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THE AUTOMORPHISM GROUPS OF SPACES AND FIBRATIONS

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This paper deals with the automorphism group of fibrations $f: X \to Y$, where X and Y are simply connected CW-complexes with either a finite number of homology groups or homotopy groups. It is proved that the automorphism groups of such fibrations are finitely presented, and that in case X and Y are H_0-spaces the image of the obvious map $\mathrm{Aut}(f) \to \mathrm{Aut}(H^*(f, Z))$ has finite index in $\mathrm{Aut}(H^*(f, Z))$. It is also proved that in case that Y belongs to the genus of X, $\mathrm{Ker}(\mathrm{Aut}\, X \to \mathrm{Aut}\, X_p)$ is isomorphic to $\mathrm{Ker}(\mathrm{Aut}\, Y \to \mathrm{Aut}\, Y_p)((\ )_p$-localization of p).

Introduction. Let X, Y be spaces and let $f: X \to Y$ be a fibration. This work concerns the group $\mathrm{Aut}\, X$ of homotopy classes of self equivalences of X as well as the group $\mathrm{Aut}(f)$ of homotopy classes of pairs $(h, k) \in \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ which satisfy $fh \sim kf$. Throughout this paper all spaces considered are of the homotopy type of nilpotent CW-complexes of finite type, and all, except those which appear in Chapter four, are of the homotopy type of simply connected CW-complexes, which are either finite dimensional or with a finite number of homotopy groups.

We use the notations of Wilkerson [8]. We recall that a space X is called an *H_0-space* if $H^*(X, Q)$ is an exterior algebra on odd dimensional generators, that the *genus of* X is the set $G(X)$ of homotopy types of spaces Y with $Y_p \approx X_p$ for every prime p, and that the ele. ments $[f']$ of *the genus of a fibration* $f: X \to Y$ are equivalence classes of homotopy classes f' which satisfy: For every prime p there exist homotopy equivalences $h_p: X'_p \to X_p$, $k_p: Y'_p \to Y_p$ satisfying $f_p h_p \sim k_p f'_p$.

Concerning $\mathrm{Aut}\, X$ and $\mathrm{Aut}(f)$ we are interested in the following questions:

(a) Is the group $\mathrm{Aut}(f)$ finitely presented? i.e., can Theorem B in Wilkerson [8] be generalized to $\mathrm{Aut}(f)$?

(b) What is the relation between:

(1) $\mathrm{Aut}\, X$ and $\mathrm{Aut}\, H^*(X, Z)$ where X is an H_0-space.

(2) $\mathrm{Aut}(f)$ and $\mathrm{Aut}\, H^*(f, Z)$ where f is an H_0-fibration, i.e., f is a fibration between H_0-spaces.

(3) $\mathrm{Aut}\, X$ and $\mathrm{Aut}\, X'$ where X' belongs to the genus of X.

(4) $\mathrm{Aut}(f)$ and $\mathrm{Aut}(f')$ where f' belongs to the genus of f.

The answer to question (a) is given by:

MAIN THEOREM. Let X, Y be simply connected CW-complexes and let $F \to X \xrightarrow{f} Y$ be a fibration. Then:

(a) Aut(f) is commensurable with an arithmetic subgroup of Aut(f_0), where $f_0: X_0 \to Y_0$ is the rationalization of f.

(b) Aut(f) is finitely presented, and

(c) Aut(f) has only a finite number of finite subgroup up to conjugation.

One of the results of this theorem is:

COROLLARY 2.8. *Let X be a simply connected finite CW-complex and let $G \subsetneqq \mathrm{Aut}\, X$ be a finitely generated subgroup. If $H_*(X, Z)$ is torsion free then the centralizer of G is finitely presented.*

Concerning question (b) we obtain the following interesting results:

PROPOSITION 3.2. *Let X, Y be H_0-spaces and let $f: X \to Y$ be a fibration. Then:*

(a) *The map* $[Y, X] \to \mathrm{Hom}(H_*(Y, Z), H_*(X, Z)$ *is finite to one.*

(b) $\mathrm{Im}\,(\mathrm{Aut}\, X \to \mathrm{Aut}\, H^*(X, Z)$ *is a subgroup of finite index.*

(c) *The kernel of the obvious map* $\mathrm{Aut}(f) \to \mathrm{Aut}\, H^*(f, Z)$ *is finite and its image is a subgroup of finite index in* $\mathrm{Aut}\, H^*(f, Z)$.

(d) *For any pair* $(h, k) \in \mathrm{Aut}\, H^*(f, Z)$ *there exists a pair* $(\tilde{h}, \tilde{k}) \in$ Aut(f) *and an integer m, so that $H^*(h, Z) = h^m$ and $H^*(k, Z) = k^m$.*

PROPOSITION 4.6. *Let X be an H_0-space either with a finite number of homology groups or with a finite number of homotopy groups. If $H^*(X, Z)$ is torsion free then* $\mathrm{Ker}(\mathrm{Aut}\, X \to \mathrm{Aut}\, X_q)q \in P$ *(P — the set of primes) is a direct product of finite p-groups, $p \neq q$.*

PROPOSITION 4.7. *Let X, Y be nilpotent spaces with a finite number of homology groups and let $f: X \to Y$ be a fibration. Then for every prime p and for every fibration $f': X' \to Y'$, which belongs to the genus of f,* $\mathrm{Ker}(\mathrm{Aut}(f) \to \mathrm{Aut}(f_p))$ *is isomorphic to* $\mathrm{Ker}(\mathrm{Aut}(f') \to \mathrm{Aut}(f'_p))$.

As a consequence of Propositions 3.6 and 4.7 we obtain:

PROPOSITION 3.7. *Let X, μ_0 be an H_0-space. Suppose $H^*(\mu_0, Q)$ is primitively generated, then the number of equivalence classes of H-structure on X for which $H^*(\mu, Q)$ is equivalent to $H^*(\mu_0, Q)$ is finite.*

COROLLARY 4.9. *Let X, Y be H_0-spaces either with a finite number* *of homology groups or with a finite number of homotopy groups and let* $f: X \to Y$ *be a fibration. Then for every fibration* f' *which belongs to the genus of* f, $\mathrm{Ker}(\mathrm{Aut}(f) \to \mathrm{Aut}\, H_*(f, Z))$ *is isomorphic to* $\mathrm{Ker}(\mathrm{Aut}(f') \to \mathrm{Aut}\, H_*(f', Z))$.

PROPOSITION 4.10. *Let* f *and* f' *be as in Corollary* 4.9. If $\mathrm{Aut}(f)$ *is finite, then* $\mathrm{Aut}(f)$ *is isomorphic to* $\mathrm{Aut}(f')$.

The paper is organized as follows:

In section one the relation between automorphism groups and rational equivalence is studied. The main result is proved in section two. In section three, the special properties of H_0-spaces and the results of section one are used to draw conclusions on the automorphism groups of H_0-spaces and fibrations. In the last section, section four, the relation between automorphism groups and genus is studied.

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1. Automorphism groups and rational equivalence.

LEMMA 1.1. *Let* X, Y, X', Y' *be simply connected finite type CW-complexes,* $f: X \to Y$, $f': X' \to Y'$ *be fibrations and* F *and* F' *be simple CW-complexes with* $\pi_* F$ *and* $\pi_* F'$ *finite dimensional and finite. Define* S *to be the set of homotopy classes of pairs* (φ, ψ) *satisfying*:

(a) $\varphi: X \to X'$ *and* $\psi: Y \to Y'$ *are maps with homotopy theoretic fibers* F *and* F', *respectively.*

(b) $f'\varphi \sim \psi f$.

