PACIFIC JOURNAL OF MATHEMATICS
Vol. 96, No. 2, 1981

DERIVATIONS OF OPERATOR ALGEBRAS INTO SPACES OF UNBOUNDED OPERATORS

ATSUSHI INOUE, SCHÔICHI ÔTA AND JUN TOMIYAMA

This paper is to study the spatiality of unbounded derivations in operator algebras. Let $\mathscr{M}$ be a von Neumann algebra (C^*-algebra) on a Hilbert space $\mathfrak{H}$ and δ be an unbounded derivation in $\mathscr{M}$. In this paper, extending δ to a derivation $\hat{\delta}$ of $\mathscr{M}$ into a certain space of unbounded operators, we study the spatiality of δ by investigating the property of $\hat{\delta}$.

1. Introduction. Unbounded derivations in operator algebras (C^*-algebras and von Neumann algebras) have recently been investigated by many authors, since they are appeared as infinitesimal generators of strongly continuous one-parameter groups of *-automorphisms on C^*-algebras [see; 12]. In particular, the infinitesimal generator mentioned above is implemented by a symmetric operator by giving some representation of its C^*-algebra on a Hilbert space, and there exist many closed derivations in C^*-algebras which possess such a property [2]. In this point of view, we shall study the spatiality of unbounded derivations in operator algebras (see [2]; Problem). Our method is, roughly speaking, to examine the spatiality of an unbounded derivation δ in an operator algebra $\mathscr{M}$ by extending δ to a derivation of $\mathscr{M}$ into some space of unbounded operators containing $\mathscr{M}$.

Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a *-derivation in $\mathscr{M}$ with σ-strongly dense domain $\mathscr{D}(\delta)$. Let $\mathscr{D}$ be a dense subspace of $\mathfrak{H}$. We introduce various locally convex topologies in the space $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ which is the set of all linear operators T of $\mathscr{D}$ into $\mathfrak{H}$ with $\mathscr{D}(T^*) \supset \mathscr{D}$, and extend δ to a $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ assuming corresponding continuity of δ in these topologies.

We shall then examine under what conditions the continuous $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ with some specified topology is spatial, i.e., there exists an element H of $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ such that $\hat{\delta}(A)\xi = [H, A]\xi = \{HA - AH\}\xi$ for all $A \in \mathscr{M}$ and $\xi \in \mathscr{D}$. We call the dense subspace $\mathscr{D}$ countably dominated by a sequence $\{T_n\}$ of closed operators if $\mathscr{D} = \bigcap_{n=1}^{\infty} \mathscr{D}(T_n)$ and $\| T_n\xi \| \leqq \| T_{n+1}\xi \|$ for each $\xi \in \mathscr{D}$ and $n = 1, 2, \cdots$.

Our first result (Theorem 4.11) shows that if $\mathscr{M}$ is a left von Neumann algebra of a Hilbert algebra $\mathfrak{A}$ with identity and $\mathscr{D}$ is countably dominated by $\{T_n\}$ of closed operators then $\hat{\delta}$ is spatial.

The second purpose of this paper is to show (Theorem 4.15) that if $\mathscr{M}$ has certain property (Definition 4.2) and $\mathscr{D}$ is countably dominated by $\{T_n\}$ of closed operators $\eta\mathscr{M}'$ then $\hat{\hat{\delta}}$ is a spatial $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$.

2. Spaces of unbounded operators. Let $\mathfrak{H}$ be a Hilbert space with inner product $(\,|\,)$ and let $\mathscr{D}$ be a dense subspace of $\mathfrak{H}$. We denote by $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ (resp. $\mathscr{L}_c(\mathscr{D}, \mathfrak{H})$) the space of all (resp. closable) linear operators of $\mathscr{D}$ into $\mathfrak{H}$ and by $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ the space of operators A in $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ for which there exists the adjoints A^* whose domains $\mathscr{D}(A^*)$ contain $\mathscr{D}$. For each $T \in \mathscr{L}(\mathscr{D}, \mathfrak{H})$ we define

$$\|A\|_T = \sup_{\xi \in \mathscr{D}} \frac{\|A\xi\|}{\|T\xi\|}, \quad A \in \mathscr{L}(\mathscr{D}, \mathfrak{H}),$$

where $(\lambda/0) = \infty$ for $\lambda > 0$ and $(0/0) = 0$,

$$\mathfrak{M}_T = \{A \in \mathscr{L}(\mathscr{D}, \mathfrak{H});\ \|A\|_T < \infty\}$$

and

$$\mathfrak{M}_T^{\#} = \{A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H});\ \|A\|_T < \infty\}.$$

Then it is easily seen that $\mathfrak{M}_T$ is a Banach space equipped with the norm $\|\cdot\|_T$ and $\mathfrak{M}_T^{\#}$ is a subspace of $\mathfrak{M}_T$.

The following lemma is an immediate consequence of the definitions of the spaces of $\mathfrak{M}_T$ and $\mathfrak{M}_T^{\#}$.

LEMMA 2.1. *Let T be an element of $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ such that $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$, where $\mathscr{B}(\mathfrak{H})$ denotes the algebra of all bounded linear operators on $\mathfrak{H}$. We set*

$$\mathscr{B}_T = \{\overline{AT^{-1}};\ A \in \mathfrak{M}_T\} \quad \text{and} \quad \mathscr{B}_T^{\#} = \{\overline{AT^{-1}};\ A \in \mathfrak{M}_T^{\#}\}.$$

Then the map $\phi: A \to \overline{AT^{-1}}$ is an isometric isomorphism of the Banach space $\mathfrak{M}_T$ onto the Banach space $\mathscr{B}(\mathfrak{H})$.

LEMMA 2.2. *Let $\mathfrak{H}$ be a Hilbert space with inner product $(\,|\,)$. If there exists a sequence $\{T_n\}$ of closed operators on $\mathfrak{H}$ such that*

(1) $\mathscr{D} = \bigcap_{n=1}^{\infty} \mathscr{D}(T_n)$ *is dense in* $\mathfrak{H}$;

(2) $\|T_n\xi\| \leqq \|T_{n+1}\xi\|$ *for all* $\xi \in \mathscr{D}$ *and* $n = 1, 2, \cdots$, *then* $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H}) = \bigcup_{n=0}^{\infty} \mathfrak{M}_{T_n}^{\#}$ *where* $T_0 = I$.

Proof. For each $\xi \in \mathscr{D}$ we set

$$\|\xi\|_{T_n} = \|T_n\xi\| \quad \text{for} \quad n = 0, 1, 2 \cdots.$$

We consider the locally convex topology $t_{\{T_n\}}$ on $\mathscr{D}$ generated by family of the seminorms $\|\cdot\|_{T_n}(n=0,1,2,\cdots)$. Suppose that $\{\xi_k\}$ is a Cauchy sequence in $(\mathscr{D}, t_{\{T_n\}})$. Then we have

$$\lim_{k,l\to\infty}\|\xi_k-\xi_l\|=0 \quad\text{and}\quad \lim_{k,l\to\infty}\|T_n\xi_k-T_n\xi_l\|=0 \qquad \text{for}\quad n=1,2,\cdots .$$

Since T_n is a closed operator, it follows that $x\in\mathscr{D}(T_n)$ and $\lim_{k\to\infty}T_n\xi_k=T_nx$ for $n=1,2,\cdots$. Hence we have $x\in\bigcap_{n=1}^{\infty}\mathscr{D}(T_n)=\mathscr{D}$ and $\lim_{k\to\infty}T_n\xi_k=T_nx$ for $n=1,2,\cdots$. This implies that $(\mathscr{D}, t_{\{T_n\}})$ is a Fréchet space.

