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A STRUCTURAL CRITERION FOR THE EXISTENCE OF INFINITE SIDON SETS

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Let G be any compact connected group with dual hypergroup $\hat{G}$. We establish in this paper a criterion by which the existence of an infinite Sidon set in $\hat{G}$ can be decided from the structure of G.

1. Introduction. Let G be any compact connected group with dual hypergroup $\hat{G}$. We establish in this paper a criterion by which the existence of an infinite Sidon set in $\hat{G}$ can be decided from the structure of G (see § 6 below).

Since Sidon [18] proved his famous result about Hadamard sets for the circle group, a recurring theme in the literature has been proof of existence or nonexistence of Sidon sets for more or less special compact groups G. For compact infinite abelian groups, existence was established by Hewitt and Zuckerman [10] (see [9, (37.15)]). This has been extended by Hutchinson [13] in the following form: if $\hat{G}$ has an infinite subset S whose elements are representations all of the same degree then S contains an infinite Sidon subset.

On the other hand, in [9], Hewitt and Ross showed nonexistence for $SU(2)$, thereby mounting the first attack on compact connected Lie groups G. Cecchini's result [3] shows that in fact every Sidon set for such G *must* have bounded degree. It is then not hard to show (see [13]) that nonexistence for such a G is equivalent to semisimplicity.

Semisimplicity is also equivalent to there being at most finitely many elements of $\hat{G}$ of any given degree, i.e., to G being *tall*.

For compact totally disconnected G, Hutchinson showed in [15] that tallness was again equivalent to nonexistence. Unfortunately, Figà-Talamanca and Rider [6] had given an example of a tall connected group with an infinite Sidon set, so that tallness was not a valid general criterion.

We show herein that the group $\prod_{n=2}^{\infty} PSU(n)$ admits no infinite Sidon set, though it is connected and is not semisimple. We also lay the general problem for compact connected groups to rest, by means of our criterion in § 6.

The case of a general compact group G remains open, to the best of our knowledge.

2. Mise-en-scène for Sidon sets. (2.1) *Notation.* If G is a compact group, $\hat{G}$ will denote a maximal set of pairwise inequivalent continuous unitary irreducible representations of G. The degree of a representation $\sigma \in \hat{G}$ will be denoted by d_σ. The space of trigonometric polynomials f on G whose Fourier transforms $\hat{f}$ vanish off P will be denoted $T_P(G)$. The properties of the norm $\|A\|_{\phi_1} = tr|A|$ on the space $M_n(C)$ of $n \times n$ complex matrices are developed in Hewitt and Ross [9, Appendix D].

For any $P \subseteq \hat{G}$ the *Sidon constant* $\kappa(P)$ is defined by

$$\kappa(P) = \sup\{\|\hat{f}\|_1 = \sum_{\sigma \in P} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} : f \in T_P(G),\ \|f\|_\infty \leqq 1\} .$$

One says that P is *Sidon* if $\kappa(P) < \infty$. For a singleton we replace $\kappa(\{\sigma\})$ by $\kappa(\sigma)$. The set P is *local Sidon* if its *local Sidon constant* $\kappa_0(P) = \sup\{\kappa(\sigma) \mid \sigma \in P\}$ is finite.

(2.2) *Two lemmas.* The following two lemmas will be needed later. The proofs are easy, and we omit them.

LEMMA 1. *Let $\phi: G \to H$ be a continuous epimorphism of compact groups. Let $E \subseteq \hat{H}$, and let $E\circ\phi = \{\sigma\circ\phi \mid \sigma \in E\} \subseteq \hat{G}$. Then E is a Sidon set for H if and only if $E\circ\phi$ is a Sidon set for G, and indeed $\kappa(E\circ\phi) = \kappa(E)$.*

LEMMA 2. *Let $P \subseteq \hat{G}$ be Sidon, and let $Q \subseteq \hat{G}$ be finite. Then $P \cup Q$ is Sidon.*

We note in passing that for finite Q the inequality

$$\kappa(Q) \leqq (\sum_{\sigma \in Q} d_\sigma^2)^{1/2}$$

holds. It is not known to us whether Lemma 2 holds for an arbitrary Sidon set Q. It does if Q is Sidon and $\{d_\tau \mid \tau \in Q\}$ is bounded (Bożejko [2]).

3. Mise-en-scène for compact Lie groups. We now summarize briefly the facts we need from the theory of representations of compact Lie groups. This information may be gleaned from Dieudonné [4], Humphreys [11], and Price [16].

(3.1) Suppose that G is a connected, simply-connected compact Lie group. Denote by $\mathfrak{g}$ the Lie algebra of G, and by $\mathfrak{g}_C$ the complexification of $\mathfrak{g}$. Then $\mathfrak{g}$ is semisimple (Dieudonné [4, (21.6.9)]), and so also is $\mathfrak{g}_C$.

Fix a maximal torus T of G. The Lie algebra $\mathfrak{t}$ of T, regarded as a Lie subalgebra of $\mathfrak{g}$, is a maximal commutative subalgebra of

$\mathfrak{g}$. Its complexification $\mathfrak{t}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$.

(3.2) Let Φ denote the set of roots of $\mathfrak{g}_C$ relative to $\mathfrak{t}_C$, and let (,) be the inner product induced by the Killing form on the $\boldsymbol{R}$-linear hull of Φ. For $\alpha \in \Phi$, put $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Fix a basis $\{\alpha_1, \cdots, \alpha_l\}$ of Φ (l is called the *rank* of G, $\mathfrak{g}$ or $\mathfrak{g}_C$). The *weights* λ are defined by requiring $(\lambda, \alpha^\vee) \in \boldsymbol{Z}$ for all $\alpha \in \Phi$. These comprise a lattice Λ, which is ordered by: $\lambda_1 \prec \lambda_2$ if $\lambda_2 - \lambda_1 = m_1\alpha_1 + \cdots + m_l\alpha_l$ with nonnegative integers $m_1, \cdots, m_l$.

(3.3) The set of *dominant weights* Λ^+ consists of those weights λ satisfying $(\lambda, \alpha_k^\vee) \geqq 0 \neq (k = 1, 2, \cdots, l)$. The *fundamental weights* $\lambda_1, \cdots, \lambda_l$ are defined by $(\lambda_j, \alpha_k^\vee) = \delta_{jk}$. Each $\lambda \in \Lambda^+$ can be written $n_1\lambda_1 + n_2\lambda_2 + \cdots + n_l\lambda_l$ with $n_1, \cdots, n_l$ nonnegative integers.

(3.4) If $\lambda \in \Lambda^+$ there is a finite-dimensional irreducible Lie representation $\phi_\lambda: \mathfrak{g}_C \to \mathfrak{g}'(H_\lambda)$ with $\prec -$ maximal weight λ, acting on a $\boldsymbol{C}$-vector space H_λ of dimension $d_\lambda < \infty$. There is a unique continuous representation $\sigma_\lambda: G \to GL(H_\lambda)$ of G such that

$$\sigma_\lambda(\exp_G(X)) = \exp(\phi_\lambda(X)) \text{ for all } X \in \mathfrak{g} ,$$

where $\exp_G: \mathfrak{g} \to G$ is the exponential mapping. Since G is compact, σ_λ is unitary with respect to a suitable inner product on H_λ.

