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THE SPECTRAL DENSITY OF A STRONGLY MIXING STATIONARY GAUSSIAN PROCESS

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Let w be a nonnegative integrable weight function on the real line R such that $(\log w)/(1+x^2)$ is also integrable. Let F_T and P_T denote, respectively, the closed linear spans in $L^2(R, wdx)$ of $\{e^{iax}: a \geqq T\}$ and $\{e^{iax}: a \leqq T\}$. Let $\theta(T)$ denote the angle between P_0 and F_T. The problem considered here is that of describing those weights w for which $\theta(T) \to \pi/2$ as T tends to infinity (such weights arise as the spectral densities of strongly mixing stationary Caussian processes). Some necessary conditions on w are given for $\theta(T) \to \pi/2$, and a construction is given to show that w may have arbitrarily wild oscillatory discontinuities even if $\theta(T) \to \pi/2$. Another measure of the interdependence of P_0 and F_T is introduced: let $\theta*(T)$ denote the angle between $P_T \ominus (P_T \cap F_0)$ and $F_0 \ominus (P_T \cap F_0)$. A complete structural char acterization is given of those weights w for which both $\theta(T)$ and $\theta*(T)$ tend to $\pi/2$. Moreover, it is shown that if either $\theta(T)$ or $\theta*(T)$ is eventually positive and the other tends to $\pi/2$, then they both do.

Let W denote the class of weights w for which $\theta(T)$ tends to $\pi/2$. These weights arise as the spectral densities for stationary Gaussian processes which satisfy a certain strong mixing condition. Helson and Sarason studied the analogous class of weights on the unit circle, which correspond to discrete-time processes. In [10] and [17] they give a structual characterization of these weights as those of the from $|P|^2 e^{u+\tilde{v}}$, where P is a polynomial and where u and v are continuous functions on the unit circle ($\tilde{v}$ denotes a harmonic conjugate of v). Since the entire functions of exponential type are related to the bounded uniformly continuous functions on R rougly in the same way as polynomials are related to continuous functions on the unit circle, it is tempting to conjecture that W is precisely the class W_1 of weights w which have the form $w = |B|^2 e^{u+\tilde{v}}$, where B is an entire function of exponential type and is square-summable on R, and where u and v are in BUC, the space of bounded uniformly continuous functions on R. The class W_1 is discussed in § 4. It is easy to show that $W_1 \subset W$, but whether or not $W \subset W_1$ remains unanswered. A necessary and sufficient condition for a weight function in W to be in W_1 is given in Theorem 2 of that section; it is hoped that further investigation of that condition will lead to an answer. Section 6 contains some partial results in this direction.

The class W_* of weights for which $\theta^*(T) \to \pi/2$ is taken up in § 5.

2. Some preliminaries. Let L^∞ denote the space of essentially bounded measurable functions on the real line, and let H^∞ denote the space of functions in L^∞ whose Poisson extensions into the upper half-plane are analytic. This section will outline some facts about the closed algebras between H^∞ and L^∞ which will be needed either in substance or for motivation in the following sections.

Let A be a closed subalgebra of L^∞ which contains H^∞. It was shown by Marshall and Chang in [16] and [4] that A is generated by H^∞ and the complex conjugates of the inner functions which are invertible in A; such algebras have been named Douglas algebras. The main algebra we will consider is $H^\infty[e^{-ix}]$, the closed algebra generated by H^∞ and the function e^{-ix}. In [19], Sarason showed that this algebra equals the algebraic sum $H^\infty + BUC$, where BUC denotes the space of bounded uniformly continuous function on $\boldsymbol{R}$; BUC is also the C^*-algebra generated by the inner functions which are invertible in $H^\infty + BUC$. It turns out that this form holds for all Douglas algebras. In [3], Chang proved the first two parts of the following theorem which will be used in proving Theorem 5. Part (iii) was proved by Chang and Marshall in [4].

THEOREM. *Let A be a closed subalgebra of L^∞ which contains H^∞, and let C_A denote the C^*-algebra generated by the inner functions which are invertible in A. Then the following statements are true:*

(i) $A = H^\infty + C_A$

(ii) $A \cap \bar{A} = L^\infty \cap (C_A + \tilde{C}_A)$ *where $\bar{A}$ denotes the space of complex conjugates of functions in A, and $\tilde{C}_A$ denotes the space of harmonic conjugates of functions in C_A.*

(iii) *If f is a function in C_A, then* $\operatorname{dist}(f, H^\infty) = \operatorname{dist}(f, H^\infty \cap C_A)$, *where* $\operatorname{dist}(f, S) = \inf\{\|f - g\|_\infty : g \in S\}$.

The role of the conjugation operator in the study of Douglas algebras is tied largely to results discovered by Fefferman and Stein in [9]. If f is a locally integrable function and I a finite interval, we let $f_I = |I|^{-1}\int_I f(x)dx$, where $|I|$ denotes the length of I. A function is said to be of bounded mean oscillation, or to lie in BMO, if the quantity $\|f\|_* = \sup_I |I|^{-1}\int_I |f(x) - f_I|dx$ is finite (the supremum being taken over all finite intervals). If functions in BMO which differ by a constant are identified, then BMO becomes a Banach space, with norm $\| \ \|_*$, which Fefferman and Stein identified in the above paper as the dual of H^1. They also showed that the functions f in BMO are precisely those of the form $f = u + \tilde{v}$ where u and v are in L^∞, and that the conjugation operator is a bounded map from L^∞ into BMO, which is a fact we will need later.

Another important class of functions is VMO, the functions of vanishing mean oscillation. A function f in BMO is said to be in VMO if the numbers $M_a(f) = \sup_{|I|\leq a} |I|^{-1}\int_I |f(x) - f_I|dx$ tend to zero as a tends to zero. It is easy to see that a uniformly continuous function in BMO belongs to VMO. It is also well known that the conjugation operator preserves certain smoothness properties of functions (see [5]), so it is not surprising that if u and v are uniformly continuous functions in BMO, then $u + \tilde{v}$ is in VMO. In [19], Sarason proved that the converse is also true. He also showed that VMO plays an important role in the structure of $H^\infty + BUC$, namely that $(H^\infty + BUC) \cap \overline{H^\infty + BUC} = L^\infty \cap VMO$. In [3], Chang associated to each Douglas algebra A a space VMO_A which is a generalized version of VMO. (Roughly speaking, functions in VMO_A look locally like VMO functions where the functions in C_A are nicly behaved.) Furthermore, it is shown there that $VMO_A \cap L^\infty = A \cap \bar{A}$ and $VMO_A = C_A + \tilde{C}_A$.