Suppose $S\in\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})$. We show that the graph of S: $G(S)\equiv\{\langle\xi, S\xi\rangle;\ \xi\in\mathscr{D}\}$ is closed in $(\mathscr{D}, t_{\{T_n\}})\times\mathfrak{H}$. Suppose that a sequence $\{\langle\xi_n, S\xi_n\rangle\}$ in $G(S)$ converges to an element $\langle\xi, y\rangle$ of $\mathscr{D}\times\mathfrak{H}$. It then follows that $\xi_n-\xi\in\mathscr{D}$, $\lim_{n\to\infty}\|\xi_n-\xi\|=0$ and $\lim_{n\to\infty}\|S(\xi_n-\xi)-(y-S\xi)\|=0$. Since S is closable, we have $y=S\xi$. This implies that $G(S)$ is closed in $(\mathscr{D}, t_{\{T_n\}})\times\mathfrak{H}$. By the closed graph theorem it follows that the map $S\colon(\mathscr{D}, t_{\{T_n\}})\to\mathfrak{H}$ is continuous. Hence there exist a number n and a constant $\gamma>0$ such that

$$\|S\xi\|\leqq\gamma\|T_n\xi\| \quad\text{for all}\quad \xi\in\mathscr{D}.$$

Therefore, $S\in\mathfrak{M}^{\#}_{T_n}$. This implies that $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})=\bigcup_{n=0}^{\infty}\mathfrak{M}^{\#}_{T_n}$.

DEFINITION 2.3. Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathfrak{H}$. If there exists a sequence $\{T_n\}$ of closed operators in $\mathfrak{H}$ such that $\mathscr{D}=\bigcap_{n=1}^{\infty}\mathscr{D}(T_n)$ and $\|T_n\xi\|\leqq\|T_{n+1}\xi\|$ for all $\xi\in\mathscr{D}$ and $n=1,2,\cdots$, then $\mathscr{D}$ is said to be countably dominated by $\{T_n\}$. If there exists a sequence $\{S_n\}$ in $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})$ such that $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})=\bigcup_{n=1}^{\infty}\mathfrak{M}^{\#}_{S_n}$ and $\|S_n\xi\|\leqq\|S_{n+1}\xi\|$ for all $\xi\in\mathscr{D}$ and $n=1,2,\cdots$, then $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})$ is said to be countably dominated by $\{S_n\}$.

REMARK. (1) Lemma 2.2 implies that if a pre-Hilbert space $\mathscr{D}$ is countably dominated then $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})$ is also countably dominated.

(2) It will be seen, by a simple calculation, that if $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})=\bigcup_{n=1}^{\infty}\mathfrak{M}^{\#}_{S_n}$ for $S_n\in\mathscr{L}^{\#}(\mathscr{D})\equiv\mathscr{L}^{\#}(\mathscr{D},\mathscr{D})(n=1,2,\cdots)$, then $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})$ is countably dominated.

Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathfrak{H}$. We now introduce some locally convex topologies on $\mathscr{L}^{\#}(\mathscr{D},\mathfrak{H})$. We put

$$P_{\xi,x}(A)=|(A\xi|x)|\,,$$
$$P_{\xi}(A)=\|A\xi\|\,,$$

where $A\in\mathscr{L}(\mathscr{D},\mathfrak{H})$, $\xi\in\mathscr{D}$ and $x\in\mathfrak{H}$. The locally convex topology on $\mathscr{L}(\mathscr{D},\mathfrak{H})$ generated by the seminorms $\{P_{\xi,\eta}(\cdot);\ \xi,\eta\in\mathscr{D}\}$ (resp.

$\{P_{\xi,x}(\cdot);\ \xi \in \mathscr{D},\ x \in \mathfrak{H}\}$, $\{P_\xi(\cdot);\ \xi \in \mathscr{D}\}$) is said to be the weak topology (resp. quasi-weak topology, strong topology) and is simply denoted by $t_w^{\mathscr{D}}$(resp. $t_{qw}^{\mathscr{D}}$, $t_s^{\mathscr{D}}$).

Let $\mathfrak{H}_\infty$ be the Hilbert direct sum of the Hilbert spaces $\mathfrak{H}_n \equiv \mathfrak{H}(n = 1, 2, \cdots)$ and let

$$\mathscr{D}_\infty(\mathscr{D}) = \{\{\xi_n\} \in \mathfrak{H}_\infty;\ \xi_n \in \mathscr{D} \quad \text{for} \quad n = 1, 2, \cdots \\ \text{and} \quad \sum_{n=1}^{\infty} \| A\xi_n \|^2 < \infty \quad \text{for all} \quad A \in \mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})\} .$$

We set

$$P_{\{\xi_n\},\{x_n\}}(A) = \left| \sum_{n=1}^{\infty} (A\xi_n | x_n) \right| ,$$
$$P_{\{\xi_n\}}(A) = \left[\sum_{n=1}^{\infty} \| A\xi_n \|^2 \right]^{1/2} ,$$

where $A \in \mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})$, $\{\xi_n\} \in \mathscr{D}_\infty(\mathscr{D})$ and $\{x_n\} \in \mathscr{D}_\infty$. We equip $\mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})$ with the locally convex topology $t_{\sigma w}^{\mathscr{D}}$(resp. $t_{q\sigma w}^{\mathscr{D}}$, $t_{\sigma s}^{\mathscr{D}}$) induced by the seminorms $\{P_{\{\xi_n\},\{\eta_n\}}(\cdot);\ \{\xi_n\}, \{\eta_n\} \in \mathscr{D}_\infty(\mathscr{D})\}$ (resp. $\{P_{\{\xi_n\},\{x_n\}}(\cdot);\ \{\xi_n\} \in \mathscr{D}_\infty(\mathscr{D}),\ \{x_n\} \in \mathfrak{H}_\infty\}$, $\{P_{\{\xi_n\}}(\cdot);\ \{\xi_n\} \in \mathscr{D}_\infty(\mathscr{D})\}$). The topology $t_{\sigma w}^{\mathscr{D}}$ (resp. $t_{q\sigma w}^{\mathscr{D}}$, $t_{\sigma s}^{\mathscr{D}}$) is said to be the σ-weak topology (resp. quasi-σ-weak topology, σ-strong topology) on $\mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})$.

We next define the uniform topology and the quasi-uniform topology. A subset $\mathfrak{M}$ of $\mathscr{D}$ is said to be $\mathscr{D}$-bounded if

$$\sup_{\xi \in \mathfrak{M}} \| A\xi \| < \infty \quad \text{for each} \quad A \in \mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H}) .$$

We then define

$$P_{\mathfrak{M}}(A) = \sup_{\xi\ \eta \in \mathfrak{M}} |(A\xi | \eta)| ,$$
$$P^{\mathfrak{M}}(A) = \sup_{\xi \in \mathfrak{M}} \| A\xi \| ,$$

where $\mathfrak{M}$ is $\mathscr{D}$-bounded and $A \in \mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})$. The locally convex topology generated by the seminorms $\{P_{\mathfrak{M}}(\cdot);\ \mathfrak{M}$ is $\mathscr{D}$-bounded$\}$ (resp. $\{P^{\mathfrak{M}}(\cdot);\ \mathfrak{M}$ is $\mathscr{D}$-bounded$\}$) is said to be the uniform topology (resp. quasi-uniform topology) on $\mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})$ and is simply denoted by $t_u^{\mathscr{D}}$ (resp. $t_{qu}^{\mathscr{D}}$).

