B(\alpha^\nu, H_\nu) &= h_B(\alpha^{\nu'}, H_{\nu'}) + h_B(\alpha_{\nu\nu'}, H_\nu/H_{\nu'}) \\ &> 2\varepsilon, \text{ a contradiction} . \end{aligned}$$

Thus $h_B(\alpha^\nu, H_\nu) \to 0$, and $h_B(\alpha_\nu, G_\nu) \to h_B(\alpha, G)$.

If $h(\alpha, G)$ is infinite but $h_B(\alpha_\nu, G_\nu) \leqq M < \infty$ for all ν, we similarly arrive at a contradiction. Once again, write

$$h_B(\alpha, G) = h_B(\alpha_\nu, G_\nu) + h_B(\alpha^\nu, H_\nu) ,$$

which forces $h_B(\alpha^\nu, H_\nu) = \infty$ for all ν. However for $\nu' \geqq \nu$,

$$h_B(\alpha_\nu, G_\nu) = h_B(\alpha_\nu, G_{\nu'}) + h_B(\alpha_{\nu\nu'}, H_\nu/H_{\nu'}) ,$$

and by the first part of the proof

$$\lim_{\nu'} h_B(\alpha_{\nu\nu'}, H_\nu/H_{\nu'}) = h_B(\alpha_\nu, H_\nu) = \infty .$$

Hence $h_B(\alpha_\nu, G_\nu)$ is not bounded, and we have established

$$h_B(\alpha, G) = \lim h_B(\alpha_\nu, G_\nu) .$$

Finally, we must deal with h_∞. But is easy, since

$$\begin{aligned} h_\infty(\alpha, G) &= h_\infty(\alpha_\nu, G_\nu) + h_\infty(\alpha^\nu, H_\nu) \\ &= h_\infty(\alpha_\nu, G_\nu) , \end{aligned}$$

as h_∞ vanishes on any compact group.

12.1. *Some Questions.* For any l.c.a. group G, $\operatorname{Aut}(G)$ is itself a topological group (though it need not be locally compact). The topology is described e.g., in [2] or [6]. Is the map $\alpha \in \mathrm{Aut}(G) \to h(\alpha, G) \in [0, \infty]$ continuous? If G is a compactly generated abelian Lie group, from the explicit formula for $h(\alpha, G)$ we can answer in the affirmative.

12.2. In view of Theorem 4 (vi) one might expect other relationships between the entropy of $\alpha \in \mathrm{Aut}(G)$ and $\hat{\alpha} \in \mathrm{Aut}(\Gamma)$ to hold. Even in the case of compact G we do not know of any such results. If G is compact and Γ discrete, the adjoint $\hat{\alpha}$ of an $\alpha \in \mathrm{Aut}(G)$ extends to a homeomorphism $\bar{\hat{\alpha}}$ of the Stone-Čech compactification $\bar{\Gamma}$. Is there a $\bar{\hat{\alpha}}$-invariant measure ω on $\bar{\Gamma}$ for which the measure-theoretic entropy $h_\omega(\bar{\hat{\alpha}}, \bar{\Gamma})$ equals the (Kolmogorov-Sinai) entropy $h(\alpha, G)$? We end the discussion with a counterexample which answers the question in the negative if we replace measure-theoretic entropy by topological entropy on $\bar{\Gamma}$ (see [7] for appropriate definitions).

Let $G = \prod_{i=-\infty}^{\infty} (Z_n)_i$, where $(Z_p)_i = Z_p$, the integers modulo p, for all $i \in Z$, and $\alpha \in \mathrm{Aut}(G)$ the bilateral shift. Then $h(\alpha, G) = \log p$.

Claim. The topological entropy of $\bar{\hat{\alpha}}$ on the Stone-Čech compactification $\bar{\Gamma}$ is infinite.

Proof. Γ consists of all infinite sequences

$$\bar{x} = (\cdots x_k, x_{k+1}, \cdots), \ x_k \in Z_p \quad \forall k \in Z$$

and only finitely many of the x_k's are nonzero. We will say a sequence $\bar{x}$ begins at k (resp. ends at k) if $x_k \neq 0$ and $x_j = 0$ for $j < k$ (resp. $x_k \neq 0$ and $x_j = 0$ for $j > k$); if $\bar{x}$ begins at k_1 and ends at k_2, we call $k_2 - k_1 + 1$ the length of $\bar{x}$. Given an integer $N > 0$ we will define a finite cover $\xi = \{E_0, E_1, \cdots, E_N\}$ of Γ. First we define inductively a sequence of integers l_j. Set $l_0 = 1$. Suppose, for some fixed n, that l_j, $j < n$, have been defined; set

$$l_n = \inf\{l: (p-1)^2 p^{l-2} \geqq N^n \quad \text{and} \quad l > l_{n-1}\} .$$

There are $(p-1)^2 p^{l-2}$ sequences of length $l (l \geqq 2)$ beginning at 0. Set $m = N^n$. Thus it is possible to find m distinct sequences $\bar{x}^{1,n}, \cdots, \bar{x}^{m,n}$ of length l_n, beginning at 0. (How they are chosen is unimportant.) Let F_n be any one-to-one function from the integers $\{1, \cdots, m\}$ onto the set of all sequences of the form $\{i_1, \cdots, i_n: i_j \in \{1, \cdots, N\}\}$. Let $\beta: \Gamma \to \Gamma$ be the bilateral shift; $\beta\bar{x} = \bar{y}$, where $y_k = x_{k+1}$. We now define subsets $E_1^{(n)}, \cdots, E_N^{(n)} \subset \Gamma$ as follows. We will put $\beta^j \bar{x}^{i,n}$ in $E_{i_{j+1}}^{(n)}$, $0 \leqq j < n-1$, if and only if $F_n(i) = (i_1, \cdots, i_n)$, $1 \leqq i \leqq m$.

Set $E_i = \bigcup_{n\geqq 1} E_i^{(n)}$ and $E_0 = \Gamma \backslash \bigcup_{i=1}^{n} E_i$. Note that

$$\bar{x}^{i,n} \in E_{i_1}^{(n)} \cap \beta^{-1} E_{i_2}^{(n)} \cap \cdots \cap \beta^{-(n-1)} E_{i_n}^{(n)}$$
$$\subset E_{i_1} \cap \beta^{-1} E_{i_2} \cap \cdots \cap \beta^{-(n-1)} E_{i_n}$$

if and only if $F_n(i) = (i_1, \cdots, i_n)$; furthermore, $\beta^j \bar{x}^{i,n} \notin E_0$, $0 \leqq j < n$. Thus any covering of Γ by $\xi \vee \beta^{-1}\xi \vee \cdots \vee \beta^{-(n-1)}\xi$ must contain at least N^n sets.

Notice that β is the transpose of the bilateral shift α on G. We embed Γ in $\bar{\Gamma}$ in the usual way: if $A \subset \Gamma$, $\bar{A}$ is the set of all ultrafilters containing A. The operation $A \to \bar{A}$ respects finite unions and intersections, and it follows from the above that the topological entropy of $\bar{\hat{\alpha}}$ on $\bar{\Gamma}$ is at least $\log N$, for N arbitrary, hence is infinite.

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