3. The class W.

We shall only be considering integrable weights w which satisfy the condition that $(\log w)/(1 + x^2)$ is integrable, so w can be expressed in the form $w = |h|^2$ where h is an outer function in H^2, the usual Hardy space for the upper half-plane (see [5, p. 83]). For each nonnegative real number T, define the number

$$\rho(w, T) = \sup_{f_1 f_2} \left| \int_{-\infty}^{\infty} e^{iTx} f_1(x)\bar{f}_2(x)w(x)dx \right| ,$$

where f_1 and f_2 are allowed to run over the unit balls of $F_0(w)$ and $P_0(w)$, respectively. The number $\rho(w, T)$ is just the cosine of the angle $\theta(T)$ between the subspaces $P_0(w)$ and $F_T(w)$ of $L^2(\boldsymbol{R}, wdx)$. Let W denote the class of all integrable weights w for which $\rho(x, T) \to 0$ as $T \to \infty$.

The following lemma is a variation on a theme by Helson and Szegö [11].

LEMMA 1. *Let $w = |h|^2$ where h is outer in H^2. Then w is in W if and only if the function $\bar{h}/h$ belongs to the algebra $H^\infty + BUC$.*

Proof. The idea here is that $\rho(w, T)$ equals $\operatorname{dist}(e^{iTx}\bar{h}/h, H^\infty)$, the distance in L^∞ of the function $e^{iTx}\bar{h}/h$ to the space H^∞. To see this, note that the unit ball of $F_0(w)$ is the closure in $L^2(\boldsymbol{R}, wdx)$ of functions g in H^2 which satisfy $\int |gh|^2 = \int |g|^2 w \leqq 1$. Thus, we have

$$\rho(w, T) = \sup_{g_1 g_2} \left| \int_{-\infty}^{\infty} e^{iTx}(g_1 h)(g_2 h) \cdot \frac{\bar{h} \cdot h}{h \cdot h} dx \right|$$
$$= \sup_{f_1 f_2} \left| \int_{-\infty}^{\infty} e^{iTx} f_1 f_2 \frac{\bar{h}}{h} dx \right| ,$$

where f_1 and f_2 are the functions $g_1 h$ and $g_2 h$, respectively. Since h is outer, the f_i range independently over a dense subset of the unit ball of H^2, hence their product ranges over a dense subset of the unit ball of H^1. Now H^∞ is the annihilator in L^∞ of H^1, so $\rho(w, T)$ equals the norm of $e^{iTx}\bar{h}/h$ in L^∞/H^∞, which equals $\inf_{f \in H^\infty} \|e^{iTx}(\bar{h}/h) - f\|_\infty$. Thus w is in W if and only if $\text{dist}(e^{iTx}\bar{h}/h, H^\infty) \to 0$ as $T \to \infty$, end Lemma 1 is proved.

The following theorem is stated without proof in [13]. The proof given here is essentially the same as the Helson-Sarason proof in [11]; the argument is sketched below since it will be used on several occasions.

THEOREM 1. *Let $w = |h|^2$ where h is outer in H^2. Then w is in W if and only if, for every $\varepsilon > 0$, w can be written in the form $w = (1 + x^2)|B_\varepsilon|^2 e^{u_\varepsilon + \tilde{v}_\varepsilon}$ where u_ε and v_ε are real functions on R with $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty < \varepsilon$ and where B_ε is an entire function of exponential type which is bounded on the real axis and zero free in the upper half-plane.*

Proof. First suppose that $w = (1 + x^2)|B|^2 e^{u + \tilde{v}}$ where $\|u\|_\infty + \|v\|_\infty < \varepsilon$ and B is entire of exponential type and bounded on R with no zeros in the upper half-plane. Since $(\log |B|)\backslash(1 + x^2)$ is integrable on R, it follows from Nevanlinn's representation theorem ([8, p. 22]) that $e^{iTx}B$ is an outer function for some T, so

$$h = (x + i)e^{iTx}Be[(u + \tilde{v}) + i(\tilde{u} - v)]/2 ,$$

and

$$\bar{h}/h = \frac{x - i}{x + i} e^{-2iTx}[B^*(x)/B(x)]e^{i(v - \tilde{u})}$$

where $B^*(z) = \overline{B(\bar{z})}$. Furthermore, by Nevanlinna's theorem, the zeros of B^* form a Blaschke sequence for the upper half-plane, so $B^*/B = b \cdot e^{i\tau x}$ for some Blaschke product b and real number τ. The factor $e^{i\tau x}$ may be absorbed by the factor $\bar{e}^{2iTx}$, so

$$\begin{aligned}\operatorname{dist}(e^{2iTx}\bar{h}/h,\, H^\infty) &= \operatorname{dist}\left(\frac{x-i}{x+i}be^{i(v-\tilde{u})},\, H^\infty\right)\\ &= \operatorname{dist}\left(e^{i(v-\tilde{u})},\, \bar{b}\frac{x-i}{x+i}H^\infty\right)\\ &\leqq \operatorname{dist}(e^{i(v-\tilde{u})},\, H^\infty)\ .\end{aligned}$$

To get an estimate on this last quantity, let $g = e^{-(u+i\tilde{u})}$. Then g is in H^∞, and

$$\begin{aligned}\|e^{i(v-\tilde{u})} - g\|_\infty &= \|1 - g\cdot e^{-i(v-\tilde{u})}\|_\infty\\ &= \|1 - e^{-u-iv}\|_\infty\\ &\leqq [[\varepsilon\cdot(e^\varepsilon)]^2 + (e^\varepsilon - 1)^2]^{1/2}\ .\end{aligned}$$

Since this last expression tends to zero as $\varepsilon \to 0$, and $\rho(w, T)$ is a nonincreasing function of T, it follows that $\lim_{T\to\infty}\rho(w, T) = 0$.