We next define the ρ-topology and λ-topology on $\mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})$. For each $T \in \mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H})$ we put

$$\rho_T(A) = \sup_{\xi \in \mathscr{D}} \frac{|(A\xi | \xi)|}{\| T\xi \|^2} , \quad A \in \mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H}) ,$$

where $(\lambda/0) = \infty$ for $\lambda > 0$ and $0/0 = 0$, and

$$\mathfrak{N}_T^\sharp = \{A \in \mathscr{L}^\sharp(\mathscr{D}, \mathfrak{H});\ \rho_T(A) < \infty\} .$$

Then it is easily seen that $\mathfrak{N}_T^\#$ is a normed space equipped with the norm $\rho_T(\cdot)$ and $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H}) = \bigcup_{T \in \mathscr{L}^\#(\mathscr{D},\mathfrak{H})} \mathfrak{N}_T^\#$. The inductive limit topology on $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ with respect to the normed spaces $\{(\mathfrak{N}_T^\#, \rho_T(\cdot)); T \in \mathscr{L}^\#(\mathscr{D}, \mathfrak{H})\}$ (resp. $\{(\mathfrak{M}_T^\#, \|\cdot\|_T); T \in \mathscr{L}^\#(\mathscr{D}, \mathfrak{H})\}$) is said to be the ρ-topology (resp. λ-topology) on $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ and is denoted by $t_\rho^{\mathscr{D}}$ (resp. $t_\lambda^{\mathscr{D}}$).

Now one may easily see the following lemma by the definitions of the topologies.

LEMMA 2.4. *The relation among the topologies introduced here are as follows*:

$$t_\lambda^{\mathscr{D}} \geqq t_\rho^{\mathscr{D}} \geqq \left\{\begin{array}{ccccc} t_u^{\mathscr{D}} & & \leqq & & t_{qu}^{\mathscr{D}} \\ \text{\rotatebox{90}{$\leqq$}} & & & & \text{\rotatebox{90}{$\leqq$}} \\ t_w^{\mathscr{D}} & \leqq & t_{qw}^{\mathscr{D}} & \leqq & t_s^{\mathscr{D}} \\ \text{\rotatebox{90}{$\geqq$}} & & & & \text{\rotatebox{90}{$\geqq$}} \\ t_{\sigma w}^{\mathscr{D}} & \leqq & t_{q\sigma w}^{\mathscr{D}} & \leqq & t_{\sigma s}^{\mathscr{D}} \end{array}\right\} \leqq t_\lambda^{\mathscr{D}},$$

where the symbols $\tau_1 \leqq \tau_2$, $\tau_2 \geqq \tau_1$, $\begin{array}{c}\tau_1\\ \text{\rotatebox{90}{$\geqq$}}\\ \tau_2\end{array}$ *and* $\begin{array}{c}\tau_2\\ \text{\rotatebox{90}{$\leqq$}}\\ \tau_1\end{array}$ *mean the topology* τ_2 *is finer than the topology* τ_1.

REMARK. The topologies $t_u^{\mathscr{D}}$ and $t_{qu}^{\mathscr{D}}$ (resp. the topologies $t_\rho^{\mathscr{D}}$ and $t_\lambda^{\mathscr{D}}$) on $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ are generalizations of the uniform topology and quasi-uniform one (resp. the ρ-topology and λ-topology) introduced by G. Lassner [8] (resp. D. Arnal and J. P. Jurzak [1]), for an unbounded operator algebra respectively. We denote by t_u (resp. t_w, t_s, $t_{\sigma w}$, $t_{\sigma s}$) the usual uniform (resp. weak, strong, σ-weak, σ-strong) topology on $\mathscr{B}(\mathfrak{H})$. The relations between the topologies on $\mathscr{B}(\mathfrak{H})$ are as follows: $t_u^{\mathfrak{H}} = t_{qu}^{\mathfrak{H}} = t_\rho^{\mathfrak{H}} = t_\lambda^{\mathfrak{H}} = t_u$, $t_w^{\mathfrak{H}} = t_{qw}^{\mathfrak{H}} = t_w$, $t_s^{\mathfrak{H}} = t_s$, $t_{\sigma w}^{\mathfrak{H}} = t_{q\sigma w}^{\mathfrak{H}} = t_{\sigma w}$ and $t_{\sigma s}^{\mathfrak{H}} = t_{\sigma s}$.

LEMMA 2.5. *Suppose that* $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ *is countably dominated by* $\{T_n\}$ *and* $\mathfrak{N}$ *is a subset of* $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$. *Then the following statements are equivalent*:

(1) $\mathfrak{N}$ *is* $t_\rho^{\mathscr{D}}$*-bounded*;

(2) $\mathfrak{N}$ *is* $t_u^{\mathscr{D}}$*-bounded*;

(3) *there exist a number* n *and a constant* $\gamma > 0$ *such that*

$$|(A\xi \mid \xi)| \leqq \gamma \|(I + |\overline{T_n}|)\xi\| \quad \textit{for all} \quad A \in \mathfrak{N} \quad \textit{and} \quad \xi \in \mathscr{D},$$

where $\overline{T_n} = U|\overline{T_n}|$ *is the polar decomposition of* $\overline{T_n}$.

Proof. This is proved in the same way as in ([13] Lemma 2.1).

LEMMA 2.6. *Suppose that* $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{S})$ *is countably dominated by* $\{T_n\}$ *and* $\mathfrak{N}$ *is a subset of* $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{S})$. *Then the following statements are equivalent:*

(1) $\mathfrak{N}$ *is* $t_\lambda^{\mathscr{D}}$*-bounded;*

(2) $\mathfrak{N}$ *is* $t_{qu}^{\mathscr{D}}$*-bounded;*

(3) $\mathfrak{N}$ *is* $t_{\sigma s}^{\mathscr{D}}$*-bounded;*

(4) *there exists a number* n *and a constant* $\gamma > 0$ *such that*

$$\| A\xi \| \leqq \gamma \| (I + |\bar{T}_n|)\xi \| \quad \textit{for all} \quad A \in \mathfrak{N} \quad \textit{and} \quad \xi \in \mathscr{D} .$$

Furthermore, if $\mathscr{D} = \bigcap_{T \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{S})} \mathscr{D}(\bar{T})$, *then the statements* (1)$\sim$(4) *are equivalent to the following statements* (5) *and* (6):

(5) $\mathfrak{N}$ *is* $t_s^{\mathscr{D}}$*-bounded;*

(6) $\mathfrak{N}$ *is* $t_{qw}^{\mathscr{D}}$*-bounded.*

Proof. Since $t_\lambda^{\mathscr{D}} \geqq t_{qu}^{\mathscr{D}}$ and $t_\lambda^{\mathscr{D}} \geqq t_{\sigma s}^{\mathscr{D}}$, one can see the implications (4) $\Rightarrow$ (1), (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). We show the implication (3) $\Rightarrow$ (4). Suppose that the statement (4) is not true. Then there exists a sequence $\{A_n\}$ in $\mathfrak{N}$ and a sequence $\{\xi_n\}$ of nonzero elements of $\mathscr{D}$ such that

$$\| A_n\xi_n \| \geqq n^2 \| (I + |\bar{T}_n|)\xi_n \| \quad \text{for} \quad n = 1, 2, \cdots .$$