Then $\mathrm{Aut}(f)$ acts on S and $S/\mathrm{Aut}(f)$ is a finite set.

Proof. Let M be the set of triples $(\varphi, \psi, \tilde{f})$, where $F \to \tilde{X} \xrightarrow{\phi} X'$ and $F' \to \tilde{Y} \xrightarrow{\varphi} Y'$ are fibrations, $\tilde{f}: \tilde{X} \to \tilde{Y}$ a map and $\psi\tilde{f} \sim f'\varphi$. Define an equivalence relation on M by: $(\varphi, \psi, \tilde{f}) \sim (\varphi', \psi', \tilde{f}')$ $(\tilde{f}': \tilde{X}' \to \tilde{Y}')$ if and only if there exist homotopy equivalences $\alpha: \tilde{X} \to \tilde{X}'$, $\beta: \tilde{Y} \to \tilde{Y}'$ so that the following diagram homotopy commutes. For any pair $(\varphi, \psi) \in S$ there is a factorization of φ and ψ as $X \xrightarrow{i} X_\varphi \xrightarrow{\tilde{\varphi}} X'$, $Y^j \to Y_\psi \xrightarrow{\tilde{\phi}} Y'$, where i and j are homotopy equivalences, $\tilde{\varphi}$ and $\tilde{\psi}$ are fibrations and $\tilde{\varphi} i \sim \varphi$, $\tilde{\psi} i \sim \psi$. Obviously $f'\tilde{\varphi} \sim \tilde{\psi}(jfi^{-1})$ ($i^{-1} =$ the homotopy inverse of i) and therefore the triple $(\tilde{\varphi}, \tilde{\psi}, jfi^{-1}) \in M$. Changing (φ, ψ) within a homotopy class does not vary the equivalence class of the triple $(\tilde{\varphi}, \tilde{\psi}, jfi^{-1})$. Hence $S \to M$ is well

$$\begin{tikzcd}
\widetilde{X}' \arrow[d, "\varphi'"'] & \widetilde{X} \arrow[l, "\alpha"'] \arrow[r, "\tilde{f}"] \arrow[ld, "\varphi"] & \widetilde{Y} \arrow[r, "\beta"] \arrow[d, "\psi"] & \widetilde{Y}' \arrow[ld, "\psi'"] \\
X' \arrow[r, "f'"] & Y' & &
\end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{X}' \arrow[rrr, bend left, "\tilde{f}'"] & & & \widetilde{Y}'
\end{tikzcd}$$

defined.

Suppose (φ, ψ), $(\varphi', \psi) \in S$ and there exists a pair $(\alpha, \beta) \in \mathrm{Aut}(f)$ so that $\varphi\alpha \sim \varphi'$ and $\psi\beta \sim \psi'$, then the triples $(\tilde{\varphi}, \tilde{\psi}, jfi^{-1})$ and $(\tilde{\varphi}', \tilde{\psi}', j'fi'^{-1})$ are equivalent in M. Conversely, if the triples $(\tilde{\varphi}, \tilde{\psi}, jfi^{-1})$ and $(\tilde{\varphi}', \tilde{\psi}', j'fi'^{-1})$ are equivalent in M i.e., if there are homotopy equivalences $\alpha: X_\varphi \to X_{\varphi}$, $\beta: Y_\psi \to Y_{\psi'}$ so that $(j'fi'^{-1})\alpha \sim \beta(jfi^{-1})$, then $\varphi(i^{-1}\alpha^{-1}i') \sim \varphi'$ and $\psi(j^{-1}\beta j') \sim \psi'$. Thus $S/\mathrm{Aut}(f) \to M/\sim$ is well defined and monic. Therefore it is enough to prove that $M/\sim$ is finite. But by a standard Moore-Postnikov argument any element of M can be obtained as a sequence of principal fibrations (φ_n, ψ_n) with fibers $K(\pi_n X, n)$ and $K(\pi_n Y, n)$, so that $f_{n-1}\varphi_n \sim \psi_n f_n$. Hence it suffices to show that for each n there is a finite number of equivalence classes of such fibrations, where the equivalence relation is defined as in M.

Suppose for the pair of k-invariants $(k, k') \in H^{n+1}(X_{n-1}, \pi_n X) \times H^{n+1}(Y_{n-1}, \pi_n Y)$ there exists $f_n: X_n \to Y_n$ so that $f_{n-1}\varphi_n \sim \psi_n f_n$. Assume also that $\varphi'_n: X'_n \to X_{n-1}$ and $\psi'_n: Y'_n \to Y_{n-1}$ are fibers of k and k', respectively, and there exists $f'_n: X'_n \to Y'_n$ satisfying $\psi'_n f'_n \sim f_{n-1}\varphi'_n$.

Consider the following diagram

$$\begin{tikzcd}
X'_n \arrow[d, "\varphi'_n"'] \arrow[rrr, bend left, "f'_n"] & X_n \arrow[l, dashed, "\alpha"'] \arrow[r, "f_n"] \arrow[ld, "\varphi_n"] & Y_n \arrow[d, "\psi_n"] & Y'_n \arrow[l, dashed, "\beta"'] \arrow[ld, "\psi'_n"] \\
X_{n-1} \arrow[r, "f_{n-1}"] & Y_{n-1} & &
\end{tikzcd}$$

There exist homotopy equivalences $\alpha: X_n \to X'_n$, $\beta: Y'_n \to Y_n$ so $\varphi'_n\alpha \sim \varphi_n$ and $\psi_n\beta \sim \psi'_n$. The map $\beta f'_n\alpha$ is a lift of f_{n-1}, hence the finiteness of the group π_* (fiber ψ) and the number of stages implies the finiteness of $M/\sim$.

LEMMA 1.2. *Let* $f: X \to Y$ *and* $f': X' \to Y'$ *be as in Lemma* 1.1, *and let* M *and* M' *be simple connected* CW*-complexes with* $H_*(M, Z)$ *and* $H_*(M', Z)$ *finite dimensional and finite. Define* S *to be the set of homotopy classes of pairs* (φ, ψ) *satisfying*:

(a) $\varphi: X \to X'$ *and* $\psi: Y \to Y'$ *are maps satisfying* $X' U$ Cone$U(\varphi)$ *is homotopy equivalent to* M *and* $Y' \cup$ Cone (ψ) *is homotopy equivalent to* M'.

(b) $f'\varphi \sim \psi f$.

Then Aut(f') acts on S and S/Aut(f') is a finite set.

Proof. Dual to the proof of 1.1.

THEOREM 1.3. *Let* X, Y, X', Y' *be simply connected finite type* CW*-spaces which are either* H_**-finite dimensional or* π_* *finite dimensional, and let* $f: X \to Y$ *and* $f': X' \to Y'$ *be fibrations.*

Suppose $\varphi: X \to X'$ *and* $\psi: Y \to Y'$ *are rational equivalences satisfying* $f'\varphi \sim \psi f$. *Then* Aut(f) *and* Aut(f') *are commensurable groups.*

Proof. Let $\Delta(\varphi, \psi) \subseteqq$ Aut$(f) \times$ Aut(f') be the set of pairs $((h, k), (h', k')) \in$ Aut$(f) \times$ Aut(f') for which the diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[rrr, bend left, "h"] \arrow[d, "f"] & X' \arrow[r, "h'"] \arrow[d, "f'"] & X' \arrow[d, "f'"] & X \arrow[l, "\varphi"'] \arrow[d, "f"] \\
Y \arrow[r, "\psi"] \arrow[rrr, bend right, "k"'] & Y' \arrow[r, "k'"] & Y' & Y \arrow[l, "\psi"']
\end{tikzcd}$$

commutes, and let Stab$(\varphi, \psi$, Aut$(f'))$ be the image in Aut(f') of the second projection map on $\Delta(\varphi, \psi)$. We shall show that Aut(f) and Aut(f') are commensurable with $\Delta(\varphi, \psi)$.