(3.5) The set $\{\sigma_\lambda: \lambda \in \Lambda^+\}$ is a maximal set of pairwise inequivalent continuous unitary irreducible representations of G. We may thus take $\hat{G}$ to be this set.

Weyl's dimension formula states:

$$d_\lambda = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)} ,$$

where $\delta = \lambda_1 + \cdots + \lambda_l$ and $\Phi^+ = \{\alpha \in \Phi \mid \alpha \succ 0\}$.

(3.6) The representation theory of a connected compact Lie group G reduces to the simply-connected case via the *structure theorem* (Price [16, (6.4.5)]). This asserts that G is isomorphic to $(T_0 \times \tilde{G})/K$, where T_0 is the identity component of the center of G and $\tilde{G}$ is a connected, simply-connected compact Lie group, and where K is a finite subgroup of the center of $T_0 \times \tilde{G}$.

(3.7) A connected compact Lie group is called *semisimple* if its Lie algebra $\mathfrak{g}$ (equivalently $\mathfrak{g}_C$) is semisimple. This is equivalent to $T_0 = \{1\}$ in the notation of (3.6), i.e., to G having finite center. When G is semisimple, and π, $\tilde{G} \to G$ the surjection implicit in (3.6), then the map $\sigma \mapsto \sigma \circ \pi$ identifies $\hat{G}$ with a subset Λ_1 of Λ^+, where Λ^+ is the set of dominant weights of $\tilde{G}$.

(3.8) A connected compact Lie group G is called *simple* if its Lie algebra $\mathfrak{g}$ is simple. Equivalently, $\mathfrak{g}_C$ is simple. For if $\mathfrak{g}$ is simple, then so is $\mathfrak{g}_C$ because it is semisimple, and any compact real form of $\mathfrak{g}_C$ must be isomorphic to $\mathfrak{g}$. The reverse implication holds for any Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$.

4. The key estimates. In this section we derive inequalities for Sidon constants to be used later.

(4.1) *A special coordinate function.* Let G be a simply-connected connected (and therefore semisimple) compact Lie group, and let Λ^+ be the set of dominant weights for G. For each fixed $\lambda \in \Lambda^+$ let $B_\lambda = \{\zeta_1^\lambda, \cdots, \zeta_{d_\lambda}^\lambda\}$ be an orthonormal basis for H_λ, and let $\sigma_{jk}^\lambda(y) = \langle \sigma_\lambda(y)\zeta_k^\lambda, \zeta_j^\lambda \rangle$ for $y \in G$ and $j, k = 1, \cdots, d_\lambda$. Thus $(\sigma_{jk}^\lambda)_{j,k=1}^{d_\lambda}$ is the matrix of $\sigma_\lambda(y)$ with respect to B_λ. We may choose B_λ so that $(\sigma_{jk}^\lambda(x))$ is a diagonal matrix for each $x \in T$, let us say $(\sigma_{jk}^\lambda(x)) = \mathrm{diag}\,(\chi_1^\lambda(x), \cdots, \chi_{d_\lambda}^\lambda(x))$. The maps $\chi_j^\lambda \colon T \to \boldsymbol{T}$ are characters of T, and so for each $\lambda \in \Lambda^+$ and $j \in \{1, \cdots, d_\lambda\}$ there is a unique linear map $\mu_j^\lambda \colon \mathfrak{t}_C \to \boldsymbol{C}$ such that

$$\chi_j^\lambda(\exp_G(X)) = \exp(\mu_j^\lambda(X)) \quad (X \in \mathfrak{t}) .$$

Since $\sigma_\lambda(x)\zeta_j^\lambda = \chi_j^\lambda(x)\zeta_j^\lambda$ for all $x \in T$, it follows that

$$\phi_\lambda(X)\zeta_j^\lambda = \mu_j^\lambda(X)\zeta_j^\lambda \quad (X \in \mathfrak{t}) ,$$

with $\{\zeta_1^\lambda, \cdots, \zeta_{d_\lambda}^\lambda\}$ a corresponding set of weight vectors. Reordering the basis B_λ if necessary, we may assume that μ_1^λ is the maximal weight λ of ϕ_λ. Since the weight space of the maximal weight λ is one-dimensional, we have $\mu_j^\lambda \lneqq \lambda$ for $j = 2, \cdots, d_\lambda$.

For each $\lambda \in \Lambda^+$ choose a basis and order it in the above manner. For $r \in \{1, \cdots, l\}$ write ψ_r instead of σ_{11}^λ when λ is the fundamental weight λ_r. Since $\sigma_{11}^\lambda(\exp_G(X)) = \exp(\lambda(X))$ for $X \in \mathfrak{t}$, it is clear that if $\lambda = n_1\lambda_1 + n_2\lambda_2 + \cdots + n_l\lambda_l$, then $\sigma_{11}^\lambda(y) = \psi_1(y)^{n_1} \cdots \psi_l(y)^{n_l}$ for all $y \in G$ of the form $\exp_G(X)$, $X \in \boldsymbol{t}$. In fact we have the following

PROPOSITION. *With notation as above, the identity*

$$(*) \qquad \sigma_{11}^\lambda = \psi_1^{n_1}\psi_2^{n_2} \cdots \psi_l^{n_l}$$

holds on G.

The proof is to be found in Giulini and Travaglini [8].

(4.2) Let G be as in the previous section. We now obtain lower bounds for the (local) Sidon constants $\kappa(\sigma)$ for $\sigma \in \hat{G}$.

PROPOSITION. *Let $\lambda = n_1\lambda_1 + \cdots + n_l\lambda_l \in \Lambda^+$, and let κ_λ be the Sidon constant of the corresponding representation σ_λ. Then for any $\varepsilon \in]0, 1/2[$ we have, provided $\lambda \neq 0$,*

$$\kappa_\lambda \geqq \frac{\varepsilon}{2} \frac{d_\lambda^{\frac{1}{2}-\varepsilon}}{M_\lambda^{\frac{1}{2}+\varepsilon}}$$
$$\geqq \frac{\varepsilon}{2} \frac{d_\lambda^{\frac{1}{2}-\varepsilon}}{|\Phi^+|^{\frac{1}{2}+\varepsilon}} ,$$

where M_λ is the number of positive roots $\alpha \in \Phi^+$ such that $(\lambda, \alpha) \neq 0$.

Proof. Retaining the notation of (4.1), let $f = \sigma_{11}^\lambda$. Then the orthogonality relations yield $\int_G |f(x)|^2 dx = 1/d_\lambda$. For $s \in N$ we have $f^s = (\psi_1^{n_1} \cdots \psi_l^{n_l})^s = \psi_1^{sn_1} \cdots \psi_l^{sn_l} = \sigma_{11}^{s\lambda}$, where $s\lambda = sn_1\lambda_1 + \cdots + sn_l\lambda_l$. Hence $\int_G |f(x)|^{2s} dx = 1/d_{s\lambda}$. On the other hand, since $f \in T_{\sigma_\lambda}(G)$, we have $\|f\|_{2s} \leqq 2\kappa_\lambda \sqrt{s} \|f\|_2$ (see Hewitt and Ross [9, (37.10) and (37.25)]). Hence we have

$$\kappa_\lambda \geqq \frac{1}{2\sqrt{s}} \frac{(1/d_{s\lambda})^{1/(2s)}}{(1/d_\lambda)^{1/2}} = \frac{1}{2\sqrt{s}} \left[\frac{d_\lambda^s}{d_{s\lambda}} \right]^{1/(2s)} .$$