Putting

$$\eta_n = \frac{\xi_n}{n\|(I + |\bar{T}_n|)\xi_n\|} \quad \text{for} \quad n = 1, 2, \cdots ,$$

we have

$$\| A_n\eta_n \| \geqq n \quad \text{and} \quad \| T_n\eta_n \| < \frac{1}{n} .$$

We now show $\{\eta_n\} \in \mathscr{D}_\infty(\mathscr{D})$. Since $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{S}) = \bigcup_{n=1}^{\infty} \mathfrak{M}_{T_n}^{\#}$, it follows that for each $A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{S})$ there exists a number k and a constant $\gamma > 0$ such that

$$\| A\xi \| \leqq \gamma \| T_k\xi \| \quad \text{for all} \quad \xi \in \mathscr{D} .$$

Then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \| A\eta_n \|^2 &\leqq \gamma \sum_{n=1}^{\infty} \| T_k\eta_n \|^2 \\
&\leqq \gamma \left\{ \sum_{n=1}^{k-1} \| T_k\eta_n \|^2 + \| T_k\eta_k \|^2 + \| T_k\eta_{k+1} \|^2 + \cdots \right\} \\
&\leqq \gamma \left\{ \sum_{n=1}^{k-1} \| T_k\eta_n \|^2 + \| T_k\eta_k \|^2 + \| T_{k+1}\eta_{k+1} \|^2 + \cdots \right\}
\end{aligned}$$

$$\leqq \gamma \left\{ \sum_{n=1}^{k-1} \| T_k \eta_n \|^2 + \frac{1}{k^2} + \frac{1}{(k+1)^2} + \cdots \right\}$$
$$< \infty .$$

This means $\{\eta_n\} \in \mathscr{D}_\infty(\mathscr{D})$. Furthermore, we have

$$\sup_{A \in \mathfrak{N}} P_{\{\eta_n\}}(A) = \sup_{A \in \mathfrak{N}} \left[\sum_{n=1}^{\infty} \| A\eta_n \|^2 \right]^{1/2}$$
$$\geqq \| A_n \eta_n \| \geqq n .$$

This contradicts that $\mathfrak{N}$ is $t_{\sigma s}^{\mathscr{D}}$-bounded. This comqletes the proof of the implication (3) $\Rightarrow$ (4).

The implication (2) $\Rightarrow$ (4) is proved in the same way as in ([13] Lemma 2.2).

If $\mathscr{D} = \bigcap_{T \in \mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})} \mathscr{D}(\bar{T})$, the equivalence of the statements (1)$\sim$(6) follows from ([1] Proposition 1.6).

3. Extension of derivations.

Let $\mathscr{M}$ be a C^*-algebra (or a von Neumann algebra). A linear map $\delta: \mathscr{D}(\delta) \subset \mathscr{M} \to \mathscr{M}$ is said to be a *$*$-derivation in $\mathscr{M}$ if it satisfies the following conditions:*

(1) the domain $\mathscr{D}(\delta)$ of δ is a dense $*$-subalgebra of $\mathscr{M}$ (i.e., $\mathscr{D}(\delta)$ is norm-dense if $\mathscr{M}$ is a C^*-algebra, and weak-dense if $\mathscr{M}$ is a von Neumann algebra);

(2) $\delta(AB) = \delta(A)B + A\delta(B)$ for each $A, B \in \mathscr{D}(\delta)$;

(3) $\delta(A^*) = \delta(A)^*$ for each $A \in \mathscr{D}(\delta)$.

We begin with the following lemma.

LEMMA 3.1. *Let $\mathscr{M}$ be a unital C^*-algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation in $\mathscr{M}$ with domain $\mathscr{D}(\delta)$. If there exists a dense subspace $\mathscr{D}$ of $\mathfrak{H}$ such that $\mathscr{M}\mathscr{D} \subset \mathscr{D}$ and δ is a continuous map of $(\mathscr{D}(\delta), t_u)$ into $(\mathscr{M}, t_{qu}^{\mathscr{D}})$, then δ is extended to a continuous linear map $\hat{\delta}$ of $(\mathscr{M}, t_u)$ into $(\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H}), t_{qu}^{\mathscr{D}})$ such that*

(1) $\hat{\delta}(AB)\xi = \hat{\delta}(A)B\xi + A\hat{\delta}(B)\xi$;

(2) $\hat{\delta}(A)^*\xi = \hat{\delta}(A^*)\xi$;

(3) $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$

for each $A, B \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$. Namely, the following diagram holds:

$$\begin{array}{lcl} \hat{\delta};\ (\mathscr{M}, t_u) & \xrightarrow{\textit{continuous}} & (\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H}), t_{qu}^{\mathscr{D}}) \\ \uparrow \quad \cup & & \cup \\ \delta;\ (\mathscr{D}(\delta), t_u) & \xrightarrow[\textit{continuous}]{} & (\mathscr{M}, t_{qu}^{\mathscr{D}}) . \end{array}$$

By Lemma 3.1 we define a derivation of a C^*-algebra into a space of unbounded operators as follows:

DEFINITION 3.2. Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathfrak{H}$ and let $\mathscr{M}$ be a unital C^*-algebra acting on $\mathfrak{H}$ with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. A linear map δ of $\mathscr{M}$ into $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ is said to be a derivation of $\mathscr{M}$ into $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ if

$$\delta(AB)\xi = \delta(A)B\xi + A\delta(B)\xi \quad \text{for each} \quad A, B \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D}\,.$$

In particular, a derivation δ is said to be a *-derivation if the range of δ is contained in $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ and

$$\delta(A)^*\xi = \delta(A^*)\xi \quad \text{for each} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D}\,.$$

If a derivation δ of $\mathscr{M}$ into $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ is a continuous map of $(\mathscr{M}, \tau_1)$ into $(\mathscr{L}(\mathscr{D}, \mathfrak{H}), \tau_2)$, where τ_1 and τ_2 are topologies on $\mathscr{M}$ and $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ respectively, then it is said to be $(\tau_1 \to \tau_2)$-continuous.

We also have the following result:

LEMMA 3.3. *Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation in $\mathscr{M}$. If δ is $(t_w \to t_{qw}^{\mathscr{D}})$-continuous* (resp. $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$*-continuous*), *then δ is extended to a $(t_w \to t_{qw}^{\mathscr{D}})$-continuous* (resp. $(t_s \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$*-continuous*) *$*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ satisfying $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$ for each $A \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$.*

DEFINITION 3.4. Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation of a C^*-algebra $\mathscr{M}$ on $\mathfrak{H}$ into $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$. If $\delta(\mathscr{M}) \subset \mathfrak{M}_T^\dagger$ for some $T \in \mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$, then δ is said to be a $*$-derivation of $\mathscr{M}$ into $\mathfrak{M}_T^\dagger$. If there exists an element T of $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ such that $\delta(\mathscr{M}_u)$ is a bounded subspace of the normed space $\mathfrak{M}_T^\dagger$, where $\mathscr{M}_u$ is the set of all unitary operators in $\mathscr{M}$, then δ is said to be quasi-bounded.