Suppose now that w is a weight in W. Then, if $\varepsilon > 0$ is given, for some positive T, there exists a function A in H^∞ such that $\bar{h}/h = e^{-iTx}A\cdot e^{s+it}$ where s and t are real functions with $\|s\|_\infty + \|t\|_\infty < \varepsilon/2$. Thus the inequality

$$0 < |h(x)|^2 e^{-s-\tilde{t}} = e^{-iTx}A(x)h^2(x)e^{-\tilde{t}(x)+it(x)}$$

holds almost everywhere on R. There is no harm in assuming that $\varepsilon < \pi/2$. This insures that $Re[\overline{e^{\tilde{t}+it}}] \geqq 0$ so that the last factor on the right is actually the boundary function for a function in $(z+i)^2\cdot H^1$ (see [5, p. 34]). So

$$S(z) = e^{-iTz}A(z)h^2(z)e^{-\tilde{t}(z)+it(z)}$$

is a function which is analytic in the upper half-plane, positive a.e. on the real axis, and is in $H^{1/2}$ of every half-disk with diameter on the real axis. Using a fact about analytic continuation noted by Helson and Sarason (seen [14]), we can analytically continue S across the real axis by reflection across the diameters of arbitrarily large half-disks. Thus S is an entire function. That it is of exponential type follows from a theorem of Krein which says that an entire function which is of bounded characteristic in both upper and lower half-planes is of exponential type. However, the following direct estimate obtained by Koosis in [14] is more useful:

$$|S(z)| \leqq C\cdot(1 + |z|^2)e^{|T\operatorname{Im} z|}\ ,$$

where C is a constant independent of z. Now, since $(\log S)/(1+x^2)$ is summable, S can be factored as $S(x) = B_1^*(x)B_1(x)$ on the real axis, where B_1 is entire of exponential type at most $T/2$ and has no zeros in the upper half-plane (see [2, p. 125]). If B_1 has no zeros, then $B_1(x) = e^{a+cx}$ for some constants a and c. But the constant c must be purely imaginary for $(\log|B_1|)/(1 + x^2)$ to be integrable, and in this case, we can assume that B_1 is constant. So, without loss of generality, B_1 has a root z_0. Let $B_2 = B_1/(z - z_0)$. Then for real x, we have $|B_2(x)|^2 = |S(x)/(x - z_0)|^2$ which is bounded on $\boldsymbol{R}$ by the Koosis estimate. Furthermore,

$$w = |h|^2 = S(x)e^{s+\tilde{t}} = (1 + x^2)\cdot|B_2(x)|^2 e^{r+s+\tilde{t}}$$

where $r = \log|x - z_0/x - i|^2$. Since the function e^r is in BUC, it can be uniformly approximated by entire functions of exponential type (see [2, p. 249]). Thus, we can write $e^r = |B_3|^2 e^{s_1}$ where B_3 is entire of exponential type and $\|s_1\|_\infty < \varepsilon/2$. Putting all this together yields

$$w = (1 + x^2)\cdot B_2 B_2^* B_3 B_3^* e^{s_1+s+\tilde{t}} .$$

Now $B_2 B_2^* B_3 B_3^*$ can be factored as BB^* on the x-axis where B is an entire function of the desired type. Setting $u_\varepsilon = s_1 + s$ and $v_\varepsilon = t$ gives the desired result.

The following corollary gives local versions of properties stated in [10] for weights w in W. Corollary 2 will be used in proving Theorem 3.

COROLLARY 1. *If w is in W, there is a unique sequence (r_n) of real numbers such that, if B_0 is the Hadamard product with zeros (r_n), then the following are true*:

(1) *$w = |B_0|^2 e^f$ where f is a function which is of vanishing mean oscillation on every finite interval.*

(2) $\lim_{|J|\to 0, J\subset I} \sup |J|^{-2}\left[\int_J (w/|B_0|^2)dx\right]\left[\int_J (|B_0|^2/w)dx\right]=1$ *uniformly for every finite interval I*.

(3) *$w/|B_0|^2$ has an antiderivative which is uniformly smooth on every finite interval I*, i.e.,

$$w_I(a) = \sup_{x\in I, |h|\leqq a} \left|\int_x^{x+h} w/|B_0|^2 - \int_{x-h}^x w/|B_0|^2\right| \Big/ \left|\int_{x-h}^{x+h} w/|B_0|^2\right|$$

tends to zero uniformly as a tends to zero for each finite interval I.

(4) *w cannot have a jump discontinuity.*

Proof. Suppose that $w = (1 + x^2)|B_1|^2 e^{u_1+\tilde{v}_1}$ and $w = (1 + x^2)|B_2|^2 e^{u_2+\tilde{v}_2}$ are two representations of w given by Theorem 1. Then $|B_1/B_2|^2$ and $|B_2/B_1|^2$ are both locally summable, provided that $\|v_1 - v_2\|_\infty < \pi/2$. Thus, if v_1 and v_2 are small enough, B_1 and B_2 must have the same real zeros, counting multiplicity. Let B_0 denote the Hadamard product with these zeros. Then, for every small positive ε, you can write $f = \log(w/|B_0|^2) = \log|B_\varepsilon/B_0|^2 + \log(1 + x^2) + u_\varepsilon + \tilde{v}_\varepsilon$ where $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty < \varepsilon$. Now, the first two terms on the right are continuous; let g denote their sum. Then it is clear that, for any finite interval I,

$$\begin{aligned}\limsup_{\substack{|J|\to 0\\ J\subset I}} |J|^{-1}\int_J |f - f_J| &\leqq \limsup_{\substack{|J|\to 0\\ J\subset I}} \Big\{|J|^{-1}\int_J |g - g_J| \\ &\qquad + |J|^{-1}\int_J |(u_\varepsilon + \tilde{v}_\varepsilon) - (u_\varepsilon + \tilde{v}_\varepsilon)_J|\Big\} \\ &\leqq 0 + \|u_\varepsilon + \tilde{v}_\varepsilon\|_* \\ &\leqq \|u_\varepsilon\|_* + \|\tilde{v}_\varepsilon\|_* .\end{aligned}$$

Since the conjugation operator is a bounded map from L^∞ into BMO, it follows that $\|u_\varepsilon\|_* + \|\tilde{v}_\varepsilon\|_* \leqq K(2\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty)$. So, $\limsup_{|J|\to 0, J\subset I} 1/|I| \int_I |f - f_I| \leqq 2K\cdot\varepsilon$, for some absolute constant K and arbitrary ε. This means that f is in $VMO(I)$, and (1) is true. The equivalence of properties (1) and (2) and that (2) implies (3) were established by elementary methods in [19]. Property (4) is an easy consequence of property (1).