Let S' be the set of homotopy classes of pairs of the form $(h'\varphi h, k'\psi k)$ where $(h, k) \in$ Aut(f) and $(h', k') \in$ Aut(f'). Then S' is a subset of S of Lemma 1.1, and hence S'/Aut(f) is a finite set. But Aut(f') acts on S'/Aut(f), i.e., there is a map

$$\eta: \text{Aut}(f') \longrightarrow \text{Aut}(S'/\text{Aut}(f)) .$$

Then the group Stab$(\varphi, \psi$, Aut$(f'))$ contains the kernel of η, and therefore the fact that Aut$(S'$/Aut$(f))$ is a finite set implies that Stab$(\varphi, \psi$, Aut$(f'))$ has finite index in Aut(f').

On the other hand, the fact that φ and ψ are rational equivalences implies that the kernel of the map $\Delta(\varphi, \psi) \to \mathrm{Aut}(f')$ is finite. Hence $\Delta(\varphi, \psi)$ and $\mathrm{Aut}(f')$ are commensurable groups. The proof that $\Delta(\varphi, \psi)$ and $\mathrm{Aut}(f)$ are commensurable is dual.

NOTATION. For a fibration $f\colon X \to Y$ denote by $\mathrm{Aut}_X(f)$ the group of homotopy classes of self homotopy equivalences $k\colon Y \to Y$ satisfying $kf \sim f$, and by $\mathrm{Aut}_Y(f)$ the group of homotopy classes of self homotopy equivalences $h; X \to X$ which satisfy $fh \sim f$.

COROLLARY 1.4. *Let f, f', φ and ψ be as in Theorem* 1.3. *Then* $\mathrm{Aut}_X(f)$ *is commensurable with* $\mathrm{Aut}_{X'}(f')$ *and* $\mathrm{Aut}_Y(f)$ *is commensurable with* $\mathrm{Aut}_{Y'}(f')$.

THEOREM 1.5. *Let X, Y, X', Y' be simply connected finite type CW-spaces and let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be fibrations. Suppose f_0 is homotopy equivalent to f_0'. Then* $\mathrm{Aut}(f)$ *and* $\mathrm{Aut}(f')$ *are commensurable groups.*

Proof. Since f_0 is homotopy equivalent to f_0' there exists a commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "f"] & X_0 \arrow[d, "f_0"] & X' \arrow[l, "\varphi'"'] \arrow[d, "f'"] \\
Y \arrow[r, "\psi"] & Y_0 & Y' \arrow[l, "\psi'"']
\end{tikzcd}$$

where the horizontal maps are rationalizations.

Let Y'' be a simply connected CW-complex which satisfies: There exist rational equivalences $\alpha\colon Y'' \to Y$, $\alpha'\colon Y'' \to Y'$ and $\psi''\colon Y'' \to Y_0$ so that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[dr, "\psi"'] & Y'' \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] \arrow[d, "\psi''"] & Y' \arrow[dl, "\psi'"] \\
& Y_0 &
\end{tikzcd}$$

(By Wilkerson [8] such a space exists.)

Consider the following diagram:

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[ddr, "\varphi"] & & X'' \arrow[r, "\beta"] \arrow[ddl, "\varphi''"'] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\
Y \arrow[ddr, "\psi"'] & & Y'' \arrow[ll, "\alpha"'] \arrow[r, "\alpha'"] \arrow[ddl, "\psi''"] & Y' \\
 & X_0 \arrow[d, "f_0"] & & \\
 & Y_0 & &
\end{tikzcd}$$

where X'' is the pullback of $Y'' \xrightarrow{\alpha} Y' \xrightarrow{f'} X'$ and $\varphi'' \colon X' \to X_0$ is a rationalization, which satisfies $f_0\varphi'' \sim \psi''f''$. (The existence of such a rational equivalence follows from the above three diagrams.)

Since φ and φ'' aae rationalizations there exists a bouquet of spheres VS^{n_i} and maps $X \xleftarrow{\gamma} VS^{n_i} \xrightarrow{\gamma''} X''$ so that $\pi_*\gamma \otimes Q$ and $\pi_*\gamma'' \otimes Q$ are epimorphisms and $\varphi''\gamma'' \sim \varphi\gamma$. Therefore the commutativity of the two parallelograms and the triangle, in the last diagram, implies that $\psi\alpha f''\gamma'' \sim \psi f\gamma$. Consequently there exists a map $\delta\colon VS^{n_i} \to VS^{n_i}$ so that $\alpha f''\gamma''\delta \sim f\gamma\delta$.

Consider the cofibration $VS^{n_i} \xrightarrow{\lambda} VS^{n_i} \xleftarrow{j} C_\lambda$ where $\mathrm{Im}(\pi_*\lambda \otimes Q) = \mathrm{Ker}(\pi_*(\gamma\delta) \otimes Q) = \mathrm{Ker}(\pi_*(\gamma''\delta) \otimes Q)$. There exist maps $\varepsilon\colon C_\lambda \to X$, $\varepsilon''\colon C_\lambda \to X''$ so that $\varepsilon j \sim \gamma\delta$, $\varepsilon'' j \sim \gamma''\delta$ and $\varphi''\varepsilon'' \sim \varphi\varepsilon$. Consequently the considerations of the previous paragraph imply the existence of a map $\mu\colon VS^{n_i} \to VS^{n_i}$ and rational equivalences $\phi\colon C_{\lambda\mu} \to X$, $\phi''\colon C_{\lambda\mu} \to X''$ ($C_{\lambda\mu}$ — the cofibre of $\lambda\mu$), so that $\alpha f''\phi'' \sim f\phi$. Hence Theorem 1.3 implies that $\mathrm{Aut}(f)$ and $\mathrm{Aut}(f')$ are both commensurable with $\mathrm{Aut}(f''\phi'')$ and therefore they are commensurable.

2. Proof of the main theorem. By Wilkerson [8] there are finitely generated free simplicial N^cZ groups $M.$ an $N.$ and a map $f.\colon M. \to N.$ so that $\mathrm{Aut}(f)$ can be identified with the group of loop homotopy equivalence classes of self-equivalences of $f.$, and $\mathrm{Aut}(f_0)$ can be identified with the group of loop homotopy equivalence classes of self-equivalences of $f_{0\cdot}\colon M_{0\cdot} \to N_0$. Theretore we study here these groups. We denote them by $H\,\mathrm{Aut}(f)$ and $H\,\mathrm{Aut}(f_{0\cdot})$, respectively.

Let $M_{0\cdot}$ and $N_{0\cdot}$ be finitely generated N^cQ groups. Denote by $\mathrm{Aut}(M_{0\cdot})_0\,\mathrm{Aut}(N_{0\cdot})_0)$ the group of simplicial automorphisms of $M_{0\cdot}(N_{0\cdot})$ and by $\mathrm{Aut}(M_{0\cdot})_1(\mathrm{Aut}(N_{0\cdot})_1$ the set of automorphisms of $M_{0\cdot} \otimes \varDelta(1)$ $(N_{0\cdot} \otimes \varDelta(1))$ lying over the identity on $\varDelta(1)$. The face maps d_0, d_1: $\mathrm{Aut}(M_{0\cdot})_1 \to \mathrm{Aut}(M_{0\cdot})_0$ and d_0', d_1': $\mathrm{Aut}(N_{0\cdot})_1 \to \mathrm{Aut}(N_{0\cdot})_0$.

