SOME TOPOLOGICAL PROPERTIES OF SPACES OF MEASURES

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Let X be a completely regular space and $M_\tau(X)$, $M_t(X)$ and $M_c(X)$ the spaces of the τ-additive measures, tight measures and measures with compact support on X, endowed with the weak topology. The aim of this paper is to study topological properties that devolve from X to $M_\tau(X)$, $M_t(X)$ and $M_c(X)$ or their positive cones $M_\tau^+(X)$, $M_t^+(X)$ and $M_c^+(X)$. It is proved that if X is paracompact (resp. Lindelöf) and Čech complete, then $M_\tau^+(X)$ and $M_t^+(X)$ have the same properties, but $M_c^+(X)$ does not (unless X is compact). If X is realcompact then $M_c(X)$ has the same property, but $M_\tau(X)$ and $M_t(X)$ need not. However, if X is realcompact paracompact, then $M_\tau(X)$ is realcompact.

Let X be a completely regular space and $C(X)$ the space of bounded real-valued continuous functions on X with the supremum norm. The spaces $M_\sigma(X)$, $M_\tau(X)$, $M_t(X)$ and $M_c(X)$ of measures on X are defined as subsets of the dual of $C(X)$ ([22]) and can be described using the Stone-Čech compactification of X ([13]). Identifying each point x of X with the point mass ε_x, X can be considered as a closed subset of $M_s(X)$ and $M_s^+(X)$ (for $s = \tau$, t or c) endowed with the weak topology which is defined by $C(X)$. Our purpose is to find topological properties that devolve from X to $M_s(X)$ or $M_s^+(X)$.

Notations and preliminary results are given in § 1. In § 2 it is proved that certain topological properties which can be described by perfect functions with values in metric spaces, devolve from X to $M_s^+(X)$ but not to $M_s(X)$. For this purpose it is shown that a perfect function between two spaces X and Y induces a perfect function between $M_s^+(X)$ and $M_s^+(Y)$. Section 3 is concerned with the property of realcompactness. Using a result of Corson, it is proved that $M_s(X)$ is realcompact if and only if every σ-additive measure on X which on every countably generated σ-algebra of Baire sets coincides with some element of $M_s(X)$ is, in fact, an element of $M_s(X)$. Realcompactness devolves from X to $M_c(X)$, but not to $M_\tau(X)$ and $M_t(X)$.

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1. **Preliminaries and notations.** A basic reference for the

theory of measures on topological spaces is Varadarajan [22]. All topological spaces considered in this paper are assumed to be completely regular (and, of course, Hausdorff). Let X be a basic fixed completely regular space. We denote by $C(X)$ the space of bounded continuous real-valued functions on X, endowed with the supremum norm. A linear functional φ on $C(X)$ is

(a) σ-additive, if $\varphi(f_n) \to 0$ for every sequence $\{f_n\}$ in $C(X)$ which is decreasing and pointwise convergent to zero;

(b) τ-additive, if $\varphi(f_\alpha) \to 0$ for every net $\{f_\alpha\}$ in $C(X)$ which is decreasing and pointwise convergent to zero;

(c)   tight, if $\varphi(f_\alpha) \to 0$ for every uniformly bounded net $\{f_\alpha\}$ in $C(X)$ which converges to zero uniformly on compact subsets.

According to the Aleksandrov representation theorem [22, Part I, Theorem 6], the dual of $C(X)$ is identified by an order preserving isometry with the space $M(X)$ of (finitely additive Baire) measures on X. A measure μ on X is said to be σ-additive, τ-additive or tight if the functional

$$\varphi(f) = \int_X f\,d\mu , \quad f \in C(X)$$

has these properties. The spaces of the σ-additive, τ-additive and tight measures on X are denoted by $M_\sigma(X)$, $M_\tau(X)$ and $M_t(X)$ respectively. The elements of $M_\sigma(X)$ are exactly the countably additive measures defined on the σ-algebra $\mathscr{B}(X)$ of all Baire sets of X (see [22, Part I, Theorem 18]). We recall that the collection of Baire sets of X is the σ-algebra generated by the zero sets of real-valued continuous functions on X.

The space of measures with compact support is denoted by $M_c(X)$. The elements of $M_c(X)$ correspond to the functionals on $C(X)$ with the property: $\varphi(f_\alpha) \to 0$ for every net $\{f_\alpha\}$ in $C(X)$ which converges to zero uniformly on compact subsets. Usually we do not distinguish between the functional and the associated measure.

We have $M(X) \supset M_\sigma(X) \supset M_\tau(X) \supset M_t(X) \supset M_c(X)$. The subsets of positive measures in these spaces are denoted by $M^+(X)$, $M_\sigma^+(X)$, $M_\tau^+(X)$, $M_t^+(X)$ and $M_c^+(X)$.

If $\mu \in M(X)$, $\bar\mu$ denotes the corresponding regular Borel measure on the Stone-Čech compactification βX of X (via the isometry of $C(X)$ and $C(\beta X)$).

THEOREM 1.1 (Knowles [13]).   *If $\mu \in M^+(X)$, then*

(i)   *$\mu \in M_\sigma^+(X)$ if and only if $\bar\mu$ vanishes on all zero sets of βX contained in $\beta X - X$;*

(ii)   *$\mu \in M_\tau^+(X)$ if and only if $\bar\mu$ vanishes on all compact sets contained in $\beta X - X$;*

(iii) $\mu \in M_t^+(X)$ *if and only if for every* $\varepsilon > 0$ *there is a compact set* $K \subset X$ *such that* $\bar{\mu}(\beta X - K) < \varepsilon$.