Now $d_\lambda = \prod_{\alpha \in \Gamma^+} (\lambda + \delta, \alpha)/(\delta, \alpha)$ (Weyl's formula), and $(s\lambda + \delta, \alpha)/(\delta, \alpha) \leqq s(\lambda + \delta, \alpha)/(\delta, \alpha)$. Since $(s\lambda + \delta, \alpha)/(\delta, \alpha) = (\lambda + \delta, \alpha)/(\delta, \alpha)$ when $(\lambda, \alpha) = 0$, we obtain the estimate $d_{s\lambda} \leqq s^{M_\lambda} d_\lambda$. Hence

$$\kappa_\lambda \geqq \frac{d_\lambda^{1/2 - 1/(2s)}}{2s^{1/2 + M_\lambda/(2s)}} .$$

Let $\varepsilon \in]0, 1/2[$ be given, and let A be the integer satisfying $A - 1 < 1/2\varepsilon \leqq A$. Let $s = M_\lambda A$ ($s \neq 0$ since $\lambda \neq 0$). Since $1/(2M_\lambda A) \leqq 1/(2A) \leqq \min\{1/2, \varepsilon\}$ we have

$$\kappa_\lambda \geqq \frac{d_\lambda^{\frac{1}{2} - 1/(2M_\lambda A)}}{2(M_\lambda A)^{\frac{1}{2} + 1/(2A)}} \geqq \frac{d_\lambda^{\frac{1}{2}-\varepsilon}}{2AM_\lambda^{\frac{1}{2}+\varepsilon}} > \frac{\varepsilon}{2} \frac{d_\lambda^{\frac{1}{2}-\varepsilon}}{M_\lambda^{\frac{1}{2}+\varepsilon}} ,$$

the last since $A < 1/2\varepsilon + 1 < 1/\varepsilon$.

(4.3) Using now the structure theorem for compact connected Lie groups, we are able to give estimates for all local Sidon constants in such groups.

PROPOSITION. *Let G be a connected compact Lie group. Then there is a constant $C_0 > 0$ (depending only on G) such that*

$$\kappa(\sigma) \geqq \frac{C_0 d_\sigma^{1/2}}{ln(d_\sigma)}$$

for any $\sigma \in \hat{G}$, $d_\sigma \neq 1$.

Proof. By (3.6), G is a quotient of a group $T_0 \times \tilde{G}$, where T_0 is compact and abelian, and $\tilde{G}$ is a compact connected simply-connected Lie group. Let $\pi: T_0 \times \widetilde{G} \to G$ be the quotient map. Then by Hewitt and Ross [9, (27.43)], if $\sigma \in \widehat{G}$ then $\sigma \circ \pi$ has the form $(x, y) \mapsto \chi(x)\tau(y)$ for $x \in T_0$ and $y \in \widetilde{G}$, where $\chi: T_0 \to T$ is a character of T_0 and $\tau \in (\widetilde{G})\hat{}$. Clearly $T_{\sigma\circ\pi}(T_0 \times \widetilde{G})$ consists precisely of all functions f of the form $f(x, y) = \chi(x)g(y)$, with $g \in T_\tau(\widetilde{G})$.

It is clear that $\|\hat{f}\|_1 = \|\hat{g}\|_1$ and $\|f\|_\infty = \|g\|_\infty$ for such f and g. Hence $\kappa(\sigma\circ\pi) = \kappa(\tau)$, and so by Lemma 1 of (2.2) and Proposition 4.2 with $\varepsilon = 1/ln(d_\sigma)$, a constant C_0 exists for which the inequality holds for all σ with $d_\sigma \geqq 8$. Since $\kappa(\sigma) \geqq 1$, we may diminish C_0 if necessary so that the inequality holds for $1 < d_\sigma < 8$ as well.

(4.4) *Estimates independent of G.* The constant C_0 appearing in (4.3) depends on the group. We give, after a preliminary lemma, an estimate valid for all such G simultaneously.

LEMMA. *Let $\mathfrak{g}_C$ be simple of rank l. If $0 \neq \lambda = \sum_{i=1}^{l} n_i\lambda_i \in \Lambda^+$, then $d_\lambda \geqq l^3/8$ except when either*

(i) *$\mathfrak{g}_C$ is of type A_l and $\lambda \in \{\lambda_1, \lambda_l, \lambda_2, \lambda_{l-1}, 2\lambda_1, 2\lambda_l, \lambda_1 + \lambda_l\}$ or*

(ii) *$\mathfrak{g}_C$ is of type B_l, C_l or D_l and $\lambda \in \{\lambda_1, \lambda_2, 2\lambda_1\}$.*

Proof. We may clearly assume $l \geqq 3$. From the Weyl dimension formula it is obvious that

(a) $d_\lambda \geqq d_\mu$ if $\mu = m_1\lambda_1 + \cdots + m_l\lambda_l \in \Lambda^+$ and $n_j \geqq m_j$ for all j.

Assume first that $\mathfrak{g}_C$ is of type A_l. The set of positive roots of $\mathfrak{g}_C$ is $\{\alpha_i + \cdots + \alpha_j \mid 1 \leqq i \leqq j \leqq l\}$, and $\check{\alpha}_j = \alpha_j$ for each j (see Tits [19]). So Weyl's formula becomes

$$d_\lambda = \prod_{1\leqq i\leqq j\leqq l}\left[\frac{n_i + \cdots + n_j + j - i + 1}{j - i + 1}\right].$$

From this it is easy to see that

(b) $d_\lambda = d_{\lambda'}$ if $\lambda' = n_l\lambda_1 + \cdots + n_1\lambda_l$, and

(c) $d_\lambda = \begin{bmatrix} l+1 \\ j \end{bmatrix}$ if $\lambda = \lambda_j$, $1 \leqq j \leqq l$.

So let $\lambda = n_1\lambda_1 + \cdots + n_l\lambda_l \in \Lambda^+\backslash\{0\}$ be given. If λ is not one of the weights listed in (i) above, then (a), (b) and (c) imply that $d_\lambda \geqq d_\mu$ for some $\mu \in \{\lambda_3, \lambda_1 + \lambda_2, \lambda_1 + \lambda_{l-1}, 2\lambda_1 + \lambda_l, \lambda_2 + \lambda_{l-1}, 3\lambda_1, 2\lambda_2\}$. Direct calculation yields that the smallest such d_μ is $\begin{bmatrix} l+1 \\ 3 \end{bmatrix} > l^3/8$.

Next suppose that $\mathfrak{g}_C$ is of type B_l, C_l, or D_l. If

$$\begin{aligned} d_\lambda &\geqq \min\{d_{\lambda_{l-1}}, d_{\lambda_l}\} \\ &\geqq 2^{l-1} \quad \text{(see Tits [19, p. 30 ff])} \\ &\geqq l^3/8\,. \end{aligned}$$

Now notice (see Bourbaki [1, p. 250 ff]), that for each pair (i, j) with $1 \leqq i \leqq j \leqq l$ (except $(i, j) = (l-1, l)$ in case D_l) Φ^+ contains $m(\alpha_i^\vee + \cdots + \alpha_j^\vee)$ with $m = 1, 2$ or $1/2$. Thus in Weyl's formula a factor $(n_i + \cdots + n_j + j - i + 1)/(j - i + 1)$ appears for each such pair (i, j). Consequently, in the B_l, C_l, D_l case, if $\lambda = \sum_{i=1}^{l} n_i\lambda_i$ with $n_{l-1} = n_l = 0$, then

(d) d_λ is at least the degree of the corresponding weight in A_l. Thus if λ is not listed in (ii), and hence is not listed in (i), we have $d_\lambda \geqq l^3/8$ by the A_l case.