COROLLARY 2. *Let $w = |h|^2$ be in W. Then for every x in $\boldsymbol{R}$, $\lim_{y\to 0} |(\bar{h}/h)(x + iy)| = 1$ (by $(\bar{h}/h)(z)$ we mean the Poisson extension of $\bar{h}/h$ into the upper half-plane) and this convergence is uniform on bounded subsets of $\boldsymbol{R}$.*

Proof. First of all, if k is a bounded function on $\boldsymbol{R}$, it may be extended harmonically into the upper half-plane by $k(x + iy) = (P_y * k)(x)$ where $P_y(t) = y/\pi(t^2 + y^2)$. If f and g are bounded functions on the line and f is continuous, then on every finite interval J,

$$\limsup_{y\to 0^+}\sup_{x\in J} |f(x + iy)\cdot g(x + iy) - (fg)(x + iy)| = 0 .$$

A proof of thit can be found in [18]. Let $\varepsilon > 0$ be given and J be fixed. Then we can write $\bar{h}/h = [(x - i)/(x + i)]e^{-2iTx}b(x)e^{i(v(x) - \tilde{u}(x))}$ as in the first paragraph of the proof of Theorem 1. Now write $\bar{h}/h = fg$ where $g = e^{i(v - \tilde{u})}$ and $f = [(x - i)/(x + i)]/e^{-2iTx}b(x)$. Then f is continuous on $\boldsymbol{R}$, so by the remark at the beginning of this paragraph, it will suffice to show that $\lim_{y\to 0^+}\sup_{x\in R} |1 - |g(x + iy)|| < 10\varepsilon$: this implies the inequality

$$1 - 10\varepsilon \leqq \liminf_{y\to 0^+}\inf_{x\in J}\left|\frac{\bar{h}}{h}(x + iy)\right| \leqq \limsup_{y\to 0^+}\sup_{x\in J}\left|\frac{\bar{h}}{h}(x + iy)\right| \leqq 1 + 10\varepsilon$$

for arbitrary ε. To get the desired inequality for g, write $g(x) = k(x)e^{u(x) + iv(x)}$ where $k = e^{-u - i\tilde{u}}$, which is in H^∞. Then $e^{-\varepsilon} \leqq |k(x)| \leqq e^{\varepsilon}$, and $|1 - e^{u(x)+iv(x)}| \leqq 4\varepsilon$, so

$$|k(x + iy) - g(x + iy)| = \left|\int_{-\infty}^{\infty} k(t)\{1 - e^{u(t)+iv(t)}\}P_y(x - t)dt\right| \leqq e^{\varepsilon}4\varepsilon .$$

The desired inequality now follows easily for $\varepsilon < 1/2$.

4. The class W_1. Let W_1 denote the class of integrable weights w which can be expressed in the form $w = |B|^2 e^{u+\tilde{v}}$ where B is an entire function of exponential type which is square summable on $\boldsymbol{R}$, and where u and v are real bounded uniformly continuous functions. In this section, we show that W_1 is a subset of W, and a necessary and sufficient condition is given for a weight w in W to belong to W_1.

To see that W_1 is a subset of W let $w = |h|^2$ be of the form $w = |B|^2 e^{u+\tilde{v}}$ where B, u, and v are as above. We can assume without loss of generality that B has no zeros in the upper half-plane, so there exists a number T such that $e^{iTx}B$ is an outer function in H^2. This implies that $\bar{h}/h = e^{-2iTx}(B^*/B)\cdot e^{i(v-\tilde{u})}$; since B^*/B essentially is a Blaschke product for the upper half-plane, it will suffice, by Lemma 1, to show that $e^{i(v-\tilde{u})}$ belongs to $H^\infty + BUC$. But $e^{i(v-\tilde{u})} = e^{iv+u}\cdot e^{-u-i\tilde{u}}$. The first factor is in BUC since u and v are, and the second factor is in H^∞, so the desired result follows.

THEOREM 2. *Let w be in W, and $w = |h|^2$ where h is outer in H^2. Then the following are equivalent*:

(1) *w belongs to W_1.*

(2) *$\bar{h}/h$ can be factored as an inner function times a function which is invertible in $H^\infty + BUC$.*

Before proving Theorem 2, it should be remarked that it is unknown whether or not condition (2) holds for all w in W. It will be shown in Lemma 6 that (2) is "almost" true for every weight w in W. This will be used to get a representation of the sort that defines the class W_1, except that the entire function B will be of finite order. Next, two lemmas are given. A proof of Lemma 2 can be found in [6]. An argument from [18] is used in proof of Lemma 3.

LEMMA 2. *Let b be an inner function. Then b is invertible in $H^\infty + BUC$ if and only if b is of the form $b(x) = e^{i\sigma x}\cdot b_0(x)$ where σ is a nonnegative number and b_0 is a Blaschke product whose zero sequence (z_n) satisfies the inequality*

$$\sup_{-\infty<x<\infty} \sum \frac{\operatorname{Im} z_n}{|x - z_n|^2} < \infty .$$

LEMMA 3. *Let ϕ be a unimodular function in $H^\infty + BUC$. Then the following are equivalent:*

(1) *ϕ is invertible in $H^\infty + BUC$.*

(2) $\operatorname{dist}(\bar{\phi}, H^\infty + BUC) < 1$.