Let $\mathrm{SimpAut}(f_{0\cdot})$ denote the set of simplicial automorphisms of

$f_{0\cdot}$. Two pair $(h, k), (h', k') \in \mathrm{SimpAut}(f_{0\cdot})$ are homotopic if and only if $h' \in d_1 d_0^{-1}(h)$ and $k' \in d_1' d_0'^{-1}(k)$. Hence

$$H\,\mathrm{Aut}(f_{0\cdot}) = \mathrm{Simp\,Aut}(f_{0\cdot})/(d_1 d_0^{-1}(id) \times d_1' d_0'^{-1}(id)) \cap \mathrm{Simp\,Aut}(f_{0\cdot})\ .$$

PROPOSITION 2.1. *Let* $f_0\colon M_{0\cdot} \to N_{0\cdot}$ *be a simplicial map between finitely generated free simplicial* N^cQ *groups. There exists an affine group scheme* G *over* Q, *so that* $\mathrm{Simp\,Aut}(f_{0\cdot})$ *can be identified with the* Q*-valued points of* G.

Proof. Similar to the proof of Proposition 9.2 in Wilkerson [8].

PROPOSITION 2.2. *There is a normal closed subgroup scheme over* Q, H *of* G, *such that* $(d_1 d_0^{-1}(id) \times d_1' d_0'^{-1}(id)) \cap \mathrm{SimpAut}(f_0) = H(Q)$.

Proof. Since linear algebraic groups are closed under finite cartezian products and finite intersections, the result follows from Proposition 9.3 in Wilkerson [8].

PROPOSITION 2.3. *Let* G *and* H *be as defined above. There exists an affine group scheme* G/H *over* Q, *such that* $H\,\mathrm{Aut}(f_0) = (G/H)(Q) = G(Q)/H(Q)$.

Proof. Proposition 9.4. in Wilkerson [8], the discussion above and thefact that a subgroup of a unipotent group is unipotent, implies that H is unipotent and that $H\,\mathrm{Aut}(f_0) = G(Q)/H(Q)$. By Borel [1, 6.8], the quotient of an affine group scheme over Q by a closed normal subgroup scheme over Q is again an affine group scheme over Q. That is G/H exists. The Galois cohomology sequence [Serre] $1 \to H(Q) \to G(Q) \to G/H(Q) \to H^1(\mathrm{Gal}(\bar{Q}, Q), H) \cdots$ is an exact sequence of groups and pointed sets. Hence the fact that H is unipotent implies that $H^1(\mathrm{Gal}(\bar{Q}, Q), H) = 0$ and the result follows.

PROPOSITION 2.3′. *Let* X, Y *be simply connected finite CW-complexes and let* $f\colon X \to Y$ *be a fibration. Then* $\mathrm{Aut}(f_0)$ *is the set of* Q*-valued points of a linear algebraic group over* Q.

PROPOSITION 2.4. *Let* $M.$ *and* $N.$ *be finitely generated free simplicial nilpotent groups of class* c *and let* $f.\colon M. \to N.$ *be a simplicial map. Define* $M_L \subseteqq M_0(N_L \subseteqq N_0)$ *to be the intersection of all lattice subgroups of* $M_0.(N_0.)$ *that contain* $M.(N.)$.

Then $f.$ *induces a map* $f_L\colon M_L \to N_L$ *and* $\mathrm{Simp\,Aut}(f.)$ *has finite index in* $\mathrm{Simp\,Aut}(f_L)$.

Proof. The existence of f_L and the fact that $G \underset{\text{def}}{=} (\text{Simp Aut}(M.) \times \text{SimpAut}(N.)) \subseteqq \text{SimpAut}(M_L) \times \text{Simp Aut}(N_L)$ is a subgroup of finite index, follows from Wilkerson [8, 8.1 and 8.3]. Hence

$$G \cap \text{SimpAut}(f_L) \subseteqq \text{SimpAut}(f_L)$$

is a subgroup of finite index and it suffices to prove that $\text{SimpAut}(f_L) = G \cap \text{SimAut}(f_L)$. But this is clear, since $(h, k) \in G \cap \text{SimpAut}(f_L)$ implies that $h|M.: M. \to M.$, $k|N.: N. \to N.$ and $kf_Lh^{-1} = f_L$ and therefore $(h|M., k|N.) \in \text{Simp Aut}(f.)$.

PROPOSITION 2.5. *Let X, Y be simply connected finite CW-complexes and let $f: X \to Y$ be a fibration. There exist finite CW-complexes X' and Y' so that $H_*(X', Z)$ and $H_*(Y', Z)$ are torsion free and a fibration $f': X' \to Y'$ so that* $\text{Aut}(f')$ *and* $\text{Aut}(f)$ *are commensurable groups.*

Proof. By Theorem 1.3 it suffices to prove that there exist rational equivalences $h: X' \to X$ and $k: Y' \to Y$ so that the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "h"'] & Y' \arrow[d, "k"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

Since f is homotopic to a cellular map we can assume that f is cellular. Suppose there exists a commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "h_n"'] & Y_n \arrow[d, "k_n"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where f_n is cellular, h_n, k_n are rational equivalences and the groups $H_m(X_n, Z)$ and $H_m(Y_n, Z)$ are torsion free for $m \leqq n$.

Let $X_n^{(n)}$, $Y_n^{(n)}$ be the n-skeletons of X_n and Y_n. Since f_n is cellular f_n induces a map $f_n': X_n/X_n^{(n)} \to Y_n/Y_n^{(n)}$. Therefore the fact that $H_{n+1}(X, Z) = \pi_{n+1}(X_n/X_n^{(n)})$ and $H_{n+1}(Y, Z) = \pi_{n+1}(Y_n/Y_n^{(n)})$ implies the existence of a commutative diagram (${}^t(\ \ )$ denotes the torsion subgroup of ( ).)

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\
X_n/X_n^{(n)} \arrow[r, "f'_n"] \arrow[d] & Y_n/Y_n^{(n)} \arrow[d] \\
K(H_{n+1}X, n+1) \arrow[r, "(f'_n)_*"] \arrow[d] & K(H_{n+1}Y, n+1) \arrow[d] \\
K({}^tH_{n+1}X, n+1) \arrow[r, "(f_n)_*"] & K({}^tH_{n+1}Y, n+1)
\end{tikzcd}$$

Let X_{n+1} and Y_{n+1} be the fibers of the maps

$$X_n \longrightarrow X_n/X_n^{(n)} \longrightarrow K(H_{n+1}X, Z) \longrightarrow K({}^tH_{n+1}X, Z)$$

and

$$Y_n \longrightarrow Y_n^{(n)}/Y_n^{(n)} \longrightarrow K(H_{n+1}Y, Z) \longrightarrow K(t_{n+1}Y, Z)$$

and let $f_{n+1}\colon X_{n+1} \to Y_{n+1}$ be the iduced map. Obviously X_{n+1} is rational equivalent to X, Y_{n+1} is rational equivalent to Y and there exists a commutative diagram

$$\begin{tikzcd}
X_{n-1} \arrow[r, "f_{n+1}"] \arrow[d, "h_{n+1}"'] & Y_{n-1} \arrow[d, "k_{n+1}"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where h_{n+1} and k_{n+1} are rational equivalences and f_{n+1} is cellular. By the Serre spectral sequence $H_m(X_{n+1}, Z)$ and $H_m(Y_{n+1}, Z)$ are torsion free for $m \leqq n+1$, and the result follows.