Finally, assume that g_C is an exceptional algebra. Then by (a) we have $d_\lambda \geqq \min\{d_{\lambda_j} | 1 \leqq j \leqq l\}$, which is at least $l^3/8$ in each case (see Tits [19, p. 41 ff]).

Here now are our absolute estimates for the local Sidon constant:

PROPOSITION. *Let G be a simple compact Lie group of rank l. There is an absolute constant C ($C = 560$ will do) with the following property. Let $\sigma \in \hat{G}$ be nontrivial with $\lambda \in \Lambda^+$ its corresponding weight. Then the inequality*

$$\kappa(\sigma) \geqq l^{1/3}/C$$

holds, except when either

(i) *g_C is of type A_l and $\lambda \in \{\lambda_1, \lambda_l\}$, or*

(ii) *g_C is of type B_l, C_l, or D_l and $\lambda = \lambda_1$.*

Proof. Referring to Tits [19], we find that $|\Phi^+| \leqq 2l^2$ in all cases. If λ is not one of the weights listed in (i) and (ii) of the preceding lemma, then $d_\lambda \geqq l^3/8$. With these estimates, Proposition 4.2 with $\varepsilon = 1/30$ shows that $\kappa_\lambda \geqq l^{1/3}/240$.

If g_C is of type A_l and $\lambda \in \{\lambda_2, \lambda_{l-1}, 2\lambda_1, 2\lambda_l, \lambda_1 + \lambda_l\}$ or if g_C is of type B_l, C_l, or D_l and $\lambda \in \{\lambda_2, 2\lambda_1\}$, explicit calculations show that $d_\lambda \geqq l^2/2$ and $M_\lambda \leqq 4l$. Proposition 4.2 with appropriate $\varepsilon > 0$ again shows that $\kappa_\lambda \geqq l^{1/3}/560$.

REMARK. In fact inequalities

$$\kappa(\sigma) \geqq Cl^{1/2}/ln(l) \qquad (\text{if } l > 1)$$

and

$$\kappa(\sigma) \geqq Cd_\sigma^{1/6}/ln(d_\sigma) \quad (\text{if } \sigma \neq 1)$$

for some absolute constant $C > 0$ can be obtained by taking $\varepsilon = 1/ln(l)$ [resp. $\varepsilon = 1/ln(d_\sigma)$] in the above derivation (provided l and d_σ are $\geqq e^2$).

(4.5) There is good reason to exclude the weights in (i) or (ii) of the previous proposition. Their local Sidon constants are bounded independent of g_C, as the next two lemmas show. This fact is responsible for the Sidonicity of the Figà-Talamanca-Rider set constructed in § 5.

LEMMA 1. *Let $G = SU(n)$ or $U(n)$, where $n \geqq 2$. Then there is a constant K independent of n such that*

$$\|A\|_{\phi_1} + \|B\|_{\phi_1} \leqq K \max\{|tr(AU) + tr(B\bar{U})| : U \in G\}$$

for any $A, B \in M_n(\boldsymbol{C})$. Hence if σ denotes the self-representation of G, then $\kappa\{\sigma, \bar{\sigma}\} \leqq K$.

Proof. First note that for $z, w \in \boldsymbol{C}$ and $n \geqq 3$ we have

$$\max\{|\alpha z + \bar{\alpha} w| : \alpha \in \boldsymbol{T}\} = |z| + |w| \leqq 4 \max\{|\alpha z + \bar{\alpha} w| : \alpha^n = 1\} .$$

Hence for $n \geqq 3$ we have

$$\max\{|tr(AU) + tr(B\bar{U})| : U \in U(n)\}$$
$$\leqq 4 \max\{|tr(AU) + tr(B\bar{U})| : U \in SU(n)\} .$$

This shows that we may assume $G = U(n)$.

Now if $A, B \in M_n(\boldsymbol{C})$ we have for some $U \in U(n)$,

$$\|A\|_{\phi_1} = tr|A| = tr(AU) \leqq \frac{1}{2}\{|tr(AU) + tr(B\bar{U})| + |tr(iAU) - tr(iB\bar{U})|\} .$$

This, together with a similar inequality for $\|B\|_{\phi_1}$ shows that for $n \geqq 3$, $K = 2$ will do if $G = U(n)$, and $K = 8$ will do if $G = SU(n)$. If $n = 2$, $\kappa(\{\sigma, \bar{\sigma}\}) \leqq (d_\sigma^2 + d_{\bar{\sigma}}^2)^{1/2} = 2\sqrt{2}$.

LEMMA 2. *Let σ be the self-representation of $G = SU(n)$, $SO(n)$, or $Sp(n)$. Then*

(a) $\kappa(\sigma) = \kappa(\bar{\sigma}) = 1$ *if* $G = SU(n)$,

(b) $\kappa(\sigma) \leqq 4$ *if* $G = SO(n)$,

(c) $\kappa(\sigma) \leqq 4$ *if* $G = Sp(n)$.

Proof. (a) This follows from Hewitt and Ross [9, (D. 30)].

(b) If $A \in M_n(\boldsymbol{C})$, write $A = B + iC$, where $B, C \in M_n(\boldsymbol{R})$. Then $\max\{|tr(AO)| : 0 \in SO(n)\} \geqq \max\{|tr(BO)| : O \in SO(n)\}$, and similarly for C. By Gantmacher [7, p. 286] we can write $B = |B|O_1$, where $O_1 \in O(n)$. If n is odd then O_1 or $-O_1 \in SO(n)$, and so $\max\{|tr(BO)| : O \in SO(n)\} \geqq tr|B| = \|B\|_{\phi_1}$ (thus $\kappa(\sigma) \leqq 2$ for odd n). If n is even and $O_1 \in SO(n)$ this inequality still holds. If $\det O_1 = -1$, let $D = \operatorname{diag}(1, \cdots, 1, -1, 1, \cdots, 1)$, the "$-1$" appearing in position j, where β_j is a minimal diagonal entry of $|B|$. Then

$$\begin{aligned}\max\{|tr(BO)|: O\in SO(n)\} &\geqq |tr(BO_1^tD)| = tr(|B|D)\\ &= \beta_1+\cdots+\beta_n-2\beta_j\\ &\geqq \Big(1-\frac{2}{n}\Big)(\beta_1+\cdots+\beta_n)\\ &= \Big(1-\frac{2}{n}\Big)\|B\|_{\phi_1}\end{aligned}$$

similarly for C. Hence for $n \geqq 4$ we have

$$\|A\|_{\phi_1} \leqq \|B\|_{\phi_1} + \|C\|_{\phi_1} \leqq 2\frac{n}{n-2}\max\{|tr(AO)|: O\in SO(n)\}\,.$$

Thus, once $n \geqq 4$, $\kappa(\sigma) \leqq 4$ holds, while $\kappa(\sigma) \leqq d_\sigma \leqq 4$ for $n < 4$.