(3) *$\phi = e^{-iTx}b(x)e^{i(u+\tilde{v})}$ where T is a real number, u and v are real in BUC, and b is a Blaschke product whose zero sequence (z_n) satisfies $\sup_{-\infty<x<\infty} \sum \operatorname{Im} z_n/|x - z_n|^2 < \infty$.*

Proof. That (1) implies (2) is obvious. Suppose now that $\operatorname{dist}(\bar{\phi}, H^\infty + BUC) < 1$. Then for some positive T, there is a ψ in H^∞ such that $\|e^{iTx}\bar{\phi} - \psi\|_\infty < 1$, so $\|1 - \phi e^{-iTx}\psi\|_\infty < 1$. Now $\phi e^{-iTx}\psi$ is in $H^\infty + BUC$, so, by the last inequality, it must have a logarithm in $H^\infty + BUC$. Hence, we can write $\phi e^{-iTx}\psi = e^{f+i\tilde{f}+r+is}$ where $f + i\tilde{f}$ is in H^∞, and $r + is$ belongs to BUC. Factoring ψ as a product of its inner and outer parts, we get

$$\psi = be^{\log|\psi|+i\log|\psi|^\sim} ,$$

where b is inner. Since $|\phi e^{-iTx}| = 1$ a.e., we must have $f + r = \log|\psi|$ a.e., so

$$\phi = e^{iTx}\bar{b}e^{i(\tilde{f}-\log|\psi|^\sim)+is} = e^{iTx}\bar{b}e^{i(s-\tilde{r})} .$$

Now, $e^{i(\tilde{r}-s)} = e^{r+i\tilde{r}}\cdot e^{-r-is}$, which belongs to $H^\infty + BUC$ since the first factor belongs to H^∞ and the second to BUC. Thus $\phi^{-1} = e^{-iTx}be^{i(\tilde{r}-s)}$ and $\bar{b} = e^{-iTx}\phi e^{i(\tilde{r}-s)}$ both belong to $H^\infty + BUC$. Thus conditions (1) and (2) are equivalent and imply that the function $\bar{\phi}$ satisfies (3). Therefore (1) implies that ϕ must satisfy (3) as well. It is clear that (3) implies (1), so the lemma is proved.

Proof of Theorem 2. Suppose that (2) in the statement of the theorem holds. Then by Lemma 3, $\bar{h}/h = e^{-iTx}be^{i(u+\tilde{v})}$ where b is some inner function, and where u and v are in BUC. Now e^{iu} can be uniformly approximated by entire functions of exponential type, so we can rewrite the last expression as $\bar{h}/h = e^{-iTx}\cdot F\cdot b\cdot e^{r+i(s+\tilde{v})}$ where F is entire of exponential type, and where r and s are also in BUC, with $\|s\|_\infty < \pi/2$. Then, we have

$$|h|^2e^{-r-\tilde{s}+v} = e^{-iTx}h^2\cdot F\cdot be^{-\tilde{s}+is}\cdot e^{v+i\tilde{v}} ,$$

and the proof of Theorem 1 can be adapted to show that

$$|h|^2 = (1 + x^2)|B_1|^2e^{u_1+\tilde{v}_1}$$

where u_1 and v_1 are real in BUC and B_1 is an entire function of exponential type which is bounded on R. By letting $B_2 = B_1/(z - z_0)$ for some zero z_0 of B_1, $|h|^2$ can be written as

$$|h|^2 = (1 + x^2)^2 |B_2|^2 e^{u_2 + \tilde{v}_2}$$

where $|B_2(x)|$ is now $O(1/x)$ and hence square summable on $\boldsymbol{R}$. Now, the factor

$$(1 + x^2)^2 = e^{2\log(1+x^2)} = e^{-4[\arg(x+i)]^{\sim}}$$

is of the form $e^{\tilde{t}}$ for t in BUC. So putting this together yields $w = |h|^2 = |B|^2 e^{u+\tilde{v}}$ where $B = B_2$, $u = u_2$ and $v = v_2 + t$, so w belogs to W_1.

Suppose conversely that $w = |B|^2 e^{u+\tilde{v}}$ as in (1) of Theorem 2. Then we can assume without loss of generality that B has no zeros in the upper half-plane. This means that for some T, $e^{iTx}B$ is outer in H^2, so $\bar{h}/h = e^{-2iTx}(B^*/B)e^{i(v-\tilde{u})}$. The factor B^*/B is a Blaschke product, modulo a harmless exponential factor, so $\bar{h}/h$ is of the desired form by Lemma 3.

5. The class W_*. For a nonnegative weight function w and real number T, define the number

$$\rho^*(w, T) = \sup_{f,g} \left| \int_{-\infty}^{\infty} f(x)\bar{g}(x)w(x)dx \right|$$

where f and g range, respectively, over the unit spheres of the subspaces $F_0 \ominus (F_0 \cap P_T)$ and $P_T \ominus (F_0 \cap P_T)$ of $L^2(R, wdx)$. If w is the spectral density of a stationary Gaussian process, then $\rho^*(w, T)$ measures something like the amount of dependence between "past" and "future" events conditional on the field generated by $F_0 \cap P_T$ (see Dym and McKean [8]). Let W_* denote the class of weight functions for which $\rho^*(w, T) \to 0$ as T tends to infinity. At first glance, it may seem that W is contained in W_* however, this is not the case. An example is furnished at the end of this section. The weight functions in $W \cap W_*$ have a nice form which is given in the next theorem.

THEOREM 3. *Let w be an integrable weight function on $\boldsymbol{R}$. Then the following conditions are equivalent:*

(1) *w belongs to $W \cap W_*$.*

(2) *w belongs to W and $\rho^*(w, T) < 1$ for some T.*

(3) *w belongs to W_* and $\rho(w, T) < 1$ for some T.*

(4) *$w(x) = (1 + x^2)|B(x)|^2 e^{u(x)+\tilde{v}(x)}$ where u and v are in real BUC, and where B is an entire function of finite ex-*

ponential type which is square integrable on $\boldsymbol{R}$ *and whose zero sequence* (z_n) *satisfies*

$$\sup_{-\infty<x<\infty} \sum_{n=1}^{\infty} \frac{|\operatorname{Im} z_n|}{|x-z_n|^2} < \infty .$$

Before proving Theorem 3, two lemmas will be stated and proved.

LEMMA 4. *Let* A *and* B *be two closed subspaces of a Hilbert space* H. *Let* $\rho(A, B)$ *denote the cosine of the minimum angle between* A *and* B. *If* $H = A + B$ *and* $\rho(A, B) < 1$, *then* $H = A^{\perp} + B^{\perp}$ *and* $\rho(A^{\perp}, B^{\perp}) = \rho(A, B)$.