PROPOSITION 2.6. *Let $M.$ and $N.$ be finitely generatad connected minimal simplicial N^cZ groups and let $f.\colon M. \to N.$ be a simplicial map. Then $H\,\mathrm{Aut}(f.)$ is an arithmetic subgroup of $H\,\mathrm{Aut}(f_0\cdot)$.*

Proof. Since $\mathrm{Simp\,Aut}(f) \subseteqq \mathrm{Simp\,Aut}(f_L)$ is a subgroup of finite index, the theorem follows from Theorem 9.8 in [8] by replacing $N.$ by $f.$, N_L by f_L and $d_1d_0^{-1}(id)$ by $(d_1d_0^{-1}(id) \times d'_1d'_0{}^{-1}(id) \cap \mathrm{Simp\,Aut}(f_0\cdot)$.

Proof of the main theorem. By Proposition 2.5 we can assume that $H_*(X, Z)$ and $H_*(Y, Z)$ are torsion free. Hence by Wilkerson [8] Aut(f) can be calculated as $H\,\mathrm{Aut}(f.)$ for some $f.\colon M. \to N.$, where $M.$ and $N.$ are connected minimal free simplicial N^cZ groups. Therefore Aut(f) is an arithmetic subgroup of a linear algebraic group, and the result follows from Proposition 10.3 in [8].

COROLLARY 2.7. *Let X, Y be simply connected finite CW-complexes and let $f: X \to Y$ be a fibration. Then* $\mathrm{Aut}_Y(f)$ *is finitely presented.*

Proof. Similar to the proof of the main theorem.

COROLLARY 2.8. *Let X be a simply connected finite CW-complex and let $G \subseteqq \mathrm{Aux}\, X$ be a finitely generated subgroup. If $H_*(X, Z)$ is torsion free then the centralizer of G is finitely presented.*

Proof. Suppose G is generated by $g_1, g_2, \cdots, g_n$. Since the centralizer of G is equal to the centralizer of the set $\{g_1, g_2, \cdots, g_n\}$, the proof is similar to the proof of the main theorem.

3. Commensurability and H_0-spaces and fibrations. Let X, Y be H_0-spaces and let $f: X \to Y$ be a fibration. In this section we deal with the relation between $\mathrm{Aut}\, X$ and $\mathrm{Aux}\, H^*(X, Y)$ and between $\mathrm{Aut}(f)$ and $\mathrm{Aut}\, H^*(f, Z)$. In case X is an H-space we draw conclusions on the relation between the H-structures on X and the Hopf-algebra structures on $H^*(X, Q)$.

NOTATION. For any H_0-space X we denote $K(QH^*(X, Z)/\text{torsion})$ by $K(X)$.

PROPOSITION 3.1. *Let $f_1, f_2: X \to Y$ be fibrations. If* $\mathrm{rank}(H^*(f_1, Q))$ *is equal to* $\mathrm{rank}(H^*(f_2, Q))$ *then* $\mathrm{Aut}(f_1)$ *and* $\mathrm{Aut}(f_2)$ *are commensurable groups.*

Proof. Since $\mathrm{rank}\,(H^*(f_1, Q)) = \mathrm{rank}\,(H^*(f_2, Q))$ there exist Eilenberg-Maclane spaces K_1, K_2 and rational equivalences $\varphi_i: X \to K(X)$, $\psi_i: Y \to K(Y) (i = 1, 2)$ so that $K(X) = K \times K_1$, $K(Y) = K \times K_2$ and the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "f_i"] \arrow[d, "\varphi_i"'] & & Y \arrow[d, "\psi_i"] \\
K(X) = K \times K_1 \arrow[r, "p"'] & K \arrow[r, hook, "i"'] & K \times K_2 = K(Y)
\end{tikzcd} \qquad (i = 1, 2)$$
.

Hence $\mathrm{Aut}(f)$ and $\mathrm{Aut}(g)$ are both commensurable with $\mathrm{Aut}(ip)$ and therefore they are commensurable groups.

PROPOSITION 3.2. *Let X, Y be H_0-spaces and let $f: X \to Y$ be a fibration. Then:*

(a) *The map* $[Y, X] \to \mathrm{Hom}(H_*(Y, Z), H_*(X, Z))$ *is finite to one.*

(b) $\mathrm{Im}(\mathrm{Aut}\, X \to \mathrm{Aut}\, H^*(X, Z))$ *is a subgroup of finite index.*

(c) *The kernel of the obvious map* $\eta\colon \mathrm{Aut}(f) \to \mathrm{Aut}\, H^*(f, Z)$ *is finite and its image is a subgroup of finite index in* $\mathrm{Aut}\, H^*(f, Z)$.

(d) *For any pair* $(h, k) \in \mathrm{Aut}\, H^*(f, Z)$ *there exists a pair* $(\tilde{h}, \tilde{k}) \in \mathrm{Aut}(f)$ *and an integer* m, *so that* $H^*(\tilde{h}, Z) = h^m$ *and* $H^*(\tilde{k}, Z) = k^m$.

Proof. (a) Let $\varphi\colon X \to K(X)$ be a rational equivalence which represents generators of $H^*(X, Z)$/torsion. Since $H^*(f, Z) = H^*(g, Z)$ $(f, g\colon Y \to X)$ implies that $\varphi f \sim \varphi g$, the result follows from the fach that any map $h\colon Y \to K(X)$ has only a finite number of lifts to a map $\tilde{h}\colon Y \to X$, which satisfies $\varphi\tilde{h} \sim h$.

(b) Let $\varphi\colon X \to K(X)$ be as in (a). By Wilkerson [8] $\mathrm{Im}(\mathrm{Aut}(\varphi) \xrightarrow{proj} \mathrm{Aut}(K(X)) \to \mathrm{Aut}(H^*(K(X), Z)/\mathrm{torsion}) \to \mathrm{Aut}(H^*(X, Z)/\mathrm{torsion})$ is a subgroup of finite index in $\mathrm{Aut}(H^*(X, Z)/\mathrm{torsion})$. Hence the result follows from the fact that $\mathrm{Im}(\mathrm{Aut}\, X \to \mathrm{Aut}(H^*(X, Z)/\mathrm{torsion}))$ contains the image of the above map.

(c) The fact that Kern is a finite group follows from part (a).

Let $G = \mathrm{Im}(\mathrm{Aut}\, X \times \mathrm{Aut}\, Y \to \mathrm{Aut}\, H^*(X, Z) \times \mathrm{Aut}\, H^*(Y, Z))$. By part (b) G is a subgroup of finite index in $\mathrm{Aut}\, H^*(X, Z) \times \mathrm{Aut}\, H^*(Y, Z)$, hence $G \cap \mathrm{Aut}\, H^*(f, Z) \subseteqq \mathrm{Aut}\, H^*(f, Z)$ is a subgroup of finite index and it suffices to prove that $\mathrm{Im}\, \eta \subseteqq G \cap \mathrm{Aut}\, H^*(f, Z)$ is a subgroup of finite index.

Let $(h, k) \in G \cap \mathrm{Aut}\, H^*(f, Z)$. There exists a pair $(\tilde{h}, \tilde{k}) \in \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ satisfying $H^*(\tilde{h}, Z) = h$, $H^*(\tilde{k}, Z) = k$ and $H^*(\tilde{k}^{-1} f \tilde{h}, Z) = H^*(f, Z)(\tilde{k}^{-1}$ – the homotopy inverse of $\tilde{k}$). Therefore the fact that there is only a finite number of maps $f_1, f_2, \cdots, f_n$ which satisfy $H^*(f_i, Z) = H^*(f, Z)$ implies that $\mathrm{Im}\, \eta \subseteqq G \cap \mathrm{Aut}\, H^*(f, Z)$ is a subgroup of finite index, and the proof of part (c) is complete.