(c) Recall that $Sp(n)$ consists of all $2n \times 2n$ unitary matrices of the form $T = \begin{bmatrix} U & V \\ -\bar{V} & \bar{U}\end{bmatrix}$ where $U, V \in M_n(\boldsymbol{C})$. If $A \in M_{2n}(\boldsymbol{C})$, write $A = \begin{bmatrix} X & Y \\ Z & W\end{bmatrix}$ with $X, Y, Z, W \in M_n(\boldsymbol{C})$. Then taking $T \in Sp(n)$ as above with $V = 0$, we have

$$\begin{aligned}\max\{|tr(AT)|: T\in Sp(n)\} &\geqq \max\{|tr(XU + W\bar{U})|: U \in U(n)\}\\ &\geqq \frac{1}{2}(\|X\|_{\phi_1} + \|W\|_{\phi}) \text{ by Lemma 1.}\end{aligned}$$

Similarly, taking $T \in Sp(n)$ as above with $U = 0$, we find

$$\max\{|tr(AT)|: T\in Sp(n)\} \geqq \frac{1}{2}(\|Y\|_{\phi_1} + \|Z\|_{\phi_1})\,.$$

There follows

$$\begin{aligned}\|A\|_{\phi_1} &\leqq \|X\|_{\phi_1} + \|Y\|_{\phi_1} + \|Z\|_{\phi_1} + \|W\|_{\phi_1}\\ &\leqq 4\max\{|tr(AT)|: T\in Sp(n)\}\,.\end{aligned}$$

5. Sidon sets on products. In § 6 below we shall need the existence of infinite Sidon sets for certain infinite products. This we now establish.

(5.1) Let G be a compact connected group which is the cartesian product $\prod_{\alpha\in A} G_\alpha$ of normal subgroups G_α, each of which is isomorphic to one of the four groups $SU(n_\alpha)$, $Sp(n_\alpha)$, $SO(n_\alpha)$, or $Spin(n_\alpha)$ for some $n_\alpha \in \boldsymbol{N}$. Let $\sigma_\alpha \in \hat{G}_\alpha$ be the self-representation in the first three cases, and in the fourth case let $\sigma_\alpha = \tau_\alpha \circ \psi_\alpha$, where $\psi_\alpha: Spin(n_\alpha) \to SO(n_\alpha)$ is the covering map and τ_α is the self-representation of $SO(n_\alpha)$. Let $P = \{\sigma_\alpha \circ \pi_\alpha \mid \alpha \in A\} \subseteq \hat{G}$, where $\pi_\alpha: G \to G_\alpha$ is the canonical projection.

Taking notice of the fact that the decomposition $\prod_{\alpha\in A} G_\alpha$ of such a group is unique up to reindexing, we see that the set $P \cup$

$\bar{P} \subseteq \hat{G}$ depends only on G. We shall call $P \cup \bar{P} \cup \{1\}$ the *Figà-Talamonca-Rider set* (FTR set) of G, since these authors in [6] constructed what is essentially the set P in the case in which $G_\alpha = SU(n_\alpha)$ for each α.

We shall proceed to show that the FTR set (and hence P) is a Sidon set. We shall also give a description of all local Sidon sets for such a group G, in terms of the FTR set.

PROPOSITION 5.2. *Let G be the product $\prod_{\alpha \in A} G_\alpha$ of a family of compact groups, for each $\alpha \in A$ let $P_\alpha \subseteq \hat{G}_\alpha$ be a Sidon set, and let $\pi_\alpha: G \to G_\alpha$ be the canonical projection. Then the set $P = \bigcup_{\alpha \in A} (P_\alpha \circ \pi_\alpha) \subseteq \hat{G}$ is a Sidon set if and only if $\sup_\alpha \kappa(P_\alpha) < \infty$.*

Proof. If P is Sidon, then so is $P_\alpha \circ \pi_\alpha \subseteq P$, and hence by Lemma 1 of (2.2), $P_\alpha \subseteq \hat{G}_\alpha$ is Sidon with $\kappa(P_\alpha) = \kappa(P_\alpha \circ \pi_\alpha) \leqq \kappa(P) < \infty$. Thus $\sup_\alpha \kappa(P_\alpha) \leqq \kappa(P) < \infty$.

Suppose conversely that $\sup_\alpha \kappa(P_\alpha) = K < \infty$. Applying Lemma 2 of (2.2) with $Q = \{1\}$, we may assume that $1 \notin P_\alpha$ for each α. The union $\bigcup_\alpha (P_\alpha \circ \pi_\alpha)$ is then disjoint. Let $f \in T_P(G)$. Then $f = \sum_\alpha f_\alpha$, with $f_\alpha \neq 0$ for only finitely many α's, and for all α, $f_\alpha \in T_{P_\alpha \circ \pi_\alpha}(G)$, so that $f_\alpha = g_\alpha \circ \pi_\alpha$ for some $g_\alpha \in T_{P_\alpha}(G_\alpha)$.

Now for each $\alpha \in A$, there exists an element $x_\alpha \in G_\alpha$ such that $\|\hat{g}_\alpha\|_1 \leqq K \|g_\alpha\|_\infty = K |g_\alpha(x_\alpha)|$, and so $\|\hat{f}\|_1 = \sum_\alpha \|\hat{f}_\alpha\|_1 = \sum_\alpha \|\hat{g}_\alpha\|_1 \leqq K \sum_\alpha |g_\alpha(x_\alpha)|$. It is elementary that for some $\theta \in \{1, i, -1, -i\}$ we have

$$\sum_{\alpha \in A} |g_\alpha(x_\alpha)| \leqq 4 Re(\theta \sum_{\alpha \in A_1} g_\alpha(x_\alpha)) ,$$

where $A_1 = \{\alpha \in A \mid Re[\theta g_\alpha(x_\alpha)] > 0\}$. Since $\theta g_\alpha \in T_{P_\alpha}(G_\alpha)$ and $1 \notin P_\alpha$, we have $\int_{G_\alpha} \theta g_\alpha(x) dx = 0$, whence $Re(\theta g_\alpha(y_\alpha)) \geqq 0$ for some $y_\alpha \in G_\alpha$ (we may take $y_\alpha = x_\alpha$ if $\alpha \in A_1$). There follows

$$\begin{aligned} \|\hat{f}\|_1 &\leqq 4K Re(\theta \sum_{\alpha \in A_1} g_\alpha(y_\alpha)) \\ &\leqq 4K Re(\theta \sum_{\alpha \in A} g_\alpha(y_\alpha)) \\ &= 4K Re(\theta f(y)), \text{ where } y = (y_\alpha) \in G \\ &\leqq 4K \|f\|_\infty . \end{aligned}$$

This shows that P is Sidon.

COROLLARY 5.3. *Let G be as in* (5.1) *and let $P_0 \subseteq \hat{G}$ be its* FTR *set. Then P_0 is a Sidon set.*

Proof. With the notation of (5.1), we have $P_0 = P \cup \bar{P} \cup \{1\}$.

Hence the statement follows from Lemma 2 of (2.2), (5.2) and Lemmas 1 and 2 of (4.5), since $\sigma_\alpha = \bar{\sigma}_\alpha$ save when $G_\alpha = SU(n_\alpha)$.

Note that the Sidonicity of P was established by Hutchinson ([12, (9.5)], [13]) by methods different from ours.

(5.4) Next we prepare the way for our description of all local Sidon sets for G with three lemmas.