A (nonzero) measure is 2-valued if its range is $\{0, 1\}$. The τ-additive 2-valued measures on X are exactly the point masses ε_x, $x \in X$.

On the space $M(X)$ we consider the weak topology $\sigma(M(X), C(X))$. This space is completely regular and can be identified with the space $M(\beta X)$ by the homeomorphism $\mu \to \bar{\mu}$. Also, the function $x \to \varepsilon_x$ is a homeomorphism of X and $\{\varepsilon_x : x \in X\}$ which is closed in $M_\tau^+(X)$.

The following theorem is one of the deepest results obtained by Varadarajan in [22]. A simpler proof can be found in [5].

THEOREM 1.2 ([22, Part II, Theorem 13]). *The space* $M_\tau^+(X)$ *is metrizable if and only if* X *is metrizable.*

Using 1.2 Varadarajan also proves the following.

THEOREM 1.3 ([22, Part II, Theorems 14, 18]). *The space* $M_\tau^+(X)$ *is separable (resp. complete) metrizable if and only if* X *is separable (resp. complete) metrizable.*

Let X, Y be completely regular spaces and $f: X \to Y$ a continuous function. We define $f_*: M(X) \to M(Y)$ by $f_*(\mu)(g) = \mu(g \circ f)$ for every $g \in C(Y)$. It is immediate that f_* is continuous (with respect to the weak topologies $\sigma(M(X), C(X))$ and $\sigma(M(Y), C(Y))$) and $f_*(M_s(X)) \subset M_s(Y)$ for $s = \sigma, \tau, t$ or c. If $\mu \in M_\sigma(X)$, then $f_*(\mu)$ coincides with the classical image measure, i.e.,

$$f_*(\mu)(B) = \mu(f^{-1}(B)) \quad \text{for all} \quad B \in \mathscr{B}(Y)$$

(cf. [7, p. 163] and [19, Theorem 3.1 (c)]).

2. **Perfect functions.** A continuous function $f: X \to Y$ is called perfect if f is onto, closed and $f^{-1}(\{y\})$ is compact for every $y \in Y$. The following theorem will be used to show that some topological properties related to Cech completeness and M-property (defined below) devolve from X to $M_s^+(X)$ for $s = \tau, t$ or c.

THEOREM 2.1. *Let* $f: X \to Y$ *be continuous onto. Then,* $f_*: M_s^+(X) \to M_s^+(Y)$ *(for* $s = \tau, t$ *or* c*) is perfect if and only if* f *is perfect.*

We shall use the following two lemmas.

LEMMA 2.2. *If* $f: X \to Y$ *is continuous, onto, then* $f_*: M(X) \to$ $M(Y)$ *is onto and* $f_*(M^+(X)) = M^+(Y)$.

This is a well-known application of Hahn-Banach theorem.

LEMMA 2.3. *Let* $f: X \to Y$ *be a perfect function and* $\mu \in M^+(X)$. *If* $f_*(\mu) \in M_s^+(Y)$ *then* $\mu \in M_s^+(X)$, *for* $s = \tau, t$ *or* c.

Proof. Let $\bar{f}: \beta X \to \beta Y$ be the continuous extension of f. We observe that $\overline{f_*(\mu)} = \bar{f}_*(\bar{\mu})$ and, by the uniqueness of the regular Borel extension of Baire measures on compact spaces, $\overline{f_*(\mu)}(B) = \bar{\mu}(\bar{f}^{-1}(B))$ for all Borel sets B in βY. We also note that $\bar{f}(\beta X - X) \subset \beta Y - Y$ since f is perfect (cf. [1, Lemma 9.2]).

Now assume that $f_*(\mu) \in M_\tau(Y)$. For every compact set $K \subset \beta X - X$ we have $f(K) \subset \beta Y - Y$ and, by Theorem 1.1 (ii),

$$0 = \overline{f_*(\mu)}(\bar{f}(K)) = \bar{\mu}(\bar{f}^{-1}(\bar{f}(K)))$$
$$\geqq \bar{\mu}(K) \geqq 0 \; .$$

Therefore $\mu \in M_\tau^+(X)$.

The proof for $s = t$ or c is similar.

Proof of Theorem 2.1. We have $f_*(M_s^+(X)) \subset M_s^+(Y)$, so $f_*:$ $M_s^+(X) \to M_s^+(Y)$ is well-defined.

Suppose that f_* is perfect. Identifying X with the closed subset $\{\varepsilon_x: x \in X\}$ of $M_s^+(X)$, f is the restriction of f_* to this set, hence it is perfect.

Conversely, assume that f is perfect. Clearly $f_*: M_s^+(X) \to M_s^+(Y)$ is continuous and Lemmas 2.2 and 2.3 combine to show that f_* is onto.

To show that f_* is closed let F be a closed subset of $M_s^+(X)$, $\{\mu_\alpha\}$ a net in F and $\nu \in M_s^+(Y)$ such that $f_*(\mu_\alpha) \to \nu$. We have $\|\mu_\alpha\| = \mu_\alpha(1) = f_*(\mu_\alpha)(1) \to \nu(1)$. Now by Alaoglu-Bourbaki theorem $\{\mu_\alpha\}$ has a cluster point $\mu \in M^+(X)$. The continuity of f_* implies $f_*(\mu) = \nu$ and, by Lemma 2.3, $\mu \in M_s^+(X)$. Since F is closed in $M_s^+(X)$ we have $\mu \in F$ and $\nu \in f_*(F)$. Therefore $f_*(F)$ is closed in $M_s^+(Y)$.