(d) Suppose $(h, k) \in \mathrm{Aut}\, H^*(f, Z)$. We have to show that there exists an integer m so that $(h^m, k^m) \in \mathrm{Im}\,(\mathrm{Aut}\,(f) \to \mathrm{Aut}\, H^*(f, Z))$. Since h and k are automorphisms, this follows immediately from the fact that $\mathrm{Im}(\mathrm{Aut}(f) \to \mathrm{Aut}\, H^*(f, Z))$ is a subgroup of finite index in $\mathrm{Aut}\, H^*(f, Z)$.

COROLLARY 3.3. *Let* X *be an* H_0*-space. Suppose* $h, k \in \mathrm{Aut}\, X$ *satisfy* $H^*(h, Z) = H^*(k, Z)$. *Then there exists an integer* m *so that* $h^m \sim k^m$. *Consequently* $h \in \mathrm{Aut}\, X$ *is of finite order if and only if* $H^*(h, Z)$ *is.*

Proof. The pair $(H^*(h, Z), H^*(k, Z)) \in \mathrm{Aut}(H^*(1_X, Z))$, hence the result follows from part (d) in Proposition 3.2.

COROLLARY 3.4. *Suppose* X, Y *are* H_0*-spaces,* $f\colon X \to Y$ *a fibration* *and* $(h, k) \in \mathrm{Aut}(f)$. *Then:*

(a) $H^*(f, Z)$ *is monic and the order of* h *is finite implies that the order of* k *is finite.*

(b) $H^*(f, Z)$ *is epic and the order of* k *is finite implies that the order of* h *is finite.*

Proof. (a) Obviously, the order of h is finite implies that the order of $H^*(k, Z)$ is finite. Hence the result follows from Corollary 3.3.

(b) Similar to (a).

COROLLARY 3.5. *Let* X, Y *and* f *be as in Corollary* 3.4. *Then:*

(a) $H^*(f, Z)$ *is monic and* $\mathrm{Aut}\, X$ *is finite implies that* $\mathrm{Aut}\,(f)$ *is finite.*

(b) $H^*(f, Z)$ *is epic and* $\mathrm{Aut}\, Y$ *is finite implies that* $\mathrm{Aut}(f)$ *is finite.*

Proof. (a) $(h, k_1), (h, k_2) \in \mathrm{Aut}(f)$ and $H^*(f, Z)$ is monic implies that $H^*(k_1, Z) = H^*(k_2, Z)$. Therefore the fact that the kernel of the map $\mathrm{Aut}\, Y \to \mathrm{Aut}\, H^*(f, Z)$ is finite implies that for each $h \in \mathrm{Aut}\, X$ there exist, at most, a finite number of $k \in \mathrm{Aut}\, Y$, so that the pair $(h, k) \in \mathrm{Aut}(f)$. Hence $\mathrm{Aut}(f)$ is a finite group.

(b) Similar to (a).

In order to draw conclusions from Proposition 3.2 to the case that X is an H-space we need the following definitions:

DEFINITION. Let X be an H-space and let μ_1, μ_2 be two H-structures on X.

(a) We say that μ_1 is equivalent to μ_2 if there exists a homotopy equivalence $h: X \to X$, so that $h\mu_1 \sim \mu_2(h \times h)$.

(b) We say that $H^*(\mu_1, Z)$/torsion is equivalent to $H^*(\mu_2, Z)$/torsion if there exists a map $h \in \mathrm{Aut}(H^*(X, Z)/\text{torsion})$ so that

$$(h_* \otimes h_*)H^*(\mu_1, Q) = H^*(\mu_2, Q)h_* .$$

(c) We say that $H^*(\mu_1, Q)$ is equivalent to $H^*(\mu_2, Q)$ if there exists a map $h \in H^*(X, Q)$ so that $(h \otimes h)H^*(\mu_1, Q) = H^*(\mu_2, Q)h$.

PROPOSITION 3.6. *Let* X, μ_0 *be an* H*-space. Then the number of equivalence classes of* H*-structures* μ *on* X, *for which* $H^*(\mu, Z)/tor$*sion is equivalent to* $H^*(\mu_0, Z)/torsion$ *is finite.*

Proof. Let η: $\mathrm{Aut}\, X \to \mathrm{Aut}\, H^*(X, Z)$/torsion be the obvious map. By Proposition 3.2(b) $\mathrm{Im}\, \eta \subseteqq \mathrm{Aut}\, H^*(X, Z)$/torsion is a subgroup of finite index. Assume that the index is n and that $h_1, h_2, \cdots, h_{n-1} \in$ Aut($H^*(H, Z)$/torsion) satisfy

$$\mathrm{Aut}(H^*(X, Z)/\mathrm{torsion}) = \mathrm{Im}\,\eta \cup h_1 \mathrm{Im}\,\eta \cup \cdots \cup h_{n-1} \mathrm{Im}\,\eta \ .$$

Let μ_1, μ_2 be H-structures on X and let $h, h' \in \mathrm{Aut}(H^*(X, Z))$/torsion satisfy:

$$H^*(\mu_0, Q)(h_i h)_* = ((h_i h)_* \otimes (h_i h)_*) H^*(\mu_1, Q) \ ,$$

and

$$H^*(\mu_0, Q)(h_i h')_* = ((h_i h')_* \otimes (h_i h')_*) H^*(\mu_2, Q) \ ,$$

where $h = H^*(\tilde{h}, Z)$/torsion and $h' = H^*(\tilde{h}', Z)$/torsion. Then:

$$H^*(\mu_2, Z)/\mathrm{torsion} = H^*(\tilde{h}'\tilde{h}^{-1}\mu_1(\tilde{h}'\tilde{h}^{-1} \times \tilde{h}'\tilde{h}^{-1}))/\mathrm{torsion} \text{ i.e., } \mu_1$$

is equivalent to an H-structure μ' which satisfies $H^*(\mu', Z)/\mathrm{torsion} = H^*(\mu_2, Z)$/torsion. Consequently the results follows from the fact that for any H-structure μ on X, the number of H-structures μ' which satisfy $H^*(\mu', Z)/\mathrm{torsion} = H^*(\mu, Z)$/torsion is finite (this follows from Proposition 3.2(a)).

PROPOSITION 3.7. *Let X, μ_0 be an H-space. Suppose $H^*(\mu_0, Q)$ is primitively generated, then the number of equivalence classes of H-structures μ on X for which $H^*(\mu, Q)$ is equivalent to $H^*(\mu_0, Q)$ is finite.*

Proof. By Proposition 3.6 the number of equivalence classes of H-structures μ on X, for which $H^*(\mu, Z)$/torsion is equivalent to $H^*(\mu_0, Z)$/torsion is finite. Hence it suffices to prove that the number of equivalence classes of comultiplications $H^*(\mu, Z)$/torsion ($\mu: X \times X \to X$ an H-structure) for which $H^*(\mu, Q)$ is equivalent to $H^*(\mu_0, Q)$ is finite.