Proof. Let f be a unit vector in $A^{\perp}$. Let $\rho = \rho(A, B) < 1$ and ρ_1 denote the length of the projection of f on $B^{\perp}$. By assumption, $f = a + b$ where a belongs to A and b belongs to B. Then the vectors f, a and b determine a right triangle whose hypotenuse has length $\|b\|$. It follows from elementary geometry that $\|b\|^2 \leqq 1/(1-\rho^2)$. Now, write $f = g + h$ with g in B and h in $B^{\perp}$. It follows that $\|h\| = \rho_1$ and $\|g\| = (1-\rho_1^2)^{1/2}$. Thus, $1 = \langle f, f\rangle = \langle b, f\rangle = \langle b, g\rangle \leqq \|b\|\,\|g\| \leqq (1-\rho_1^2)^{1/2}(1-\rho^2)^{-1/2}$. This implies that $\rho_1 \leqq \rho < 1$. Hence, $\rho(A^{\perp}, B^{\perp}) \leqq \rho(A, B) < 1$ so $A^{\perp} + B^{\perp}$ is a closed sum and $(A^{\perp} + B^{\perp})^{\perp} \subseteqq A \cap B = \{0\}$. This shows that $H = A^{\perp} + B^{\perp}$, and by symmetry, $\rho(A, B) \leqq \rho(A^{\perp}, B^{\perp})$ so the lemma is proved.

LEMMA 5. *Let* $w = |h|^2$. *Then* $\rho^*(w, T) = \operatorname{dist}(e^{iTx}h/\bar{h}, H^{\infty})$, *so* w *is in class* W_* *if and only if* $h/\bar{h}$ *is in* $H^{\infty} + BUC$.

Proof. Let $M_T = F_0 \cap P_T$. It was noted in [7] that M_T is the orthogonal complement in $L^2(R, wdx)$ of $N_T = (e^{iTx}/\bar{h})H^2 + (1/h)\overline{H^2}$ and that the cosine of the minimum angle between the two summands of N_T equals $\operatorname{dist}(e^{iTx}h/\bar{h}, H^{\infty})$. Let A denote the first summand of N_T and B the second, so $\rho(A, B) = \operatorname{dist}(e^{iTx}h/\bar{h}, H^{\infty})$. Then a function f belongs to $N_T \ominus A$ if and only if f is orthogonal to M_T and $\int_{-\infty}^{\infty} \bar{f}(e^{iTx}/\bar{h})g\bar{h}h\,dx = 0$ for every function g in H^2. This last condition implies that $\bar{f}e^{iTx}h$ is in H^2, so f is in $(e^{iTx}/\bar{h})\overline{H^2}$ which equals P_T. Thus, $N_T \ominus A = P_T \ominus M_T$. A similar argument shows that $N_T \ominus B = F_0 \ominus M_T$. An application of Lemma 4 to the Hilbert space N_T now shows that

$$\begin{aligned}\rho(A, B) &= \rho(N_T \ominus A, N_T \ominus B)\\ &= \rho(P_T \ominus (F_0 \cap P_T), F_0 \ominus (F_0 \cap P_T)) = \rho^*(w, T) ,\end{aligned}$$

provided that either $\rho(A, B)$ or $\rho^*(w, T)$ is less than 1. This proves the lemma.

Proof of Theorem 3. Let $w = |h|^2$. If any of the first three conditions of Theorem 3 are satisfied, then it follows from Lemmas 3 and 5 that the function $\bar{h}/h$ is invertible in $H^\infty + BUC$ and has the form $\bar{h}/h = e^{-iTx}be^{i(u+\tilde{v})}$ where b is a Blaschke product for the upper half-plane whose zero sequence (z_n) satisfies $\sup_{-\infty<x<\infty}\sum \operatorname{Im} z_n/|x - z_n|^2 < \infty$, and where u and v are real in BUC. Repeating the argument used to prove Theorem 1 shows that (4) holds since the nonreal zeros of the entire function B are determined by the zero sequence of the Blaschke product b. Conversely, suppose that (4) holds. It can be assumed without loss of generality that B has no zeros in the upper half-plane, so $\bar{h}/h = e^{iTx}[(x-i)/(x+i)]B^*(x)/B(x)e^{i(v-\tilde{u})}$ for some number T. It follows from Lemma 3 that $\bar{h}/h$ is invertible in $H^\infty + BUC$ and that the first three conditions of Theorem 3 must be satisfied, so the theorem is proved.

To see that W is not contained in W_*, take an integrable weight of the form $w(x) = |B(x)|^2$ where B is an entire function of finite exponential type which is square integrable on the real axis and has zeros with arbitrarily small (nonzero) imaginary part (see [8, p. 315]). Then, $\bar{h}/h = e^{-iTx}b(x)$ for some number T and Blaschke product b which is not invertible in $H^\infty + BUC$. It is then true then $\rho(w, T) = 0$ but w fails to belong to W_*. It is interesting to contrast this with the situation for weight functions on the unit circle. If w is a nonnegative integrable function on the unit circle whose logarithm is also integrable, then, $w = |h|^2$ for some outer function in H^2 of the unit circle; the numbers $\rho(w, N)$ and $\rho^*(w, N)$ are defined in a corresponding way for each nonnegative integer N. Then $\rho(w, N)$ tends to zero if and only if the function $\bar{h}/h$ belongs to the algebra $H^\infty + C$ (where C denotes the space of continuous functions on the unit circle), but $\rho^*(w, N)$ tends to zero whenever $\rho(w, N)$ does. This follows from a lemma due to Sarason (see [3, Theorem 2]) which states that if u is a unimodular function on the unit circle and $\operatorname{dist}(u, H^\infty) = 1$ but $\operatorname{dist}(u, H^\infty + C) < 1$, then $\bar{u}$ belongs to $H^\infty[u]$.

6. Some necessary conditions for w to be in W.

We begin with a lemma which shows that condition (2) in Theorem 2 is almost satisfied by any w in W.