It remains to show that $f_*^{-1}(\{\nu\})$ is compact for every $\nu \in M_s^+(Y)$. Clearly $f_*^{-1}(\{\nu\})$ is closed in $M_s^+(X)$ and by Lemma 2.3 it is closed in $M^+(X)$. For every $\mu \in f_*^{-1}(\{\nu\})$, we have $\|\mu\| = \mu(1) = f_*(\mu)(1) = \nu(1)$, so $f_*^{-1}(\{\nu\})$ is norm bounded. It follows that $f_*^{-1}(\{\nu\})$ is compact.

REMARK. Theorem 2.1 is not correct for the function $f_*: M_s(X) \to$ $M_s(Y)$, as simple examples show that $f_*^{-1}(\{\nu\})$ need not be norm bounded. Also, Theorem 2.1 is not correct for $s = \sigma$. Indeed, by [23, p. 189] there is a perfect function f from a realcompact space

X onto a space Y which is not realcompact. Then the set $F = \{\varepsilon_x : x \in X\}$ is closed in $M_\sigma^+(X)$, but $f_*(F) = \{\varepsilon_y : y \in Y\}$ is not closed in $M_\sigma^+(Y)$.

A space X is *Cech complete* if X is a G_δ set in βX. According to a theorem of Frolik (cf. [1, Theorem 9.4]), a space X is paracompact and Cech complete if and only if there is a perfect function from X onto a complete metric space. It follows that X is Lindelöf and Cech complete if and only if there is a perfect function from X onto a separable complete metric space. Using these results, Varadarajan's metrization theorem and Theorem 2.1 we have the following.

COROLLARY 2.4. *The space $M_s^+(X)$ (for $s = \tau$ or t) is*
(i) *paracompact and Cech complete, or*
(ii) *Lindelöf and Cech complete,*
if and only if X has property (i) *or* (ii) *respectively.*

Proof. First we observe that for a Cech complete space X we have $M_\tau(X) = M_t(X)$ since X is a Borel set in βX (cf. [13, Theorem 3.4]). If X has property (i) (resp. (ii)), then, by the above, there is a perfect function f from X onto a complete (resp. separable complete) metric space Y. By Theorem 2.1, the function $f_* : M_\tau^+(X) \to M_\tau^+(Y)$ is perfect, where $M_\tau^+(Y)$ is complete (resp. separable complete) metrizable space (Theorem 1.3). It follows that $M_\tau^+(X) = M_t^+(X)$ is paracompact (resp. Lindelöf) and Cech complete.

The converse follows from the fact that X is homeomorphic to a closed subset of $M_s^+(X)$.

A space X is called an *M-space* if there is a quasi-perfect function from X onto a metric space. (A continuous function $f : X \to Y$ is quasi-perfect if f is onto, closed and $f^{-1}(\{y\})$ is countably compact for every $y \in Y$). According to a theorem of Morita a space X is a paracompact *M*-space if and only if there is a perfect function from X onto a metric space (see Morita [18] and the reference given there for the theory of *M*-spaces). It follows that X is a Lindelöf *M*-space if and only if there is a perfect function from X onto a separable metric space. Analogously we have the following.

COROLLARY 2.5. *The space $M_s^+(X)$ (for $s = \tau$, t or c) is*
(i) *paracompact M-space, or*
(ii) *Lindelöf M-space,*
if and only if X has property (i) *or* (ii) *respectively.*

The following proposition which appears in Gröming [6] gives information about the topological properties which devolve from X to $M_s^+(X)$, for $s = \tau$ or t. For the sake of completeness we include a proof which is shorter than that of Grömig.

PROPOSITION 2.6. *The space X^N is homeomorphic to a closed subset of $M_\tau^+(X)$ which is contained in $M_t^+(X)$.*

Proof. We define

$$h: (\beta X)^N \to M^+(\beta X) \quad \text{by} \quad h((x_n)_{n \in N}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \varepsilon_{x_n} \ .$$

It is clear that h is $1 - 1$ and $h(X^N) \subset M_t^+(X)$.

To see that h is continuous, let $\{(x_n^\alpha)_{n \in N}\}_{\alpha \in A}$ be a net in $(\beta X)^N$ and $(x_n)_{n \in N} \in (\beta X)^N$ such that $x_n^\alpha \to x_n$ for every $n \in N$. Let $\mu_\alpha = h((x_n^\alpha)_{n \in N})$ and $\mu = h((x_n)_{n \in N})$. We shall show that $\mu_\alpha \to \mu$. Clearly $\mu_\alpha(\beta X) \to \mu(\beta X)$ and by [22, Part II, Theorem 2], it is enough to show that $\liminf \mu_\alpha(U) \geqq \mu(U)$ for every cozero set U of βX. Let $C = \{n \in N : x_n \in U\}$. For every finite set $F \subset C$ there is $\alpha_F \in A$ such that $x_n^\alpha \in U$ for all $\alpha > \alpha_F$ and $n \in F$ (since $x_n^\alpha \to x_n$). Then

$$\mu_\alpha(U) \geqq \sum_{n \in F} \frac{1}{2^n} \quad \text{for all} \quad \alpha > \alpha_F \ .$$

Therefore

$$\liminf \mu_\alpha(U) \geqq \sum_{n \in F} \frac{1}{2^n} \quad \text{for all finite} \quad F \subset C \ ,$$

that is, $\liminf \mu_\alpha(U) \geqq \sum_{n \in C} 1/2^n = \mu(U)$.