Let A be the set of the comultiplications $\nu: H^*(X, Z)/\mathrm{torsion} \to H^*(X, Z)/\mathrm{torsion} \otimes H^*(X, Z)$/torsion which satisfy: There exists a multiplication $\mu: X \times X \to X$ so that $\nu = H^*(\mu, Z)$/torsion and $H^*(\nu, Q)$ is equivalent to $H^*(\mu_0, Q)$. Denote by $\varphi: A \to \mathrm{Hom}(H^*(X, Q), H^*(X, Q) \otimes H^*(X, Q))$ and by $\eta: \mathrm{Aut}(H^*(X, Z)/\mathrm{torsion}) \to \mathrm{Aut}\, H^*(X, Q)$ the obvious maps. Since the kernels of φ and η are finite it suffices to prove [2, Proof of Theorem I] that the number of equivalence classes of $\mathrm{Im}\,\varphi$ relative to the equivalence relation: $\varphi(\mu_1) \sim \varphi(\mu_2)$ if and only if there exists $h \in \mathrm{Im}\,\eta$ so that $\varphi(\mu_1)h = (h \otimes h)\varphi(\mu_2)(\mu_1, \mu_2 \in A)$ is finite.

By Curjel [2, 5.2] the fact that the groups Hom($H^*(X, Z)$/torsion, $H^*(X, Z)$/torsion) and Hom($H^*(X, Z)$/torsion, $H^*(X, Z)/\mathrm{torsion} \otimes H^*(X, Z)$/torsion) are finitely generated implies the existence of a basis $X = \{x_{ij}\}$, of $PH^*(X, \mu_0, Q)$, so that the matrix of every map

$f \in \mathrm{Hom}(H^*(X, Z)/\mathrm{torsion}, H^*(X, Z)/\mathrm{torsion})$, with respect to this basis, is integral, and the matrix of every map

$$g \in \mathrm{Hom}(H^*(X, Z)/\mathrm{torsion}, H^*(X, Z)/\mathrm{torsion} \otimes H^*(X, Z)/\mathrm{torsion}$$

with respect to the basis $\{x_{ij} \otimes 1, 1 \otimes x_{ij}\}$ of $PH^*(X \times X, \mu_0, Q)$ is, also, integral. In particular the matrix of every map which belongs either to $\mathrm{Im}\,\psi$ or to $\mathrm{Im}\,\eta$, with respect to the above bases, is integral. Hence the result follows from the following theorem of Samelson-Leray:

THEOREM OF SAMELSON-LERAY [3, 3 Exp 2]. *Let A be an algebra over the integers. Suppose that A has no generators in even dimensions. Then all the associative comultiplications on A are equivalent.*

4. Genus and automorphism.

Let X and Y be nilpotent CW-complexes of finite type and let $f: X \to Y$ be a fibration. Denote by $G(X)$ the genus of X and by $G(f)$ the genus of f.

In this section we investigate the relations between $\mathrm{Aut}\,X$ and $\mathrm{Aut}\,X'$ where $X' \in G(X)$ and between $\mathrm{Aut}(f)$ and $\mathrm{Aut}(f')$ where $f' \in G(f)$.

NOTATION. Let X be a nilpotent CW-complex and let $\varphi: X \to X_0$ be a rationalization. For every prime p and for every $h \in \mathrm{Aut}\,X$ denote by $(h_p)_\varphi$ the localization of h at p with respect to φ.

PROPOSITION 4.1. *Let X be a nilpotent CW-complex with a finite number of homology groups and let $p \in P(P$ — the set of primes). If $X' \in G(X)$ then* $\mathrm{Ker}(\mathrm{Aut}\,X \to \mathrm{Aut}\,X_p)$ *is isomorphic to* $\mathrm{Ker}(\mathrm{Aut}\,X' \to \mathrm{Aut}\,X_p)$.

Proof. Let $\varphi: X \to X_0$ and $\psi: X' \to X_0$ be rationalizations and let $h \in \mathrm{Ker}(\mathrm{Aut}\,X \to \mathrm{Aut}\,X_p)$. Since for every prime p and for every localization $\phi_p: X_p \to X_0$ $\phi_p(h_p)_\varphi \sim 1_{X_0}\phi_p$, there exists a unique map $h' \in \mathrm{Aut}\,X'$ so that $(h'_p)_\psi = (h_p)_\psi$ for every prime p [5, II 5.6]. Obviously $h' \in \mathrm{Ker}(\mathrm{Aut}\,X' \to \mathrm{Aut}\,X'_p)$. Hence the map η: $\mathrm{Ker}(\mathrm{Aut}\,X \to \mathrm{Aut}\,X_p) \to \mathrm{Ker}(\mathrm{Aut}\,X' \to \mathrm{Aux}\,X_p)$ defined by $\eta(h) = h'$ iff $(h'_p)_\psi = (h_p)_\varphi$ for every prime p, is a well defined homomorphism. The same considerations imply the existence of a homomorphism η': $\mathrm{Ker}(\mathrm{Aut}\,X' \to \mathrm{Aut}\,X_p) \to \mathrm{Ker}(\mathrm{Aut}\,X \to \mathrm{Aut}\,X_p)$ defined by: $\eta'(k) = k'$ iff $(k'_p)_\varphi = (k_p)_\psi$ for every prime p. Since $\eta'\eta$ and $\eta\eta'$ are identities $\mathrm{Ker}(\mathrm{Aut}\,X \to \mathrm{Aut}\,X_p)$ is isomorphic to $\mathrm{Ker}(\mathrm{Aut}\,X' \to \mathrm{Aut}\,X'_p)$.

COROLLARY 4.2. *Let X be an H_0-space with either a finite number of homology groups or a finite number of homotopy groups. Then* *for every* $X' \in G(X)$ $\mathrm{Ker}(\mathrm{Aut}\, X \to \mathrm{Aut}\, H^*(X, Q))$ *is isomorphic to* $\mathrm{Ker}(\mathrm{Aut}\, X' \to \mathrm{Aut}\, H^*(X', Q))$.

Proof. Since $X_0 = \Pi K(Q, n_j)$ the result follows from Proposition 4.1.

COROLLARY 4.3. *If* X *and* X' *are as in Corollary* 4.2 *then* $\mathrm{Ker}(\mathrm{Aut}\, X \to \mathrm{Aut}\, H_*(X, Z))$ *is isomorphic to*

$$\mathrm{Ker}(\mathrm{Aut}\, X' \longrightarrow \mathrm{Aut}\, H_*(X', Z)) .$$

Proof. Let η and η' be as in the proof of Proposition 4.1. We have to show that $\eta(\mathrm{Ker}(\mathrm{Aut}\, X \to \mathrm{Aut}\, H_*(X, Z)) \subseteqq \mathrm{Ker}(\mathrm{Aut}\, X' \to \mathrm{Aut}\, H_*(X', Z))$ and that $\eta'(\mathrm{Ker}(\mathrm{Aut}\, X' \to \mathrm{Aut}\, H_*(X', Z)) \subseteqq \mathrm{Ker}(\mathrm{Aut} X \to \mathrm{Aut}\, H_*X, Z))$.

Suppose $h \in \mathrm{Ker}(\mathrm{Aut}\, X \to \mathrm{Aut}\, H_*(X, Z))$ and $\eta(h) = h'$. The definition of η and the fact that for every prime p $H_*(h, Z) \otimes Z_{(p)} = 1(Z_{(p)}$ – the localization of Z at p) imply for every prime p $H_*(h', Z) \otimes Z_{(p)} = 1$. Hence it follows from Hilton-Mislin and Roitberg [5, I. 3.13] that $h' \in \mathrm{Ker}(\mathrm{Aut}\, X' \to \mathrm{Aut}\, H^*(X', Z))$. The proof that $\eta'(\mathrm{Ker}(\mathrm{Aut}\, X' \to \mathrm{Aut}\, H_*(X', Z))) \subseteqq \mathrm{Ker}(\mathrm{Aut}\, X \to \mathrm{Aut}\, H_*(X, Z))$ is similar.