LEMMA 6. *If $w = |h|^2$ is in W, and $G(x)$ is any unbounded increasing function on the positive reals, then it is possible to represent $\bar{h}/h$ in the following way*:

$$\bar{h}/h = b_1\bar{b}_2e^{i(s+\tilde{t})},$$

where s and t are continuous functions on $\mathbf{R}\cup\{\infty\}$, $\|s\|_\infty < \pi/2$, and *where* b_1 *and* b_2 *are inner functions with* $|b_2'(x)| = O(G(|x|))$ *as* $|x| \to \infty$.

Proof. The argument used here is a refinement of one given by Axler [1] to show that every function in L^∞ is a quotient of a function in $H^\infty + C$ and a Blaschke product, where C denotes the spece of continuous functions on $\boldsymbol{R} \cup \{\infty\}$. To begin with, $H^\infty + BUC$ can be generated by H^∞ and the complex conjugate of any Blaschke product b whose zero sequence (z_n) satisfies $0 < m \leqq \sum \operatorname{Im} z_n/|x - z_n|^2 \leqq M < \infty$ for all real values of x and some constants m and M. For definiteness, let $\operatorname{Im} z_n \geqq 1$ for all n. Now, there are functions $h_1, h_2, \cdots$ in H^∞ so that $\|\bar{h}/h - \bar{b}^n h_n\|_\infty \to 0$ as $n \to \infty$. Let $\gamma(r)$ be a function on the positive reals to be determined later, but with $0 < \gamma(0)$, and with $\gamma(r)$ increasing and unbounded. For each integer n, there is an integer $M(n)$ such that for all real x

$$\sum_{k=M(n)}^{\infty} \frac{\operatorname{Im} z_k}{|x - z_k|^2} \leqq \frac{\gamma(|x|)}{2^n}$$

and

$$\sum_{k=M(n)}^{\infty} \frac{\operatorname{Im} z_k}{|z_k|^2} \leqq 1/2^n .$$

Let

$$b_2(z) = \prod_{n=1}^{\infty} \prod_{k=M(n)}^{\infty} \left(\frac{1 - z/z_k}{1 - z/\bar{z}_k} \right) .$$

Then $b_2(z)$ converges and so is a Blaschke product. For each n, we can write $\bar{b}^n h_n = h_n \cdot a_n \bar{d}_n / b_2$ where a_n is a Blaschke product, and d_n is a finite Blaschke product, so $\bar{b}^n h_n = \psi_n / b_2$ where ψ_n is in $H^\infty + C$. Since $\bar{b}^n h_n$ converges to $\bar{h}/h$ in L^∞, ψ_n must converge to some ψ in $H^\infty + C$. Thus, $\bar{h}/h = \psi \bar{b}_2$, where the zero sequence (λ_n) of b_2 satisfies

$$\sum \frac{\operatorname{Im} \lambda_n}{|x - \lambda_n|^2} \leqq \gamma(|x|) \quad \text{for all} \quad x .$$

Now, the Blaschke product $b_2(z)$ converges uniformly on compact subsets of the strip $|\operatorname{Im} z| < 1$, so the formal differentiation

$$b_2'(x) = b_2(x) \sum_{n=1}^{\infty} \frac{2i \operatorname{Im} \lambda_n}{|x - \lambda_n|^2} \; (-\infty < x < \infty)$$

may be justified by Cauchy's integral formula for the derivative. Thus,

$$|b_2'(x)| \leqq \sum_{n=1}^{\infty} \frac{2 |\operatorname{Im} \lambda_n|}{|x - \lambda_n|^2} \leqq 2\gamma(|x|) ,$$

so we get the desired estimate by letting $\gamma(|x|) = 1 + \sup\{0, G(|x|)\}$. Since b_2 is continuous across R and $\bar{h}/h = \psi\bar{b}_2$, we must have, from Corollary 2, that ψ is bounded away from zero in a neighborhood of R. So, by a theorem of Stegenga [21], we can write $\psi = [(x+i)/(x-i)]^n b_1 e^{i(s+\tilde{t})}$ where b_1 is an inner function, n is an integer, and where s and t are C, with $\|s\|_\infty + \|t\|_\infty < \pi/2$. Absorbing the factor $[(x+i)/(x-i)]^n$ into $\bar{b}_2$ does not change the asymptotic nature of $b_2'(x)$, so the lemma is proved.

THEOREM 4. *Let w belong to W. Then w can be written in the following form:*

$$w = (1 + x^2)\cdot F \cdot e^{u+\tilde{v}} ,$$

where u and v are continuous on $\boldsymbol{R} \cup \{\infty\}$, and where F is an entire function of order at most 3 which is nonnegative, and bounded on $\boldsymbol{R}$.

Proof. From the previous lemma, we can let $\bar{h}/h = b_1\bar{b}_2 e^{i(s+\tilde{t})}$ where s and t are in C, and $|b_2'(x)| = O(x^2)$. Let $a(x) = \bar{b}_2(\sqrt[3]{x})$. Then a is continuous on $\boldsymbol{R}$ and its modulus of continuity tends to zero at infinity, so by convolving with an appropriate member of Jackson's kernel $K_\lambda(x) = c_\lambda(\sin \lambda x/x)^4$, we get $a(x) = G(x)(1 + \theta(x))$ where G is entire of exponential type and θ is continuous on $R \cup \{\infty\}$ (see [20, p. 52]). Moreover, $\theta(x) \to 0$ as $x \to \infty$, and $\|\theta\|_\infty$ can be made arbitrarily small. Thus we write

$$\bar{h}/h = b_1(x)G(x^3)e^{r+i(s_1+\tilde{t}_1)}$$

where r, s_1 and t_1 are in C, and $\|s_1\|_\infty < \pi/2$. Now, by the same procedure as in the proof of Theorem 1, the desired representation may be obtained.

The next theorem gives a representation for weights in W in a closed form, but falls short of being a good generalization of the Helson-Sarason theorem.

THEOREM 5. *Let w be a weight in W. Then there is a fixed $\delta > 0$ such that w can be represented in the form $w = (1 + x^2)|B|^2 e^{u+\tilde{v}}$ where B is an entire function of exponential type which is bounded and zero free in the upper half-plane, and where $\|u\|_\infty + \|v\|_\infty < \delta$, where u and v are functions in $C(w, B)$, the C^*-algebra generated by the inner functions which are invertible in $H^\infty[e^{-ix}, B^*/B]$. Furthermore, $C(w, B) = C(w)$ does not depend upon the representation and equals BUC if and only if w also belongs to W_*.*

Before proving this theorem, the following technical lemma is needed.