Since $(\beta X)^N$ is compact, h is a topological embedding. Moreover, $h(X^N)$ is contained in the compact set $h((\beta X)^N)$, therefore

$$\text{cl}_{M_\tau^+(X)} (h(X^N)) \subset M_\tau^+(X) \cap h((\beta X)^N) = h(X^N) \ .$$

The last equality follows from Theorem 1.1 (ii). So $h(X^N)$ is closed in $M_\tau^+(X)$ and this finishes the proof.

REMARK. Every finite product of X is homeomorphic to a closed subset of $M_\tau^+(X)$ which is contained in $M_c^+(X)$. Indeed, for every $m \in N$, we define

$$h_m: (\beta X)^m \to M^+(\beta X) \quad \text{by} \quad h_m((x_i)_{1 \leq i \leq m}) = \sum_{i=1}^{m} \frac{1}{2^i} \varepsilon_{x_i} \ .$$

Then $h_m(X^m) \subset M_c^+(X)$ and we can proceed as in the proof of Proposition 2.6.

As Grömig remarked in [6, p. 401], Proposition 2.6 and the above remark yield that if P is a topological property which is hereditary on closed subsets and devolves from X to $M_\tau^+(X)$ or $M_t^+(X)$ (resp. $M_c^+(X)$), then P devolves from X to the countable (resp. finite) products of X. Moreover, we know that if X is Lindelöf, then $X \times X$ may fail to be even normal. Thus Corollaries 2.4 and 2.5 are not correct for Lindelöf or paracompact spaces and it is not incidental that the topological properties considered in these corollaries devolve from X to X^N. However, the topological properties of Corollary 2.4 as well as Cech completeness do not devolve from X to $M_c^+(X)$ as the next proposition shows. Notice that Cech completeness is hereditary on closed subsets and devolves from X to X^N (see [3, pp. 144, 145]) and, more generally, from X to $M_\tau^+(X)$ and $M_t^+(X)$ (cf. [22, Part II, Theorem 17]).

PROPOSITION 2.7. $M_c^+(X)$ *is paracompact (or Lindelöf) and Cech complete if and only if X is compact.*

Proof. If X is compact, then $M_c^+(X) = M_t^+(X)$, so $M_c^+(X)$ has the desired properties.

For the converse, we show that if $M_c^+(X)$ is Cech complete, then X is pseudocompact; this is sufficient because every pseudocompact paracompact space is compact.

Let $f : X \to R$ be a continuous function. We define

$$U_n = \{x \in X : |f(x)| > n\}$$

and

$$F_n = \{\mu \in M_c^+(X) : \mu(U_n) = 0\}$$

for $n = 1, 2, \cdots$. Each F_n is closed in $M_c^+(X)$ and $M_c^+(X) = \bigcup_{n=1}^\infty F_n$ because every compact subset of X is contained in the complement of some U_n. Since every Cech complete space satisfies the Baire category theorem (see [3, p. 145]), some F_{n_0} has nonempty interior. Then $U_{n_0} = \varnothing$ and f is bounded. This finishes the proof.

REMARK.   Cech completeness of $M_c^+(X)$ does not imply compactness of X. Indeed, if W is the space of countable ordinals, then W is locally compact and every σ-compact subset of W is relatively compact. It follows that $M_c^+(W) = M_t^+(W)$ is Cech complete, although W is not compact.

We finish this section by showing that the analogues of Corollaries 2.4 and 2.5 for $M_s(X)$ fail badly. The following lemma which is essentially due to Corson (cf. Lemma 4 and the proof of Lemma 9 in [2]) will be needed.

LEMMA 2.8. *Let M be a vector subspace of $M(X)$. Then*
(i) *M satisfies the countable chain condition; and*
(ii) *for every continuous function f from M into a separable metric space, there is a countable $K \subset C(X)$ such that*

$$\mu, \nu \in M , \quad \mu|_K = \nu|_K \Longrightarrow f(\mu) = f(\nu) .$$

(i) follows from the fact that M is homeomorphic to a dense subset of R^I for some I (take I to be an algebraic basis of $C(X)$) and that R^I satisfies the countable chain condition. The arguments used in [20] to show that every continuous function from R^I into a separable metric space is determined by countably many coordinates apply for dense subsets of R^I and this proves (ii).

THEOREM 2.9. *The following are equivalent for $s = \tau$, t or c.*
(i) *$M_s(X)$ is an M-space;*
(ii) *$M_s(X)$ is Cech complete;*
(iii) *X is finite.*

Proof. (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii) are trivial.