PROPOSITION 4.4. *Let* X *and* X' *be as in* 4.2. *If* Aut X *is finite then* Aut X *is isomorphic to* Aut X'.

Proof. Let $h \in \mathrm{Aut}\, X$ and let $\varphi: X \to X_0$ be a rationalization. Since X is an H_0-space and Aut X is finite imply that Aut X_0 is abelian. For every prime p and for every localization $\phi_p: X_p \to X_0$ $\phi_p(h_p)_\varphi \sim (h_0)_\varphi \phi_p$. Hence the proof is similar to the proof of Proposition 4.1.

NOTATIONS. Let X be an H_0-space with either a finite number of homology groups or a finite number of homotopy groups, and let $\varphi: X \to K(X)$ $(K(X)) = K(QH^*(X, Z)/\mathrm{torsion})$ be a rational equivalence. Denote by:

(a) $X(p, \varphi)$ the space which satisfies: There exists a factorization of φ

$$\begin{tikzcd} X \arrow[r, "\varphi'(p)"] \arrow[rr, bend right, "\varphi"'] & X(p, \varphi) \arrow[r, "\varphi''(p)"] & K(X) \end{tikzcd}$$

where φ' is a mod $- p$ equivalence and φ'' is a mod $P - p$ equivalence. (Such a space exists by [9, 4.3.1]).

(b) $N(X)$ – the least integer which satisfies: For every $n > N(Z)$ either $\pi_n X = 0$ or $H_n X = 0$ (π_n – if X has a finite number of homotopy groups, H_n – if X is finite dimensional).

(c) t – the least integer divisible by

$$\prod_{n \leqq N(X)} |\mathrm{torsion}(H^n(X, Z))| \cdot |\mathrm{torsion}(\pi_n(\mathrm{fiber}\ \varphi)| .$$

LEMMA 4.5. *Let X and φ be as in the notations. Then the map* $\operatorname{Aut} X(p, \varphi) \to \operatorname{Aut} X_p$ *is monic.*

Proof. Let $h \in \operatorname{Ker}(\operatorname{Aut} X(p, \varphi) \to \operatorname{Aut} X_p)$. Since $\operatorname{Ker}(\operatorname{Aut} X(p, \varphi) \to \operatorname{Aut} X_p)$ contains $\operatorname{Ker}(\operatorname{Aut} X(p, \varphi) \to \operatorname{Aut} X_0)$ and $X(p, \varphi)$ is an H_0-space, $\varphi''(p)h \sim \varphi''(p)$, i.e., for every prime p h is mod-p homotopic to the identity, hence h is homotopic to the identity [5, II 5.3].

PROPOSITION 4.6. *Let X be an H_0-space either with a finite number of homology groups or with a finite number of homotopy groups. If $H^*(X, Z)$ is torsion free, then* $\operatorname{Ker}(\operatorname{Aut} X \to \operatorname{Aut} X_q)(q \in P)$ *is a direct product of finite p-groups, $p \neq q$, p/t.*

Proof. Let $\varphi: X \to K(X)$ be a rational equivalence which represents generators of $H^*(X, Z)$. By Lemma 4.5 $\operatorname{Ker}(\operatorname{Aut} X \to \operatorname{Aut} X_p)$ is isomorphic to $\operatorname{Ker}(\operatorname{Aut} X \to \operatorname{Aut} X(p, \varphi))$. Hence the fact that X is the pullback of the maps $X(p, \varphi) \xrightarrow{p''(\varphi)} K(X)(p/t)$ [9, 4.7.2] and that $\operatorname{Ker}(\operatorname{Aut} X(p, \varphi) \to \operatorname{Aut} K(X))$ is a finite p-group [10, 2.9] implies the result.

PROPOSITION 4.7. *Let X, Y be nilpotent spaces with finite number of homology groups and let $f: X \to Y$ be a fibration. Then for every $f' \in G(f)(f': X' \to Y')$ and for every prime p,* $\operatorname{Ker}(\operatorname{Aut}(f) \to \operatorname{Aut}(f_p))$ *is isomorphic to* $\operatorname{Ker}(\operatorname{Aut}(f) \to \operatorname{Aut}(f_p))$ *is isomorphic to* $\operatorname{Ker}(\operatorname{Aut}(f') \to \operatorname{Aut}(f'_p))$.

Proof. Let $\varphi: X \to X_0$, $\psi: Y \to Y_0$, $\varphi': X' \to X_0$ and $\psi': Y' \to Y_0$ be rationalizations. Assume that f_p is the localization of f with respect to φ and ψ and that f'_p is the localization of f' with respect to φ' and ψ'. Since f'_p is homotopy equivalent to f_p one can choose decompositions of φ' and ψ'

$$\begin{tikzcd}
X' \arrow[r, "\varphi'_p"] \arrow[rr, bend right, "\varphi'"'] & X_p \arrow[r] & X_0 , & Y' \arrow[r, "\psi'_p"] \arrow[rr, bend right, "\psi'"'] & Y_p \arrow[r] & Y_0
\end{tikzcd}$$

so that $f_p\varphi'_p \sim \psi'_p f'$ [6, 2.1.2]. Consequently, the considerations of the proof of Proposition 4.1 imply that for every pair $(h, k) \in \operatorname{Aut} f$ there exists a unique pair $(h', k') \in \operatorname{Aut} f'$, which satisfies $((h'_p)_{\varphi'}, (k'_p)_{\psi'}) = ((h_p)_{\varphi}, (k_p)_{\psi})$ for every prime p and therefore $\operatorname{Ker}(\operatorname{Aut}(f) \to \operatorname{Aut}(f_p))$ is isomorphic to $\operatorname{Ker}(\operatorname{Aut}(f') \to \operatorname{Aut}(f'_p))$.

COROLLARY 4.8. *Let X, Y be as in Proposition* 4.2 *and $f: X \to Y$ be a fibration. Then for every $f' \in G(f)$* $\operatorname{Ker}(\operatorname{Aut}(f) \to \operatorname{Aut} H^*(f, Q))$ *is isomorphic to* $\operatorname{Ker}(\operatorname{Aut}(f') \to \operatorname{Aut} H^*(f', Q))$.

COROLLARY 4.9. *Let f and f' be as in Corollary* 4.8. *Then* $\operatorname{Ker}(\operatorname{Auf}(f) \to \operatorname{Aut} H_*(f, Z))$ *is isomorphic to*

$$\operatorname{Ker}(\operatorname{Aut}(f') \longrightarrow \operatorname{Aut} H_*(f', Z)) .$$

Proof. Similar to the proof of Corollary 4.3.

PROPOSITION 4.10. *Let f and f' be as in Corollary* 4.8. If $\operatorname{Aut}(f)$ *is finite, then* $\operatorname{Aut}(f)$ *is isomorphic to* $\operatorname{Aut}(f')$.

Proof. Let $\varphi: X \to X_0$ and $\psi: Y \to Y_0$ be rationalizations. Since f_0 is homotopy equivalent to f_0' one can choose rationalization $\varphi': X' \to X_0$ and $\psi': Y' \to Y_0$ so that $f_0\varphi' \sim \psi' f'$. Hence the result follows from the fact that $\operatorname{Aut} X_0$ and $\operatorname{Aut} Y_0$ are abelian groups. (The proof is similar to the proof of Proposition 4.7).

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