LEMMA 1. *Let G be the product of the family $(G_\alpha)_{\alpha\in A}$ of compact groups. For each α, let $\sigma_\alpha \in \hat{G}_\alpha$, with $\sigma_\alpha = 1$ except for finitely many α's. Let $\sigma \in \hat{G}$ be the representation of G corresponding to (σ_α) (see Hewitt and Ross [9, (27.43)]). Then $\kappa(\sigma) \geqq \prod_\alpha \kappa(\sigma_\alpha)$.*

Proof. Let $A_1 = \{\alpha \in A \mid \sigma_\alpha \neq 1\}$. For $\alpha \in A_1$ choose $f_\alpha \in T_{\sigma_\alpha}(G_\alpha)$ with $\|f_\alpha\|_\infty = 1$ and $\|\hat{f}_\alpha\|_1 = \kappa(\sigma_\alpha)$. Define $f\colon G \to \boldsymbol{C}$ by $f(x) = \prod_{\alpha\in A_1} f_\alpha(x_\alpha)$ for $x = (x_\alpha) \in G$. One sees routinely that

$$\|\hat{f}\|_1 = \prod_{\alpha\in A_1} \|\hat{f}_\alpha\|_1 = \prod_{\alpha\in A_1} \kappa(\sigma_\alpha) = \prod_{\alpha\in A} \kappa(\sigma_\alpha)\ .$$

LEMMA 2. *Let G and H be compact groups and let $P \subseteqq \hat{G}$ and $Q \subseteqq \hat{H}$ be finite. Let $P \times Q$ denote $\{\sigma \times \tau \mid \sigma \in P,\ \tau \in Q\} \subseteqq (G \times H)^\wedge$ and let $\kappa = \kappa(P \times Q)$ be its Sidon constant. Then $\kappa \geqq [\min\{\sum_{\sigma\in P} d_\sigma, \sum_{\tau\in Q} d_\tau\}]^{1/2}$. As a consequence, $P \times Q$ is never Sidon if P and Q are infinite.*

Proof. Let $s = \operatorname{card} P$, $t = \operatorname{card} Q$, let $P = \{\sigma^{(1)}, \cdots, \sigma^{(s)}\}$, $Q = \{\tau^{(1)}, \cdots, \tau^{(t)}\}$, and put $m = \sum_{\sigma\in P} d_\sigma$, $n = \sum_{\tau\in Q} d_\tau$. Put $d^{(k)} = d_{\sigma(k)}$ $(1 \leqq k \leqq s)$ and $e^{(l)} = d_{\tau(l)}$ $(1 \leqq l \leqq t)$. Fix orthonormal bases for the representation spaces of all $\sigma \in P$ and all $\tau \in Q$, and denote the corresponding coefficient functions by $(\sigma_{ij}^{(k)})_{i,j=1}^{d\{k\}}$, $(\tau_{ij}^{(l)})_{i,j=1}^{e\{l\}}$ $(1 \leqq k \leqq s,\ 1 \leqq l \leqq t)$. If $A = (a_{ij})$ is any m n complex matrix, partition the rows and columns of A using the scheme $m = d^{(1)} + d^{(2)} + \cdots + d^{(s)}$, $n = e^{(1)} + \cdots + e^{(t)}$. Thus A consists of an $s \times t$ matrix of blocks, and the (k, l)th block is a $d^{(k)} \times e^{(l)}$ matrix. Denote by α_{ij}^{kl} the (i, j)th entry of the (k, l)th block $(1 \leqq i \leqq d^{(k)},\ 1 \leqq j \leqq d^{(l)})$. For any such A, the function

$$g\colon (x, y) \longmapsto \sum_{k,l,i,j} \alpha_{ij}^{kl} \sigma_{ii}^{(k)}(x) \tau_{jj}^{(l)}(y)$$

belongs to $T_{P\times Q}(G \times H)$, and hence we have

$$\sum_{k,l,i,j} |\alpha_{ij}^{kl}| = \|\hat{g}\|_1 \leqq \kappa \max\{|\sum_{k,l,i,j} \alpha_{ij}^{kl} \sigma_{ii}^{(k)}(x) \tau_{jj}^{(l)}(y)|\colon x \in G,\ y \in H\}\ .$$

Let $p = \min\{m, n\}$ and let B be a $p \times p$ unitary matrix all of whose entries have modulus $p^{-1/2}$. Take for A above the matrix with entries

$$a_{ij} = \begin{cases} b_{ij}, & 1 \leqq i, j \leqq p \\ 0, & \text{otherwise.} \end{cases}$$

Because $|\sigma_{ii}^{(k)}(x)| \leqq 1$ for all $x \in G$ (and likewise for the τ's) the previous inequality yields

$$\begin{aligned} p^2 \cdot p^{-1/2} &\leqq \kappa \max \{|\sum_{i,j} b_{ij}\eta_j\xi_i| : \xi, \eta \in C^p, |\xi_i| \leqq 1, |\eta_j| \leqq 1, \text{ all } i, j\} \\ &= \kappa \max_{\xi,\eta} |\langle B\eta, \xi\rangle| \\ &\leqq \kappa \|B\| p^{1/2} \cdot p^{1/2} \\ &= \kappa p . \end{aligned}$$

Thus $\kappa \geqq p^{1/2}$ as claimed.

LEMMA 3. *Let G and H be compact groups. Let $P \subseteqq \hat{G}$ be finite and let $Q \subseteqq \hat{H}$ be Sidon. Then $P \times Q \subseteqq (G \times H)\hat{}$ is Sidon, with*

$$\kappa(P \times Q) \leqq (\sum_{\sigma \in P} d_\sigma^2)\kappa(Q) .$$

Proof. If f is any continuous function on G, and if $\sigma \in P$, $\tau \in Q$, then $\|\hat{f}(\sigma \times \tau)\|_{\phi_1} \leqq d_\sigma \int_G \|\hat{f}_x(\tau)\|_{\phi_1} dx$, where we write f_x for the function $y \mapsto f(x, y)$. If $f \in T_{P\times Q}(G \times H)$ then

$$\begin{aligned} \|\hat{f}\|_1 &= \sum_{P\times Q} d_{\sigma\times\tau} \|\hat{f}(\sigma \times \tau)\|_{\phi_1} \\ &= \sum_{\sigma \in P} d_\sigma \sum_{\tau \in Q} d_\tau \|\hat{f}(\sigma \times \tau)\|_{\phi_1} \\ &\leqq (\sum_{\sigma \in P} d_\sigma^2) \int_G \|\hat{f}_x\|_{\phi_1} dx \\ &\leqq (\sum_{\sigma \in P} d_\sigma^2) \int_G \kappa(Q) \|f_x\|_\infty dx \\ &\leqq (\sum_{\sigma \in P} d_\sigma^2) \kappa(Q) \|f\|_\infty . \end{aligned}$$

PROPOSITION 5.5. *Let $G = \prod_{\alpha \in A} G_\alpha$ with G_α a simple connected compact Lie group of rank l_α. Let P be any local Sidon set in $\hat{G}$. Then there is a partition $A = A_1 \cup A_2 \cup A_3$ of A and a subset P_j of $\hat{G}_j$, the dual of $G_j = \prod_{\alpha \in A_j} G_\alpha$ $(j = 1, 2, 3)$ such that*

(i) $P \subseteqq P_1 \times P_2 \times P_3$,

(ii) $P_1 = \{1\}$,

(iii) $\sup \{l_\alpha | \alpha \in A_2\} < \infty$ *and* $\sup \{d_\sigma | \alpha \in P_2\} < \infty$,

and (iv) G_3 *is as described in* (5.1) *and* P_3 *is its* FTR *set.*

Conversely, all sets P satisfying these conditions are local Sidon.