LEMMA 3.5. *Let $\mathscr{M}$ be a unital C^*-algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation in $\mathscr{M}$. If there exist a dense subspace $\mathscr{D}$ of $\mathfrak{H}$ and an element T of $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ such that $\mathscr{M}\mathscr{D} \subset \mathscr{D}$ and $\|\delta(A)\|_T \leqq \|A\|$ for all $A \in \mathscr{D}(\delta)$, then δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathfrak{M}_T^\dagger$ satisfying $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$ for each $A \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$.*

We now give some examples of quasi-bounded $*$-derivations.

EXAMPLE 3.6. Let δ be a spatial derivation in a C^*-algebra $\mathscr{M}$ acting on a Hilbert space $\mathfrak{H}$ with domain $\mathscr{D}(\delta)$, i.e., there exists a symmetric operator H on $\mathfrak{H}$ such that $\mathscr{D}(\delta)\mathscr{D}(H) \subset \mathscr{D}(H)$ and $\delta(A)\xi = i[H, A]\xi$ for each $A \in \mathscr{D}(\delta)$ and $\xi \in \mathscr{D}(H)$. If there exists a closed operator $T\eta\mathscr{M}'$ and a constant $\gamma > 0$ such that $\|H\xi\| \leqq \gamma\|T\xi\|$ for all $\xi \in \mathscr{D}(T)$, then δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}(T), \mathfrak{H})$.

2. Let $\mathscr{M}_i$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}_i$ and let δ_i be a bounded $*$-derivation on $\mathscr{M}_i (i = 1, 2, \cdots)$. Let $\mathscr{M}$ be a direct sum of the von Neumann algebras $\mathscr{M}_i$ and let $\mathfrak{H}$ be the direct sum of the Hilbert spaces $\mathfrak{H}_i$. We define

$$\mathscr{D}(\delta) = \left\{A = (A_i) \in \prod_i \mathscr{M}_i;\ A_i \neq 0 \quad \text{for only finite coordinates}\right\} ,$$
$$\delta(A) = (\delta_i(A_i)) , \quad A = (A_i) \in \mathscr{D}(\delta) .$$

Then δ is a $*$-derivation in $\mathscr{M}$ with the weakly dense domain $\mathscr{D}(\delta)$, but it is not generally bounded. However, δ is $(t_w \to t^{\mathscr{D}}_{qw})$-continuous (and $(t_s \to t^{\mathscr{D}}_s)$, $(t_u \to t^{\mathscr{D}}_u)$, $(t_u \to t^{\mathscr{D}}_{qu})$, $(t_u \to t^{\mathscr{D}}_{\lambda})$, $(t_u \to t^{\mathscr{D}}_{\rho})$-continuous), where

$$\mathscr{D} = \{(\xi_i) \in \mathfrak{H};\ \xi_i \neq 0 \quad \text{for only finite coordinates}\} .$$

Putting

$$T = (\|\delta_i\| I_i)$$

where $\|\delta_i\|$ is the norm of δ_i and I_i is the identity operator on $\mathfrak{H}_i$, we have

$$\|\delta(A)\xi\| \leqq \|A\|\,\|T\xi\| \quad \text{for each} \quad A \in \mathscr{D}(\delta) \quad \text{and} \quad \xi \in \mathscr{D} .$$

Hence, δ is extended to a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathfrak{M}^{\#}_T$.

3. Let δ be a $(t_u \to t^{\mathscr{D}}_w)$-continuous $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D})(\equiv \mathscr{L}^{\#}(\mathscr{D}, \mathscr{D}))$. If $\delta(\mathscr{M})$ is a finite dimensional subspace of $\mathscr{L}^{\#}(\mathscr{D})$, then δ is a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$.

4. Let δ be a $*$-derivation in a C^*-algebra $\mathscr{M}$ acting on a Hilbert space $\mathfrak{H}$. If there exists a densely defined closed operator T on $\mathfrak{H}$ such that $\mathscr{M}\mathscr{D}(T) \subset \mathscr{D}(T)$ and δ is $(t_u \to t^{\mathscr{D}(T)}_{qu})$-continuous (or $(t_u \to t^{\mathscr{D}(T)}_{\lambda})$-continuous), then δ is extended to a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}(T), \mathfrak{H})$. This follows immediately from Lemma 2.2.

As a slight generalization of Example 3.6, 4 we have the following result:

LEMMA 3.7. *Let $\mathscr{D}$ be a countably dominated subspace in a Hilbert space $\mathfrak{H}$ by a sequence $\{T_n\}$ of closed operators on $\mathfrak{H}$. If δ is a $(t_u \to t^{\mathscr{D}}_{qu})$-continuous (or $(t_u \to t^{\mathscr{D}}_{\lambda})$, $(t_w \to t^{\mathscr{D}}_{qw})$, $(t_{\sigma w} \to t^{\mathscr{D}}_{q\sigma w})$, $(t_s \to t^{\mathscr{D}}_s)$, $(t_{\sigma s} \to t^{\mathscr{D}}_{\sigma s})$-continuous) $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$, then δ is quasi-bounded.*

Proof. Suppose that δ is $(t_u \to t_{qu}^{\mathscr{D}})$-continuous. By the continuity of δ, $\delta(\mathscr{M}_1)$ is a bounded subset of $(\mathscr{L}^\#(\mathscr{D}, \mathfrak{H}), t_{qu}^{\mathscr{D}})$, where $\mathscr{M}_1$ is the unit ball of $\mathscr{M}$. It then follows from Lemma 2.4 that $\delta(\mathscr{M}_1)$ is a bounded subset of the normed space $\mathfrak{M}_{I+|T_n|}^\#$ for some n. This implies that δ is quasi-bounded.

4. The spatiality of quasi-bounded *-derivations. Throughout this section we may assume that $\mathscr{D}$ is a dense subspace of a Hilbert space $\mathfrak{H}$ and $\mathscr{M}$ is a unital C^*-algebra with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. Let δ be a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$, i.e., there exists an element T of $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ such that $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$ and $\delta(\mathscr{M}_u)$ is a bounded subset of the normed space $\mathfrak{M}_T^\#$.

LEMMA 4.1. *Suppose that* $\mathfrak{M}$ *is a subspace of* $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. *Then the following statements are equivalent*:

(1) f *is a* $t_{qw}^{\mathscr{D}}$*-continuous linear functional on* $\mathfrak{M}$;

(2) f *is a* $t_s^{\mathscr{D}}$*-continuous linear functional on* $\mathfrak{M}$;

(3) $f = \sum_{i=1}^n \omega_{\xi_i, x_i}$ *for* $\xi_i \in \mathscr{D}$ *and* $x_i \in \mathfrak{H}$, *where* $\omega_{\xi,x}(A) = (A\xi \mid x)$ *for* $A \in \mathscr{L}(\mathscr{D}, \mathfrak{H})$, $\xi \in \mathscr{D}$ *and* $x \in \mathfrak{H}$.

Proof. This is proved in the same way as in ([1] Theorem 1.3).

Let $T \in \mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ and $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$. Then, by Lemma 2.1 $\mathscr{B}_T^\# \equiv \{\overline{AT^{-1}};\ A \in \mathfrak{M}_T^\#\}$ is a subspace of $\mathscr{B}(\mathfrak{H})$. We denote by $\tilde{\mathscr{B}}_T^\#$ the t_w-closure of $\mathscr{B}_T^\#$ and denote by $\tilde{\mathfrak{M}}_T^\#$ the $t_{qw}^{\mathscr{D}}$-closure of $\mathfrak{M}_T^\#$ in $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. Then $\tilde{\mathscr{B}}_T^\#$ is a weakly closed subspace of $\mathscr{B}(\mathfrak{H})$ and $\tilde{\mathfrak{M}}_T^\#$ is $t_{qw}^{\mathscr{D}}$-closed subspace of $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. Furthermore, the following lemma is seen by a simple calculation.