(i) $\Longrightarrow$ (iii). There is a quasi-perfect function $F: M_s(X) \to Y$, where Y is metrizable. Since $M_s(X)$ satisfies the countable chain condition (Lemma 2.8 (i)), Y satisfies the same condition, so it is separable. By Lemma 2.8 (ii), F is determined by a countable set $K \subset C(X)$. The set

$$L = \{\mu \in M_s(X): \mu(f) = 0 \text{ for all } f \in K\}$$

is contained in the countably compact set $F^{-1}(F\{(0)\})$; also $\lambda \cdot \mu \in L$ for every $\lambda \in R$ and $\mu \in L$. It follows that $L = \{0\}$. Therefore the function

$$G: M_s(X) \longrightarrow R^K \quad \text{with} \quad G(\mu) = (\mu(f))_{f \in K}$$

is 1-1 and continuous. We define

$$H: M_s(X) \longrightarrow Y \times R^K \quad \text{by} \quad H(\mu) = (F(\mu), G(\mu)) .$$

Then $H: M_s(X) \to H(M_s(X))$ is quasi-perfect (cf. [16, Theorem 1.1]) and 1-1. Thus $M_s(X)$, being homeomorphic to a subspace of $Y \times R^K$, is metrizable.

We consider $M_s(X)$ as a dense subset of R^I, where I is an algebraic basis of $C(X)$, and let μ_0 be an element of $M_s(X)$ and $\{V_n\}$ a countable neighborhood base for μ_0 in $M_s(X)$. Then $\{\text{cl}_{R^I}(V_n)\}$ is a countable neighborhood base for μ_0 in R^I. It follows that I is at most countable. Since no Banach space is of algebraic dimension $\aleph_0$ (cf. [10, 14.32]), I is finite. Therefore X is finite.

(ii) $\Longrightarrow$ (iii). Let $F_n = \{\mu \in M_s(X): \|\mu\| \leq n\}$ for $n = 1, 2, \cdots$. Each

F_n is closed in $M_s(X)$ and $M_s(X) = \bigcup_{n=1}^{\infty} F_n$. Assume that X is infinite. Then each F_n has empty interior (cf. the proof of Theorem 16, Part II in [22]), so $M_s(X)$ is of the first category in itself and, in particular, it is not Cech complete.

3. Realcompactness. This section is concerned with the property of realcompactness in the spaces of measures. We recall that a completely regular space X is realcompact if for every $y \in \beta X - X$ there is a continuous function $f: \beta X \to [0, 1]$ such that $f(y) = 0$ and $f(x) > 0$ for all $x \in X$. The realcompactification of a space X is denoted by υX; of course X is realcompact if $X = \upsilon X$ (see [1]).

The following lemma is proved by Corson in [2] for the weak topology of a Banach space but essentially the same proof works in our case.

LEMMA 3.1 (cf. [2, Lemma 9]). *A vector subspace M of $M(X)$ is realcompact if and only if every $\mu \in M(X)$ with the property*

$$(*) \qquad \begin{cases} \textit{for every countable } K \subset C(X) \\ \textit{there is } \nu \in M \textit{ such that } \mu|_K = \nu|_K \end{cases}$$

is, in fact, in M. More precisely,

$$\upsilon M = \{\mu \in M(X) \colon \mu \textit{ satisfies } (*)\}$$

with the weak topology.

This lemma implies immediately that $\upsilon M_\sigma(X) = M_\sigma(X)$, that is, $M_\sigma(X)$ is realcompact for any space X. Moreover, if M_1, M_2 are vector subspaces of $M(X)$ and $M_1 \subset M_2$, then $\upsilon M_1 \subset \upsilon M_2 \subset M(X)$. Thus the relations between the spaces of measures imply the following diagram.

$$\begin{array}{ccccc} M_\sigma & \supset \upsilon M_\tau & \supset \upsilon M_t & \supset \upsilon M_c \\ & \cup & \cup & \cup \\ & M_\tau & \supset M_t & \supset M_c \,. \end{array}$$

Since a closed subset of a realcompact space is realcompact, in order that $M_s(X)$ (for $s = \tau$, t or c) be realcompact a necessary condition is that X be realcompact. Our purpose is to find when this condition is sufficient. First, we shall give another description of $\upsilon M_s(X)$ considering its elements as measures on X rather than linear functionals on $C(X)$.

LEMMA 3.2. *For every countable family H of bounded Baire measurable functions on X there is a countable set $K \subset C(X)$ such that, if μ and ν are σ-additive measures on X and $\mu|_K = \nu|_K$, then*

$$\int_X hd\mu = \int_X hd\nu \quad \text{for all} \quad h \in H.$$

Proof. Clearly it suffices to prove the lemma when H consists of a single bounded Baire measurable function h. This is trivial when h is continuous. In addition, if $\{h_n\}$ is a uniformly bounded sequence of Baire measurable functions which converges pointwise to some h and the conclusion is true for every h_n, then, by the Lebesgue's dominated convergence theorem, the conclusion is true for h. This completes the proof.

THEOREM 3.3. *Let* $\mu \in M_\sigma(X)$. *The following are equivalent (for $s = \tau, t$ or c):*
  (i) $\mu \in \upsilon M_s(X)$;
  (ii) *for every countable family H of bounded Baire measurable functions, there is $\nu \in M_s(X)$ such that $\displaystyle\int_X hd\mu = \int_X hd\nu$ for all $h \in H$;*
  (iii) *for every countably generated sub-σ-algebra $\mathscr{B}$ of $\mathscr{B}(X)$, there is $\nu \in M_s(X)$ such that $\mu(B) = \nu(B)$ for all $B \in \mathscr{B}$;*
  (iv) *for every continuous function f from X into a separable metric space, there is $\nu \in M_s(X)$ such that $f_*(\mu) = f_*(\nu)$.*