Proof. Each G_α is a quotient $\tilde{G}_\alpha/K_\alpha$, where $\tilde{G}_\alpha$ is a simple connected simply-connected compact Lie group. Let $\psi_\alpha: \tilde{G}_\alpha \to G_\alpha$ denote the quotient map.

Each $\sigma \in \hat{G}$ may be regarded as a family $(\sigma_\alpha)_{\alpha \in A}$, where $\sigma_\alpha \in \hat{G}_\alpha$ and where only finitely many σ_α's are nontrivial. Let $A_1 = \{\alpha \in A | \sigma_\alpha$ is trivial for every $\sigma = (\sigma_\alpha) \in P\}$. With C the constant of Proposition 4.4, let $A_2 = \{\alpha \in A \backslash A_1 | l_\alpha \leqq (C\kappa_0(P))^3\}$. Finally, let $A_3 = A\backslash(A_1 \cup A_2)$, and let $P_j = \{(\sigma_\alpha)_{\alpha \in A_j} | (\sigma_\alpha)_{\alpha \in A} \in P\} \subseteqq \hat{G}_j$ $(j = 1, 2, 3)$. Obviously (i) and (ii) hold. Also $\sup\{l_\alpha | \alpha \in A_2\} = l \leqq (C\kappa_0(P))^3$ by definition of A_2. By the classification theorem, this imples that there are only finitely many pairwise nonisomorphic groups in the set $\{G_\alpha | \alpha \in A_2\}$. Proposition 4.3 then entails $\sup\{d_{\sigma_\alpha} | \alpha \in A_2, \sigma \in P\} = d < \infty$. But for each $\alpha \in A$ there can be only finitely many $\tau \in \hat{G}_\alpha$ with $d_\tau \leqq d$, by the Weyl dimension formula (3.5). Accordingly, if $\sigma = (\sigma_\alpha) \in P$, let N_σ denote the number of $\alpha \in A_2$ with $\sigma_\alpha \neq 1$. If $\sup\{N_\sigma | \sigma \in P\} = \infty$, then the foregoing remarks show that for each $n \in N$ there exists $\sigma \in P$ with σ_α equal to the same nontrivial representation, τ say, for at least $2n$ α's in A_2. Then by Lemmas 1 and 2 of (5.4) we obtain $\kappa_0(P) \geqq \kappa(\tau \times \tau \times \cdots \times \tau)$ ($2n$ factors) $\geqq d(\tau \times \tau \times \cdots \times \tau)^{1/2}$ (n factors)$= d_\tau^{n/2} \geqq 2^{n/2}$, a contradiction. Thus $\sup\{N_\sigma | \sigma \in P\} = N < \infty$, $\sup\{d_\sigma | \sigma \in P_2\} \leqq d^N < \infty$, and (iii) holds.

Turning at last to A_3, suppose $\tau \in P$ and $\alpha \in A_3$. Then $l_\alpha > (C\kappa_0(P))^3$, and so $\tau_\alpha \circ \psi_\alpha \in \{1, \sigma_{\lambda_1}, \bar{\sigma}_{\lambda_1}\}$, since otherwise $\kappa_0(P) \geqq \kappa(\tau) \geqq \kappa(\tau_\alpha) = \kappa(\tau_\alpha \circ \psi_\alpha) \geqq l_\alpha^{1/3}/C$ by Proposition 4.4.

Moreover, $\tau_\alpha \neq 1$ holds for at most one $\alpha \in A_3$. For supposing the opposite, let τ_α and τ_β be both nontrivial with $\alpha, \beta \in A_3$, $\alpha \neq \beta$. Then again by Lemmas 1 and 2 of (5.4) we obtain $\kappa_0(P) \geqq \kappa(\tau) \geqq \kappa(\tau_\alpha \times \tau_\beta) \geqq \min\{d_{\tau_\alpha}^{1/2}, d_{\tau_\beta}^{1/2}\} \geqq \min\{l_\alpha^{1/2}, l_\beta^{1/2}\} > (C\kappa_0(P))^{3/2} \geqq \kappa_0(P)$, another contradiction. Thus P_3 is contained in the FTR set of G_3. If we redefine P_3 to be the FTR set, then (i) through (iii) remain valid while (iv) becomes valid.

The converse follows from Lemma 3 of (5.4), which shows that $P \times Q$ is local Sidon whenever $\sup\{d_\sigma | \sigma \in P\} < \infty$ and Q is local Sidon.

(5.6) In a special case, (5.5) even affords a description of all Sidon sets.

COROLLARY. *Let $G = \prod_{n=2}^\infty SU(n)$. Then $E \subseteqq \hat{G}$ is a local Sidon set if and only if it is a subset of $P' \times P''$, where for some $n_0 \geqq 1$, P' is a finite subset of $(\prod_2^{n_0} SU(n))^\wedge$ and P'' is the* FTR *set of $\prod_{n_0+1}^\infty SU(n)$. Hence E is a local Sidon set if and only if it is a Sidon set.*

Proof. The set A_2 of (5.5) must be finite, so the first assertion holds. Lemma 3 of (5.4) shows that subsets of $P' \times P''$ are always in fact Sidon.

(5.7) Specializing on another front yields a counterexample to Hutchinson's conjecture ([12, (9.1)], as abstracted in [14]). We write PSU(n) for the quotient of $SU(n)$ by its center:

COROLLARY. *If $G = \prod_{n=2}^{\infty} PSU(n)$ then G has no infinite Sidon sets, although it has infinite central Sidon sets, is connected, and is not a semisimple Lie group.*

Proof. The first statement is an application of (5.5), as again A_2 must be finite. The second was proved by Rider [17].

6. Sidon sets for compact connected groups. We give in this section our main criterion for the existence of an infinite Sidon set in the dual of a compact connected group. We give the criterion in two different forms, as Theorems 1 and 2 below.

(6.1) *Notation.* Let G be a compact connected group. Then the extended structure theorem (see Price [16, (6.5.6)]) guarantees an epimorphism $\pi: G^* \to G$, where G^* has the form $A \times G_1$, with A a connected compact abelian group and G_1 a cartesian product $\prod_{\iota \in I} S_\iota$ of compact simple simply-connected Lie groups. The kernel K of π is totally disconnected and central.

The symbols $G, G^*, \pi, A, G_1, S_\iota$ $(\iota \in I)$, and K will have these fixed meanings throughout this section.

(6.2) Here is our main result in its first form.