LEMMA 7. *Let A be a Douglas algebra and $A = H^\infty + C_A$. Then, if f is in A, and $\varepsilon > 0$, it is possible to write $f = k + g$ where k is in H^∞, g belongs to C_A, and $\|g\|_\infty < (1+\varepsilon)\|f\|_\infty$.*

Proof. If $f = k_1 + g_1$ where k_1 is in H^∞, and g_1 is in C_A, we have that $\text{dist}(g_1, H^\infty) \leqq \|f\|_\infty$, so by part (iii) of the theorem in § 2, there must be a function k_2 in $H^\infty \cap C_A$ with $\|g_1 - k_2\|_\infty \leqq \|f\|_\infty(1+\varepsilon)$. Set $k = k_1 + k_2$ and $g = g_1 - k_2$ to get the desired result.

Proof of Theorem 5. Let w belong to W. Then represent w as in Theorem 1, i.e., $w = (1+x^2)|B|^2 e^{s+\tilde{t}}$ where $\|s\|_\infty + \|t\|_\infty < \pi/2$. Then for some number T, we can assume $e^{iTx}B$ is outer in H^∞. Then $\bar{h}/h = e^{-2iTx}\cdot[(x-i)/(x+i)]\cdot b\cdot e^{i(t-\tilde{s})}$ where b is the Blaschke product associated with B. By Lemma 1, $\bar{h}/h$ is in $H^\infty[e^{-ix}]$, so $e^{i(t-\tilde{s})}$ is in $H^\infty[\bar{b}, e^{-ix}, (x+i)/(x-i)] = H^\infty[\bar{b}, e^{-ix}]$. Since $e^{s+i\tilde{s}}$ is in H^∞, it follows that $e^{s+it} = e^{s+i\tilde{s}}e^{i(t-\tilde{s})}$ is in $H^\infty[\bar{b}, e^{-ix}]$. Since $\|t\|_\infty < \pi/2$, e^{s+it} has a logarithm in $H^\infty[\bar{b}, e^{-ix}]$ by the functional calculus. By the theorem of Chang stated in § 2, this logarithm has the form $f + g$ where f is in H^∞, and g is in $C(w, B)$. By the last lemma, we may choose f and g so that $\|g\|_\infty < \pi/2$. Thus, $e^{s+it} = e^{(r+i\tilde{r})+(u+iv)}$ where $r + i\tilde{r}$ is in H^∞, and where u and v are real in $C(w, B)$ with $\|v\|_\infty < \pi/2$. Now $r = s - u$, so $e^{it} = e^{i(s-u)^\sim + iv}$, i.e., $e^{i(t-\tilde{s})} = e^{i(v-\tilde{u})}$. This can happen only if $t - \tilde{s}$ and $v - \tilde{u}$ are equal modulo 2π. It will be shown next that $e^{s+\tilde{t}} = e^{u+\tilde{v}}$ up to a constant factor. To see this, note that you can write $0 < |h|^2 e^{-u-\tilde{v}} = e^{-2iTx}h^2 b e^{-u-\tilde{v}+i(v-\tilde{u})}$ a.e., and you can carry out the analytic continuation as in Theorem 1, and get $|h|^2 = S_1 e^{u+\tilde{v}}$, where S_1 is entire, having the same zero set as $S = BB^*$: The nonreal zeros all come from b, and the real zeros must be the same because $|S|/|S_1|$ and its reciprocal are locally in $L^{1/2}$ for $\delta < \pi/2$, and the zeros have multiplicity 2. This means that S and S_1 can differ only by a factor of the form $a\cdot e^{cx}$, which, in view of the fact that $(\log S_1)/(1+x^2)$ and $(\log S)/(1+x^2)$ are integrable, must reduce to a constant. Hence, $w = (1+x^2)|B|^2 e^{s+\tilde{t}} = a(1+x^2)|B|^2 e^{u+\tilde{v}}$ as desired. To see that $C(w, B)$ really only depends on w for suitably small δ note that if

$$(1+x^2)|B_1|^2 e^{u_1+\tilde{v}_1} = (1+x^2)|B_2|^2 e^{u_2+\tilde{v}_2}$$

are two different representations of w, and, say, $\|u_i\|_\infty < 1/4 \log 2$, and $\|v_i\|_\infty < \pi/8$ (for $i = 1, 2$), then comparing the two resulting expressions obtained for $\bar{h}/h$ shows that

$$b_1\bar{b}_2 = e^{iTx}e^{i[(v_2-v_1)+(u_1-u_2)^\sim]}$$

for some T, (where b_i is the Blaschke product associated with B_i, $i = 1, 2$). By the proof of Lemma 1,

$$\operatorname{dist}(b_1\bar{b}_2, H^\infty[e^{-ix}]) < 1 .$$

Similarly,

$$\operatorname{dist}(\bar{b}_1 b_2, H^\infty[e^{-ix}]) < 1 .$$

This implies that $H^\infty[\bar{b}_1, e^{-ix}] = H^\infty[\bar{b}_2, e^{-ix}]$. The final claim follows from Lemma 3.

7. Discontinuous weights in W. Examples of unbounded weights in W can be found in [12]. The following construction shows that a weight function in W can have arbitrarily wild oscillatory discontinuities. Let f_n be the function defined on $\boldsymbol{R}$ by $f_n(x) = 1 - |x|^{1/n}$ for $|x| < 1$, and $f_n(x) = 0$ elsewhere. It is easy to check that $\|f_n\|_* < 3/n$. Furthermore, the BMO norm $\|\cdot\|_*$ is invarient under linear change of variables. Now let (r_n) be any sequence of real numbers which decreases strictly to zero, and let (a_n) be chosen so that the intervals $[r_n - a_n, r_n + a_n]$ are disjoint. Define the function

$$f(x) = \sum_{n=1}^{\infty} f_{2^n}[(x - r_n)/a_n] .$$

Then f belongs to VMO and $w = e^f/(1 + x^2)$ belongs to W but has an oscillatory discontinuity at 0 which can be made arbitrarily wild by the choice of (r_n).

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