Proof. (i) $\Rightarrow$ (ii) is immediate from Lemmas 3.1 and 3.2.
  (ii) $\Rightarrow$ (iii). Let $\mathscr{B}$ be a countably generated sub-σ-algebra of $\mathscr{B}(X)$. By [11, I, § 2, Theorem 5] there is $f: X \to R$ Baire measurable such that $\mathscr{B} = f^{-1}(\mathscr{B}(R))$. Let $\{V_n\}$ be a countable base for the topology of R. We may suppose that $\{V_n\}$ is closed under finite unions. From (ii) it follows that there is $\nu \in M_s(X)$ such that $\mu(f^{-1}(V_n)) = \nu(f^{-1}(V_n))$ for all n, so $f_*(\mu) = f_*(\nu)$. Therefore $\mu(B) = \nu(B)$ for all $B \in \mathscr{B}$.
  (iii) $\Rightarrow$ (iv). This is trivial since $f^{-1}(\mathscr{B}(Y))$ is countably generated.
  (iv) $\Rightarrow$ (i). Let $K = \{f_n : n \in N\}$ be a countable subset of $C(X)$. We define

$$f: X \longrightarrow R^N \quad \text{by} \quad f(x) = (f_n(x))_{n \in N}.$$

By (iv), there is $\nu \in M_s(X)$ such that $f_*(\mu) = f_*(\nu)$. It follows that μ and ν coincide on $f_n^{-1}(\mathscr{B}(R))$, so $\mu(f_n) = \nu(f_n)$ for all n. From Lemma 3.1 it follows that $\mu \in \upsilon M_s(X)$.

REMARK. It is clear that Theorem 3.3 is valid for any vector subspace M of $M_\sigma(X)$ in the place of $M_s(X)$. So direction (i) $\Rightarrow$ (iii) implies that the space $M_p(X)$ of perfect measures on a completely regular space X (see [14]) is always realcompact. Since $M_p \supset M_t$ we have $M_\sigma \supset M_p \supset \upsilon M_t$.
  Some well-known properties of the spaces of measures remain

valid for their realcompactifications.

COROLLARY 3.4. *If* $\mu \in \upsilon M_s(X)$ *and* $A \in \mathscr{B}(X)$, *then the measure* μ_A *defined by*

$$\mu_A(B) = \mu(A \cap B) , \quad B \in \mathscr{B}(X)$$

is in $\upsilon M_s(X)$ *(hence* μ^+, μ^- *and* $|\mu|$ *are also in* $\upsilon M_s(X)$*) for* $s = \tau, t$ *or* c.

COROLLARY 3.5. *If* $f: X \to Y$ *is continuous, then* $f_*(\upsilon M_s(X)) \subset \upsilon M_s(Y)$ *for* $s = \tau, t$ *or* c.

PROPOSITION 3.6. *If* X *is realcompact, then* $M_c(X)$ *is realcompact.*

Proof. Let μ be in $\upsilon M_c(X)$ and S_μ be the support of the corresponding measure $\bar{\mu}$ on βX. It suffices to show that $S_\mu \subset X$. Let $y \in \beta X - X$. Since X is realcompact, there is $f: X \to [0, 1]$ continuous such that $\bar{f}(y) = 0$ and $\bar{f}(x) > 0$ for all $x \in X$, where $\bar{f}: \beta X \to [0, 1]$ is the continuous extension of f. By Theorem 3.3 (iv), there is $\nu \in M_c(X)$ such that $f_*(\mu) = f_*(\nu)$. It follows that 0 is not in the support of $f_*(\mu)$, hence $y \notin S_\mu$.

The analogue of Proposition 3.6 is not correct for the spaces M_τ and M_t. First, we prove the following.

PROPOSITION 3.7. *For any cardinal* α, $\upsilon M_t(\mathbf{R}^\alpha) = \upsilon M_\tau(\mathbf{R}^\alpha) = M_\sigma(\mathbf{R}^\alpha)$.

Proof. Let $\mu \in M_\sigma(\mathbf{R}^\alpha)$ and $f: \mathbf{R}^\alpha \to Y$ a continuous function, where Y is a separable metric space. By Theorem 3.3 (iv), it suffices to show that there is $\nu \in M_t(\mathbf{R}^\alpha)$ such that $f_*(\mu) = f_*(\nu)$.

It is known (cf. [20]) that f is determined by countably many coordinates, that is, there is $A \subset \alpha$ countable such that $f = g \circ p$, where $p: \mathbf{R}^\alpha \to \mathbf{R}^A$ is the projection and g is continuous. Let $q: \mathbf{R}^A \to \mathbf{R}^\alpha$ be a continuous function such that $p \circ q$ is the identity function of $\mathbf{R}^A$. Since $\mathbf{R}^A$ is separable complete metrizable, the measure $p_*(\mu)$ is tight. If $\nu = q_*(p_*(\mu))$, then $\nu \in M_t(\mathbf{R}^\alpha)$ and, for every $A \in \mathscr{B}(Y)$, we have

$$\begin{aligned} f_*(\mu)(A) &= p_*(\mu)(g^{-1}(A)) \\ &= p_*(q_*(p_*(\mu)))(g^{-1}(A)) \\ &= p_*(\nu)(g^{-1}(A)) \\ &= f_*(\nu)(A) . \end{aligned}$$

The above proposition in conjunction with some known results yields the following (a) and (b).