THEOREM 1. *Let G be a connected compact group. Then the following are equivalent*: (i) *G has an infinite Sidon set*; (ii) *G has an infinite local Sidon set*; (iii) *there is a continuous homomorphism of G onto one of the following groups*:

(a) $\boldsymbol{T}$

(b) *an infinite cartesian power S^c of a simple Lie group S, or* (c) *a group of the form $(\prod_{k=0}^{\infty} L_k)/K_0$, where L_0 is a compact linear semisimple Lie group, $L_k = SU(n_k)$, $SO(n_k)$, or $Sp(n_k)$ for $k = 1, 2, 3, \cdots$, with $1 \leqq n_1 < n_2 < n_3, \cdots$, and where (with I denoting the appropriate identity matrix) $K_0 = \{(x_k) \in \prod_{k=0}^{\infty} L_k \mid x_0 = \lambda I$ and $x_k = \lambda^{-1} I$ for all $k \geqq 1$ and some $\lambda \in \boldsymbol{C}\}$.*

Proof. It is evident that (i) implies (ii). We show first that (ii) implies (iii). To this end, assume that G admits an infinite local Sidon set, and that there is no continuous homomorphism of G onto groups of type (a) or (b). Thus with the notation of (6.1), $A = \{1\}$, and for each $l \in \boldsymbol{N}$ there are only finitely many $\iota \in I$ such that the rank l_ι of S_ι is equal to l (see (6.3) below). By (5.5) then, there are a partition $I = B \cup C$, an infinite local Sidon set $P = \{\rho_1, \rho_2, \cdots\}$ of $\hat{G}$ and an injection $k \mapsto \iota_k$ of N into C such that $P \circ \pi \subseteq (G^*)^\wedge$ has the form $\{\rho_k \circ \pi = \tau \times \sigma_k \mid k \in N\}$, where $\tau \in (\prod_{\iota \in B} S_\iota)^\wedge$ is fixed, $B \subseteq I$ is finite, and σ_k is the canonical projection $\prod_{\iota \in C} S_\iota \to S_{\iota_k}$ followed by the representation ω_k of S_{ι_k} corresponding to the fundamental weight λ_1. Let L_k denote the image of S_{ι_k} under the representation ω_k. Then $L_k \cong SU(n_k)$, $SO(n_k)$, or $Sp(n_k)$ for some $n_k \in N$. Reducing and reordering P if necessary, we may assume that $n_1 < n_2 < \cdots$. Let $L_0 = \tau(\prod_{\iota \in B} S_\iota)$, let $\sigma\colon \prod_{\iota \in C} S_\iota \to \prod_{k=1}^\infty L_k$ be the obvious surjection, and consider the homomorphism

$$\tau \times \sigma\colon G^* = \prod_{\iota \in B} S_\iota \times \prod_{\iota \in C} S_\iota \longrightarrow \prod_{k=0}^{\infty} L_k \,.$$

If now $x = (x_\iota)_{\iota \in I} \in K$, then $1 = \rho_k \circ \pi(x) = \tau \times \sigma_k(x)$ for each $k \in N$, whence $\tau((x_\iota)_{\iota \in B}) \otimes \omega_k(x_{\iota_k}) = I$. If T and S are diagonalizable operators and $T \otimes S = I$, then $T = \lambda I$ and $S = \lambda^{-1} I$ for some $\lambda \in C$, with I denoting the identity operator on the appropriate spaces. Hence for each $x \in K$ we have $\tau \times \sigma(x) \in K_0 = \{(\xi_k) \in \prod_{k=0}^\infty L_k \mid \xi_0 = \lambda I$ and $\xi_k = \lambda^{-1} I$ for all $k \in N$ and some $\lambda \in C\}$. Hence $\tau \times \sigma$ induces a continuous homomorphism of G onto a group of type (c).

Now we assume (iii) and show that (i) holds. Thanks to Lemma 1 of (2.2) we need only show that groups of type (a), (b), and (c) have infinite Sidon sets. For groups of type (a) this is classical, and for those of type (b) we choose a fixed nontrivial $\rho \in \hat{S}$ and apply (5.2) with $P_\alpha = \{\rho\}$ for each α. Let $G = (\prod_{k=0}^\infty L_k)/K_0$ be of type (c). Let τ be the self-representation of L_0 and let π_j be the canonical projection $\prod_{k=1}^\infty L_k \to L_j$ followed by the self-representation of L_j. Then $\{\tau \times \pi_j \mid j \in N\} \subseteq (\prod_{k=0}^\infty L_k)^\wedge$ is a Sidon set by (5.3) and Lemma 3 of (5.4). Clearly $\tau \times \pi_j$ is trivial on K_0 and so induces an irreducible representation ρ_j on G. The set $\{\rho_j \mid j \in N\}$ is the infinite Sidon set we require.

(6.3) *Some alternative criteria*

PROPOSITION 1. *Let G be a compact connected group. Then the following are equivalent*:

(a) *G admits $\boldsymbol{T}$ as a homomorphic image*;

(a)′ *G admits an infinite abelian quotient*;

(a)″ $A \neq \{1\}$;

(a)‴ *G admits a nontrivial connected abelian normal subgroup.*

PROPOSITION 2. *Let G be a compact connected group. Then G is not tall if and only if* (a) *it admits an infinite abelian quotient, or* (b) *it admits an infinite power S^c of a simple Lie group S as* *a homomorphic image.*

Proof. If (a) or (b) holds, G is clearly not tall. If, conversely, G has no quotients of the stated kinds, then $A = \{1\}$. Also, since $K \subseteqq G^*$ is central, there is a surjection of G onto $G_* = G^*/Z(G^*)$. But $G^* = G_\iota = \prod_{\iota\in I} S_\iota$, so $Z(G^*) = \prod_{\iota\in I} Z(S_\iota)$, and so G_* is isomorphic to $\prod_{\iota\in I}(S_\iota/Z(S_\iota))$. By hypothesis, only finitely many of these factors lie in each isomorphism class. Consequently the same holds for the product $G^* = \prod_{\iota\in I} S_\iota$. Thus G^* is tall, and therefore G is tall.

The correspondence between the factors of G^* and of G_* noted in the last proof may be exploited once more to yield the following fact.

PROPOSITION 3. *Let G be a compact connected group. Then the following are equivalent*:

(b) *G admits an infinite power S^c of a simple Lie group S as a homomorphic image*;

(b)′ *infinitely many of the S_ι ($\iota \in I$) are pairwise isomorphic*;

(b)″ *G admits infinitely many pairwise isomorphic connected normal subgroups that are simple Lie groups.*

(6.4) When $A = \{1\}$ and I is infinite we shall call the kernel K of π *almost trivial* if there are disjoint subsets B and C of I which are finite and infinite respectively, and if there exists an element $\tau \in (\prod_{\iota\in B} S_\iota)\hat{}$ with the following property: for each $x = (x_\iota)_{\iota\in I} \in K$, there is a complex number λ such that $\tau((x_\iota)_{\iota\in B}) = \lambda I$ and $\omega_\iota(x_\iota) = \lambda^{-1} I$ for all $\iota \in C$, where ω_ι is the representation of S_ι which corresponds to the fundamental weight λ_1 (ω_ι is the self-representation unless $S_\iota = \mathrm{Spin}(n)$ for some n), and where I denotes the identity operator on the appropriate space.

If K is almost trivial, let π_0 denote the projection of G^* onto its subproduct $H = \prod_{\iota\in B\cup C} S_\iota$. Then $\pi_0(K)$ is clearly a finite cyclic central subgroup of H, and its order must divide that of $Z(S_\iota)$ for each $\iota \in C$.

We now state our main result in a second form.

THEOREM 2. *Let G be a connected compact group. Then the following are equivalent*: (i) *G has an infinite Sidon set*; (ii) *G has an infinite local Sidon set*; (iii) *either G is not tall, or G is tall and the kernel K of π is almost trivial.*

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