LEMMA 4.2. *Let* ϕ *be the isomorphism of* $\mathfrak{M}_T^\#$ *onto* $\mathscr{B}_T^\#$ *in Lemma* 2.1. *Then* ϕ^{-1} *is a continuous map of* $(\mathscr{B}_T^\#, t_w)$ *onto* $(\mathfrak{M}_T^\#, t_{qw}^{\mathscr{D}})$, *so that it is extended to a continuous linear map* $\tilde{\phi}^{-1}$ *of* $(\tilde{\mathscr{B}}_T^\#, t_w)$ *onto* $(\tilde{\mathfrak{M}}_T^\#, t_{qw}^{\mathscr{D}})$.

LEMMA 4.3. *Let* $\mathfrak{K}$ *be a subset of* $\mathfrak{M}_T^\#$ *and let* $\mathfrak{Q}$ *be the* $t_{qw}^{\mathscr{D}}$*-closed convex hull of* $\mathfrak{K}$ *in* $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. *If* $\mathfrak{K}$ *and* $\mathfrak{K}^\# \equiv \{A^\# = A^*/\mathscr{D};\ A \in \mathfrak{K}\}$ *are bounded in* $\mathfrak{M}_T^\#$, *where* $A^*/\mathscr{D}$ *is the restriction of* A^* *to* $\mathscr{D}$, *then* $\mathfrak{Q}$ *is a* $t_{qw}^{\mathscr{D}}$*-compact subset of* $\mathfrak{M}_T^\#$.

Proof. Let $\mathfrak{K}'$ be the convex hull of $\mathfrak{K}$. Then $\mathfrak{K}'$ and $(\mathfrak{K}')^\#$ are bounded in $\mathfrak{M}_T^\#$. Hence we may assume that $\mathfrak{K}$ is convex. We first show that $\mathfrak{Q}$ is a bounded subset of the normed space $\mathfrak{M}_T^\#$. By the boundedness of $\mathfrak{K}$ and $\mathfrak{K}^\#$ there exists a constant $\gamma > 0$ such that $\|A\|_T \leqq \gamma$ and $\|A^\#\|_T \leqq \gamma$ for all $A \in \mathfrak{K}$. For each $S \in \mathfrak{Q}$ there is a net $\{A_\alpha\}$ in $\mathfrak{K}$ which converges to S with respect to the topology $t_{qw}^{\mathscr{D}}$. It then follows that for each $\xi \in \mathscr{D}$ and $x \in \mathfrak{G}$

$$\begin{aligned} |(S\xi|x)| &= \lim_\alpha |(A_\alpha\xi|x)| \\ &\leqq \overline{\lim_\alpha} \|A_\alpha\xi\| \, \|x\| \\ &\leqq \gamma \|T\xi\| \, \|x\| , \end{aligned}$$

so that $\|S\|_T \leqq \gamma$. Furthermore, for each $\xi, \eta \in \mathscr{D}$ we have

$$\begin{aligned} |(S\xi|\eta)| &= \lim_\alpha |(A_\alpha\xi|\eta)| \\ &\leqq \overline{\lim_\alpha} \|A_\alpha^*\eta\| \, \|\xi\| \\ &\leqq \gamma \|T\eta\| \, \|\xi\| . \end{aligned}$$

Hence, $\eta \in \mathscr{D}(S^*)$. Thus we have $S \in \mathfrak{M}_T^\#$ and $\|S\|_T \leqq \gamma$.

We show that $\mathfrak{Q}$ is a $t_{qw}^{\mathscr{D}}$-compact subset of $\mathfrak{M}_T^\#$. In fact, $(\widetilde{\mathscr{B}_T^\#})_r \equiv \{X \in \widetilde{\mathscr{B}_T^\#};\ \|X\| \leqq \gamma\}$ is weakly compact, and so Lemma 4.2 implies that $\tilde{\phi}^{-1}((\widetilde{\mathscr{B}_T^\#})_r)$ is $t_{qw}^{\mathscr{D}}$-compact in $\widetilde{\mathfrak{M}}_T^\#$. Since $\mathfrak{Q}$ is a $t_{qw}^{\mathscr{D}}$-closed subset of $\tilde{\phi}^{-1}((\widetilde{\mathscr{B}_T^\#})_r)$, it follows that $\mathfrak{Q}$ is a $t_{qw}^{\mathscr{D}}$-compact subset of $\mathfrak{M}_T^\#$.

Notation. Let $\mathfrak{K}_\delta$ be a set $\{U^*\delta(U); U \in \mathscr{M}_u\}$ and let $\mathfrak{Q}_\delta$ be the $t_{qw}^{\mathscr{D}}$-closed convex hull of $\mathfrak{K}_\delta$ in $\mathscr{L}(\mathscr{D}, \mathfrak{G})$.

LEMMA 4.4. *$\mathfrak{Q}_\delta$ is a $t_{qw}^{\mathscr{D}}$-compact subset of $\mathfrak{M}_T^\#$.*

Proof. It is easily seen that $\mathfrak{K}_\delta$ and $\mathfrak{K}_\delta^\#$ are bounded subsets of $\mathfrak{M}_T^\#$. Hence, the lemma follows from Lemma 4.3.

Furthermore, one may easily see the following lemma.

LEMMA 4.5. *For each $U \in \mathscr{M}_u$ we define*

$$A_U(S) = U^*SU + U^*\delta(U) \quad \textit{for} \quad S \in \mathscr{L}^\#(\mathscr{D}, \mathfrak{G}) .$$

Then;

(1) *A_U is a $t_{qw}^{\mathscr{D}}$-continuous affine map of $\mathscr{L}^\#(\mathscr{D}, \mathfrak{G})$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{G})$;*
(2) *$A_U(V^*\delta(V)) = (VU)^*\delta(VU)$ for each $U, V \in \mathscr{M}_u$;*
(3) *$A_U\mathfrak{Q}_\delta \subset \mathfrak{Q}_\delta$ for each $U \in \mathscr{M}_u$;*
(4) *$A_UA_V = A_{VU}$ for each $U, V \in \mathscr{M}_u$.*

Hence, $G_{\delta,T} \equiv \{A_u; U \in \mathscr{M}_u\}$ is a semigroup of $t_{qw}^{\mathscr{D}}$-continuous affine maps of $\mathfrak{Q}_\delta$ into $\mathfrak{Q}_\delta$.

DEFINITION 4.6. If for each pair of elements $S_1 \neq S_2$ in $\mathfrak{G}_\delta$ the $t_s^{\mathscr{D}}$-closure of $\{A_U(S_1) - A_U(S_2); U \in \mathscr{M}_u\}$ does not contain 0, then $G_{\delta,T}$ is said to be noncontracting.