(a) $M_\tau(\boldsymbol{R}^c)$ and $M_t(\boldsymbol{R}^c)$ (where c is the continuum) are not real-compact, although $\boldsymbol{R}^c$ is realcompact.   Indeed, it is known that $M_\sigma(\boldsymbol{R}^c) \neq M_\tau(\boldsymbol{R}^c)$ (see [12] or [17]) and, of course, $M_\sigma(\boldsymbol{R}^c) \neq M_t(\boldsymbol{R}^c)$.

(b)   The proposition "$M_\tau(\boldsymbol{R}^{\aleph_1})$ is realcompact" is undecidable in ZFC, although $\boldsymbol{R}^{\aleph_1}$ is realcompact.   Indeed, Fremlin [4] showed that if we assume Martin's Axiom and the negation of the continuum hypothesis, then $M_\sigma(\boldsymbol{R}^{\aleph_1}) = M_\tau(\boldsymbol{R}^{\aleph_1})$.   On the other hand, if we assume the continuum hypothesis, then $M_\sigma(\boldsymbol{R}^{\aleph_1}) \neq M_\tau(\boldsymbol{R}^{\aleph_1})$ by the result mentioned in (a).

We notice that it is consistent with the negation of the continuum hypothesis for $M_\tau(\boldsymbol{R}^{\aleph_1})$ not to be realcompact (see [9]).   Also, $M_t(\boldsymbol{R}^{\aleph_1})$ is not realcompact whether or not $c = \aleph_1$ because $M_\sigma(\boldsymbol{R}^{\aleph_1}) \neq M_t(\boldsymbol{R}^{\aleph_1})$ ([4, Remark (b)]).

A space X with the countable chain condition is realcompact if and only if X is topologically complete (cf. [2, Lemma 8]).   Therefore Lemma 2.8 (i) and (b) above yield that the proposition "$M_\tau(\boldsymbol{R}^{\aleph_1})$ is topologically complete" is also undecidable in ZFC.   This shows that topological completeness is not a purely topological property.   See also [8, p. 12].

In the next theorem we show that realcompactness devolves from X to $M_\tau(X)$ and $M_t(X)$, if X is metrizable.   We recall that, by (a special form of) Katetov's theorem (cf. [1, Theorem 6.2]), a metric space X is realcompact if and only if every closed discrete subset of X has non-(Ulam-) measurable cardinal.   Also, a σ-additive measure on X is τ-additive if and only if it is supported by a closed separable subset.

THEOREM 3.8.   *If X is a realcompact metric space, then $M_\tau(X)$ and $M_t(X)$ are realcompact.*

Proof.   Let $\mu \in \upsilon M_\tau(X)$, $\mu \geq 0$.   By Corollary 3.4, in order to prove that $M_\tau(X)$ is realcompact, it suffices to show that $\mu \in M_\tau(X)$.   Let $\mathscr{U}$ be the family of all open sets of μ-measure zero.   The set $X - \cup \mathscr{U}$ satisfies the countable chain condition, so it is separable and it is enough to show that $\mu(\cup \mathscr{U}) = 0$.   By Stone's theorem there is a σ-discrete refinement $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}_n$, each $\mathscr{V}_n$ discrete and $\cup \mathscr{V} = \cup \mathscr{U}$.   If we suppose that $\mu(\cup \mathscr{U}) > 0$, then $\mu(\cup \mathscr{V}_n) > 0$ for some n.   Fix such an n.

For each $V \in \mathscr{V}_n$ we choose a point $x_V \in V$ and set $T = \{x_V : V \in \mathscr{V}_n\}$. Let $f : X \to T$ with $f|_V = x_V$ for all $V \in \mathscr{V}_n$ and $f|_{X - \cup \mathscr{V}_n}$ constant.   The cardinal of T if nonmeasurable because T is closed and discrete in X which is assumed to be realcompact.   Thus, the image measure $\kappa = f_*(\mu_{\cup \mathscr{V}_n})$ which is defined on all subsets of T and vanishes on singletons is non-atomic.   It follows (cf. [14, 2.1]) that there is $g : T \to$

[0, 1] such that $g_*(\kappa) = \alpha \cdot \lambda$, where $\alpha = \mu(\cup \mathscr{V}_n) > 0$ and λ is the Lebesgue measure on the Baire sets of [0, 1]. If $h = g \circ f$, then $h_*(\mu_{\cup \mathscr{V}_n}) = \alpha \cdot \lambda$. Since $h^{-1}(\mathscr{B}([0, 1]))$ is countably generated and $\mu_{\cup \mathscr{V}_n} \in \upsilon M_\tau(X)$ (Corollary 3.4), there is $\nu \in M_\tau(X)$ such that $h_*(\mu_{\cup \mathscr{V}_n}) = h_*(\nu)$ (Theorem 3.3). By the τ-additivity, ν is supported by $\cup \mathscr{W} \cup (X - \cup \mathscr{V}_n)$ for some countable $\mathscr{W} \subset \mathscr{V}_n$. But $h(\cup \mathscr{W} \cup (X - \cup \mathscr{V}_n))$ is countable, so $h_*(\nu)$ is supported by a countable set. This is a contradiction since $h_*(\nu) = \alpha \cdot \lambda$.

To show that $M_t(X)$ is realcompact let $\mu \in \upsilon M_t(X)$. Then $\mu \in \upsilon M_\tau(X) = M_\tau(X)$ by the above. So, μ is supported by a closed separable set S. Since $\mathscr{B}(S)$ is countably generated, it follows from Theorem 3.3, that $\mu \in M_t(X)$.