DEFINITION 4.7. Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathfrak{H}$ and let $\mathscr{M}$ be a C^*-algebra acting on $\mathfrak{H}$ with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. A $*$-derivation (resp. a derivation) δ of $\mathscr{M}$ into $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$(resp. $\mathscr{L}(\mathscr{D}, \mathfrak{H})$) is said to be spatial if there exists an element H of $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$(resp. $\mathscr{L}(\mathscr{D}, \mathfrak{H})$) such that

$$\delta(A)\xi = [H, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D} .$$

PROPOSITION 4.8. *If $G_{\delta,T}$ is noncontracting, then there exists an element S of $\mathfrak{Q}_\delta$ such that*

$$\delta(A)\xi = [S, A]\xi \quad \textit{for all} \quad A \in \mathscr{M} \quad \textit{and} \quad \xi \in \mathscr{D} ;$$

that is, δ is spatial.

Proof. We consider the locally convex space $\mathscr{X} = (\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H}), t_s^{\mathscr{D}})$. By Lemma 4.1 we have $\sigma(\mathscr{X}, \mathscr{X}^*) = t_{qw}^{\mathscr{D}}$, and hence it follows from Lemmas 4.4, 4.5 that $\mathfrak{Q}_\delta$ is a weakly compact subset of $\mathscr{X}$ and $G_{\delta,T}$ is a noncontracting semigroup of weakly continuous affine maps of $\mathfrak{Q}_\delta$ into $\mathfrak{Q}_\delta$. By Ryll-Nardzewski's fixed point theorem [9] there exists an element S_0 of $\mathfrak{Q}_\delta$ such that

$$A_U(S_0) = S_0 \quad \text{for all} \quad U \in \mathscr{M}_U .$$

Hence, putting $S = -S_0$, we have

$$\delta(A)\xi = [S, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D} .$$

COROLLARY 4.9. *Let $\mathscr{D}$ be a countably dominated subspace of a Hilbert space $\mathfrak{H}$ and let $\mathscr{M}$ be a commutative C^*-algebra acting on $\mathfrak{H}$ with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. Then there does not exist any nonzero $(t_w \to t_{qw}^{\mathscr{D}})$-continuous (or $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$, $(t_u \to t_{qu}^{\mathscr{D}})$, $(t_u \to t_\lambda^{\mathscr{D}})$-continuous) $*$-derivation in $\mathscr{M}$.*

Proof. Suppose that δ is a $*$-derivation which is continuous in one of the above topologies. It then follows from Lemma 3.3 that δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathfrak{M}_T^\sharp$ where $T \in \mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ and $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$. Since $\mathscr{M}$ is commutative, we can easily see that the semigroup $G_{\hat{\delta},T}$ is noncontracting. Hence it follows from Proposition 4.8 that there exists an element H of $\mathfrak{Q}_{\hat{\delta}}$ such that $\hat{\delta}(A)\xi = [H, A]\xi$ for all $A \in \mathscr{M}$ and $\xi \in \mathscr{D}$. By Lemma 3.3 the elements A and H commute, and so $\hat{\delta} = 0$.

LEMMA 4.10. *Let $\mathfrak{H}$ be the completion of a maximal Hilbert algebra $\mathfrak{A}$ with identity e and let $\mathscr{M}$ be the left von Neumann algebra of $\mathfrak{A}$. Let $\mathscr{D}$ be a dense subspace of $\mathfrak{H}$ such that $e \in \mathscr{D}$ and*

$\mathscr{M}\mathscr{D} \subset \mathscr{D}$ (for example, $\mathfrak{A}$ or the maximal unbounded Hilbert algebra $L_2^{\omega}(\mathfrak{A})$ [5]). If δ is a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{G})$ such that $\overline{\delta(A)}\eta\mathscr{M}$ for each $A \in \mathscr{M}$, then it is spatial.

Proof. Since δ is quasi-bounded, there is an element T of $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{G})$ such that $T^{-1} \in \mathscr{B}(\mathfrak{G})$ and $\delta(\mathscr{M}_u)$ is a bounded subset of the normed space $\mathfrak{M}_T^{\#}$. It is easily showed that $\mathfrak{A} \subset \mathscr{D}$ and $SB'\xi = B'S\xi$ for all $S \in \mathfrak{D}_\delta$, $B' \in \mathscr{M}'$ and $\xi \in \mathfrak{A}$. This implies that $G_{\delta,T}$ is non-contracting. In fact, for each pair of elements $S_1 \neq S_2$ in $\mathfrak{D}_\delta$ and $U \in \mathscr{M}_u$ we have

$$\begin{aligned} \|U^*(S_1 - S_2)Ue\| &= \|(S_1 - S_2)\overline{\pi'(u)}e\| \\ &= \|\overline{\pi'(u)}(S_1 - S_2)e\| \\ &= \|(S_1 - S_2)e\| \\ &\neq 0 , \end{aligned}$$

where π(resp. π') is the left (resp. right) regular representation of $\mathfrak{A}$ and $U = \overline{\pi(u)}$ for $u \in \mathfrak{A}$. Hence it follows from Proposition 4.8 that δ is spatial.

THEOREM 4.11. *Let $\mathscr{M}$ be the left von Neumann algebra of a maximal Hilbert algebra $\mathfrak{A}$ with identity e, $\mathfrak{G}$ the completion of $\mathfrak{A}$ and let $\mathscr{D}$ be a countably dominated subspace of $\mathfrak{G}$ by a sequence $\{T_n\}$ of closed operators such that $e \in \mathscr{D}$ and $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. If δ is a $(t_w \to t_{qw}^{\mathscr{D}})$-continuous (or $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$-continuous) $*$-derivation in $\mathscr{M}$, then it can be extended to a spatial $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{G})$.*

Proof. This follows from Lemma 3.7 and Lemma 4.10.

We next examine the spatiality of derivations of $\mathscr{M}$ into $\mathfrak{M}_T^{\#}$ when $\bar{T}\eta\mathscr{M}'$ (or $\bar{T}\eta\mathscr{M}$).

Suppose that δ is a derivation of $\mathscr{M}$ into $\mathfrak{M}_T^{\#}$, where $T \in \mathscr{L}_c(\mathscr{D}, \mathfrak{G})$ and $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{G})$. We set

$$\delta_T(A) = \overline{\delta(A)T^{-1}} \quad \text{for} \quad A \in \mathscr{M} .$$

It then follows from Lemma 2.1 that δ_T is a linear map of $\mathscr{M}$ into $\mathscr{B}(\mathfrak{G})$, and so we have the following diagram:

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\delta} & \mathfrak{M}_T^{\#} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ A & & \delta(A) \\ & \searrow^{\delta_T} & \downarrow \\ & & \overline{\delta(A)T^{-1}} \in \mathscr{B}(\mathfrak{G}) . \end{array}$$

Furthermore, we have the following result, by a simple calculation

LEMMA 4.12. *If* $T \in \mathscr{L}_c(\mathscr{D}, \mathfrak{H})$ *and* $\overline{T^{-1}} \in \mathscr{M}'$, *then the linear map* δ_T *is a derivation of* $\mathscr{M}$ *into* $\mathscr{B}(\mathfrak{H})$.

DEFINITION 4.13. A von Neumann algebra $\mathscr{M}$ on $\mathfrak{H}$ is said to have the property (C) if every derivation δ of $\mathscr{M}$ into $\mathscr{B}(\mathfrak{H})$ is inner; that is, δ is implemented by an element of $\mathscr{B}(\mathfrak{H})$.

We note [3] that if $\mathscr{M}$ is of type I or properly infinite then $\mathscr{M}$ has the property (C).