Theorem 3.8 at least for the space of τ-additive measures remains valid when X is paracompact. Indeed, the only properties of metric spaces which are actually used are paracompactness and that every open set is a Baire set. However, the latter is not essential (cf. the proof of Theorem 5.10 in [14]). In any case, this result will be deduced as corollary of the next theorem. First we give some notations and definitions related to the notion of continuous pseudometric.

We denote by $\mathscr{D}$ the family of all continuous pseudometrics on a completely regular space X. If $d \in \mathscr{D}$, let π_d be the natural projection from X onto the corresponding metric space X_d. A subset A of X is d-*discrete* if there exists $\varepsilon > 0$ such that $d(x, y) \geqq \varepsilon$ for all $x, y \in A$, $x \neq y$. As in [15], we say that X is a D_0-*space* if, for every $d \in \mathscr{D}$, all d-discrete subsets of X have nonmeasurable cardinal. It follows from Katetov's theorem that X is a D_0-space if and only if X_d is realcompact for every $d \in \mathscr{D}$ (cf. [15]).

The space $M_u(X)$ of the u-additive measures on X can be defined as follows: A measure μ on X is in $M_u(X)$ if $(\pi_d)_*(\mu) \in M_\tau(X_d)$ for all $d \in \mathscr{D}$. We have $M_o(X) \supset M_u(X) \supset M_\tau(X)$ and $M_u(X) = M_\tau(X)$ whenever X is paracompact (see [21]).

THEOREM 3.9. *For any space* X *the following are equivalent*:
(i) $M_u(X)$ *is realcompact*;
(ii) X *is a* D_0-*space*.

Proof. (i) $\Rightarrow$ (ii). The topological completion θX of X (see [1] for the definition of this term) is homeomorphic to the closed subset of $M_u(X)$ of the 2-valued u-additive measures, hence θX is realcompact. For every $d \in \mathscr{D}$, let $\bar{\pi}_d : \theta X \to X_d$ be the continuous extension of π_d. By the Katetov's theorem X_d is realcompact as a metrizable continuous image of a realcompact space.

(ii) ⟹ (i). Let $\mu \in \upsilon M_u(X)$ and $d \in \mathscr{D}$. As in Corollary 3.5 we have $(\pi_d)_*(\mu) \in \upsilon M_u(X_d)$. But $\upsilon M_u(X_d) = \upsilon M_\tau(X_d) = M_\tau(X_d)$ by Theorem 3.8. Therefore $\mu \in M_u(X)$.

Since $M_u(X) = M_\tau(X)$ for paracompact spaces and realcompact spaces are D_0-spaces, we have:

COROLLARY 3.10. *If X is a realcompact paracompact space, then $M_\tau(X)$ is realcompact.*

I don't know if the analogue of this corollary is true for the space of tight measures. (However, it is true for paracompact M-spaces).

PROPOSITION 3.11. (i) $M_u^+(X)$ *is topologically complete for any space* X; (ii) $M_c^+(X)$ *is topologically complete if (and only if) X is topologically complete.*

Proof. (i) For every $d \in \mathscr{D}$, the function $(\pi_d)_*: M^+(X) \to M^+(X_d)$ is continuous and we have

$$M_u^+(X) = \bigcap_{d \in \mathscr{D}} (\pi_d)_*^{-1}(M_\tau^+(X_d)) \ .$$

The spaces $M^+(X)$ and $M_\tau^+(X_d)$ are topologically complete since

$$M^+(X) = \bigcup_{n=1}^{\infty} \{\mu \in M^+(X): \mu(X) \leqq n\}$$

is σ-compact and $M_\tau^+(X_d)$ is metrizable (Theorem 1.2). It follows that $(\pi_d)_*^{-1}(M_\tau^+(X_d))$ is topologically complete for every $d \in \mathscr{D}$, so $M_u^+(X)$ is topologically complete as an intersection of such spaces (see [1, 1.8 and 1.10]).

(ii) If X is topologically complete, for every $y \in \beta X - X$ there is $d \in \mathscr{D}$ such that $\bar{\pi}_d(y) \in \beta X_d - X_d$, where $\bar{\pi}_d: \beta X \to \beta X_d$ is the continuous extension of π_d ([1, Theorems 3.2 and 4.4 (d)]). As in Proposition 3.6 we can show that a measure $\mu \in M^+(X)$ has compact support if and only if $(\pi_d)_*(\mu) \in M_c^+(X_d)$ for every $d \in \mathscr{D}$. Since $M_c^+(X) \subset M_\tau^+(X_d)$ is metrizable, using similar arguments as in (i), we find that $M_c^+(X)$ is topologically complete.

REMARKS. Proposition 3.11 (ii) is not much different from Proposition 3.6 since a topologically complete space of nonmeasurable cardinal is realcompact by the Shirota theorem ([1, Theorem 6.3]). However, the restriction to the subset of positive measures is necessary. Indeed, if X is a discrete space then $M_c^+(X)$ is topologically complete (Prop. 3.11 (ii)), while $M_c(X)$ is topologically complete if

and only if the cardinal of X is nonmeasurable (Lemma 2.8 (i) and Prop. 3.6).

A similar argument shows that Proposition 3.11 (i) is not correct for $M_u(X)$.

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