PROPOSITION 4.14. *Let* $\mathscr{D}$ *be a dense subspace in a Hilbert space* $\mathfrak{H}$ *and let* $\mathscr{M}$ *be a von Neumann algebra on* $\mathfrak{H}$ *with the property* (C) *and* $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. *If* δ *is a* $*$*-derivation of* $\mathscr{M}$ *into* $\mathfrak{M}_T$ *where* $T \in \mathscr{L}_c(\mathscr{D}, \mathfrak{H})$ *and* $T^{-1} \in \mathscr{M}'$, *then there exists an element* B_0 *of* $\mathscr{B}(\mathfrak{H})$ *such that*

$$\delta(A)\xi = [B_0T, A]\xi$$

for all $A \in \mathscr{M}$ *and* $\xi \in \mathscr{D}$, i.e., δ *is spatial.*

Proof. By Lemma 4.12, δ_T is a derivation of $\mathscr{M}$ into $\mathscr{B}(\mathfrak{H})$. Hence it follows by the assumption that there exists an element B_0 of $\mathscr{B}(\mathfrak{H})$ such that

$$\delta_T(A) = [B_0, A] \quad \text{for all} \quad A \in \mathscr{M}.$$

This implies that

$$\delta(A)\xi = [B_0T, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D}.$$

THEOREM 4.15. *Let* $\mathscr{M}$ *be a von Neumann algebra on a Hilbert space* $\mathfrak{H}$ *with the property* (C) *and let* δ *be a* $*$*-derivation in* $\mathscr{M}$. *Suppose that there exists a countably dominated subspace* $\mathscr{D}$ *of* $\mathfrak{H}$ *by a sequence* $\{T_n\}$ *of closed operators* $T_n \eta \mathscr{M}'$ *such that* δ *is* $(t_w \to t_{qw}^{\mathscr{D}})$*-continuous (or* $(t_s \to t_s^{\mathscr{D}})$, $(t_\sigma \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$*-continuous). Then there exists an element* B_0 *of* $\mathscr{B}(\mathfrak{H})$ *and a closed operator* $T \eta \mathscr{M}'$ *such that*

$$\delta(A)\xi = [B_0T, A]\xi \quad \textit{for all} \quad A \in \mathscr{D}(\delta) \quad \textit{and} \quad \xi \in \mathscr{D}.$$

Proof. Since $T_n \eta \mathscr{M}'$ for $n = 1, 2, \cdots$, we have $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. It follows from Lemma 3.3 that δ is extended to a $(t_w \to t_{qw}^{\mathscr{D}})$-continuous $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$. Furthermore, by Lemma 2.6 $\hat{\delta}$ is quasi-bounded, i.e., $\hat{\delta}(\mathscr{M}) \subset \mathfrak{M}^\#_{I+|T_n|}$ for some n. Hence the theorem follows from Proposition 4.14.

COROLLARY 4.16. *Let $\mathfrak{G}$ be the completion of a Hilbert algebra $\mathfrak{A}$, $\mathscr{M}$ the left von Neumann algebra of $\mathfrak{A}$ and let J be the unitary involution on $\mathfrak{A}$. Suppose that $\mathscr{M}$ has the property* (C) *and there exists a countably dominated subspace $\mathscr{D}$ of $\mathfrak{G}$ by a sequence $\{T_n\}$ of closed operators $T_n \eta \mathscr{M}$ such that $J\mathscr{D} = \mathscr{D}$. If δ is a $(t_w \to t_{qw}^{\mathscr{D}})$-continuous (or $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$-continuous) $*$-derivation in $\mathscr{M}$, then it is extended to spatial derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{G})$.*

Proof. We put

$$T'_n = JT_nJ\,, \quad n = 1, 2, \cdots .$$

It is then proved that $\mathscr{D}$ is countably dominated by the sequence $\{T'_n\}$ of closed operators $T'_n \eta \mathscr{M}'$. Hence the corollary follows from Theorem 4.15.

PROPOSITION 4.17. *Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{G}$ and let δ be a $*$-derivation in $\mathscr{M}$. If there exists a countably dominated subspace $\mathscr{D}$ of $\mathfrak{G}$ by a sequence $\{T_n\}$ of closed operators $T_n \eta \mathscr{M} \cap \mathscr{M}'$ such that δ is $(t_w \to t_{qw}^{\mathscr{D}})$-continuous, then δ is extended to a spatial $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{G})$.*

Proof. By Lemma 3.3 and Lemma 2.6, δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathfrak{M}_T^{\#}$, where $T \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{G})$ and $\overline{T^{-1}} \in \mathscr{M} \cap \mathscr{M}'$, satisfying $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$ for each $A \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$. Since $\mathscr{M}\mathscr{D} \subset \mathscr{D}$ and $\mathscr{M}'\mathscr{D} \subset \mathscr{D}$, we have $\overline{\hat{\delta}(A)} \eta \mathscr{M}$ for each $A \in \mathscr{M}$. Since $T \in \mathscr{M} \cap \mathscr{M}'$, $\hat{\delta}_T$ is a derivation of $\mathscr{M}$ into $\mathscr{M}$. Hence, there exists an element B_0 of $\mathscr{M}$ such that

$$\hat{\delta}_T(A) = [B_0, A] \quad \text{for each} \quad A \in \mathscr{M}\,,$$

so that

$$\hat{\delta}(A)\xi = [B_0T, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D}\,.$$

REFERENCES

1. D. Arnal and J. P. Jurzak, *Topological aspects of algebras of unbounded operators.* J. Functional Analysis, **24** (1977), 397-425.
2. O. Bratteli, *Unbounded derivations of operator algebras,* CNRS Colloquium on Algebras of Operators and their Applications to Mathematical Physics, Marseille, 25-29 June, 1977.
3. E. Christensen, *Extensions of derivations,* J. Functional Analysis, **27** (1978), 234-247.
4. J. Dixmier, "*Les algèbres d'opérateurs dans l'espace Hilbertien*", 2e édition, Gauthier-Villars, Paris, 1969.

5. A. Inoue, *On a class of unbounded operator algebras II*, Pacific J. Math., **66** (1976), 411–431.
6. A. Inoue, K. Kuriyama and S. Ôta, *Topologies on unbounded operator algebras*, Mem. Fac. Sci. Kyushu-Univ., **33** (1979), 355–375.
7. A. Inoue and S. Ôta, *Derivations on algebras of unbounded operators*, Trans. Math. Soc., **261** (1980), 567–577.
8. G. Lassner, *Topological algebras of operators*, Rep. Math. Phys., **3** (1972), 279–293.
9. I. Namioka and E. Asplund, *A geometric proof of Ryll-Nardzewski's fixed point theorem*, Bull. Amer. Math. Soc., **73** (1967), 443–445.
10. S. Sakai, *Derivations of W^*-algebras*, Ann. Math., **83** (1966), 273–279.
11. ———, *C^*-algebras and W^*-algebras*, Berlin-Heidelberg-New York, Springer-Verlag, 1971.
12. ———, *Recent developments in the theory of unbounded derivations in C^*-algebras*, Proceeding, Second Japan-USA Seminor, Los Angeles, April 18–22, 1977. (Lecture Notes in Math. 650, Springer-Verlag, Berlin, 1978).
13. K. Schmüdgen, *Some remarks on topologization of unbounded operator algebras*, preprint.

Received May 15, 1980.

FUKUOKA UNIVERSITY
FUKUOKA, JAPAN
KYUSHU UNIVERSITY
FUKUOKA, JAPAN
AND
NIIGATA UNIVERSITY
NIIGATA, JAPAN