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# INVERTING DOUBLE KNOTS

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**We disprove J. Montesinos's conjecture that every invertible knot in $S^3$ is strongly invertible.**

Let $K$ denote a tame, oriented knot in $S^3$, and fix an orientation of $S^3$. If there exists an orientation-preserving, PL involution of $S^3$ that inverts $K$, then $K$ is *strongly invertible*. J. Montesinos proposed this definition [5], and he has conjectured [3; Problem 1.6, p. 277] that *every* invertible knot is strongly invertible. In this paper, we disprove this conjecture; our results are as follows.

THEOREM 1. *A knot $K$ is strongly invertible if and only if each double of $K$ is strongly invertible.*

COROLLARY. *No double of a noninvertible knot is strongly invertible; hence, there exist invertible knots that are not strongly invertible.*

*Proof of Corollary.* Any double knot is invertible [6; Theorem 1, p. 235].

THEOREM 2. *If $L$ is a strongly invertible knot with exactly one maximal companion $C_L$, then $C_L$ is also strongly invertible.*

Section 1 contains a preliminary lemma. We prove Theorems 1 and 2, in §2. In §3, we give a counterexample to the converse of Theorem 2; in §4, we discuss surgery on invertible knots, give several examples, and formulate a conjecture.

I wish to thank C. Gordon and K. Murasugi for discussing this work with me.

**1. Preliminaries.** In this paper, all spaces are polyhedrons; the three-sphere has a fixed orientation; all maps are piecewise linear; all submanifolds, polyhedral; and all knots, oriented. We shall need the following lemma.

POSITIONING LEMMA. *If $K$ is a strongly invertible knot, then there exists an orientation-preserving, $K$-inverting,* PL *involution $\rho: S^3 \to S^3$ with nonempty, fixed point set $A$ and there exists a polyhedral, $\rho$-invariant 2-sphere $S$ such that* (a) *the "axis" $A$ of $\rho$ belongs to $S$,* (b) *the set $K \cap A$ contains exactly two points,* (c) *the knot $K$ is transverse with respect to the 2-sphere $S$, and* (d) *the set $K \cap S$ contains only a finite number of points.*

*Proof.* Because $K$ is strongly invertible, there exists an orientation-preserving, $K$-inverting, PL involution $\rho'\colon S^3 \to S^3$. The Lefschetz fixed-point theorem implies that $\rho'|K$ has exactly two fixed points. Because the fixed-point set (or *axis*) $A'$ of $\rho'$ is, therefore, nonempty, the axis $A'$ must be a knot [7; Theorem, p. 162]. If we set $S^3 = R^3 \cup \{\infty\}$, then, because $\rho'$ is piecewise linear and orientation-preserving, there exists a PL autohomeomorphism $\varphi$ of $S^3$ such that $\varphi\rho'\varphi^{-1}(=\rho_0)$ is a "standard" (orthogonal) $180°$-rotation and such that $\varphi(A') = (y\text{-axis}) \cup \{\infty\}(=A_0)$ [10].

Because $\varphi(K)$ is polyhedral, because the rotation $\rho_0$ takes $\varphi(K)$ onto itself, and because $\varphi(K)$ meets $A_0$ in only two points, $x_1$ and $x_2$, we can (if necessary) find a small angle $\alpha°(\alpha > 0)$ such that an $\alpha°$-rotation $\rho_\alpha$ about $A_0$ takes $\varphi(K)$ to a knot $\rho_\alpha\varphi(K)$ that is transverse to the 2-sphere $S_0(=(yz\text{-plane}) \cup \{\infty\})$ at each of the points, $x_1$ and $x_2$. We shall find a knot $K'$ (ambient isotopic to $\rho_\alpha\varphi(K)$) such that $K'$, the involution $\rho_0$ (of $S^3$) with axis $A_0$, and the 2-sphere $S_0$ satisfy the hypothesis and the conclusion of the lemma. The lemma's proof will easily follow.

Choose $\varepsilon(>0)$ so that (closed) $\varepsilon$-neighborhood $V_\varepsilon$ of $\rho_\alpha\varphi(K)$ is a solid torus; such a choice is possible, because $\rho_\alpha\varphi(K)$ is polyhedral in $S^3$. Because $\rho_\alpha\varphi(K)$ is transverse to $S_0$ at $x_1$ and at $x_2$, we can restrict $\varepsilon$ so that $V_\varepsilon \cap S_0$ contains (among other things) two disjoint meridional disks, $E_1$ and $E_2$, of $V_\varepsilon$, with $E_i \cap \rho_\alpha\varphi(K) = \{x_i\}(i = 1, 2)$. By a final restriction of $\varepsilon$, we can assume that $V_\varepsilon \cap A_0 = (E_1 \cup E_2) \cap A_0(=\text{two, disjoint arcs})$. (The constructions involved in our restrictions of $\varepsilon$ are standard, and we shall omit them.) Finally, note that $\rho_0(V_\varepsilon) = V_\varepsilon$.

The points $x_1$ and $x_2$ divide $\rho_\alpha\varphi(K)$ into two (closed) arcs, $k_1$ and $k_2$; the disks $E_1$ and $E_2$ divide $V_\varepsilon$ into (closed) 3-cells, $B_1$ and $B_2$, with $k_i$ unknotted in $B_i(i = 1, 2)$ (see [4; p. 134]). We note that $\rho_0(B_1) = B_2$ and that $B_i \cap A_0 = (E_1 \cup E_2) \cap A_0$.

Keeping $x_1$ and $x_2$ fixed, we now put $k_1$ in general position with respect to $S_0$ by an orientation-preserving autohomeomorphism $h_1\colon S^3 \to S^3$ moving each point of $k_1$ less than $\varepsilon$. We can evidently assume that $h_1|(S^3 - \operatorname{Int} B_1)$ is the identity map.

The arc $\rho_0 h_1(k_1)$ is clearly unknotted in $B_2$. Hence, there exists an orientation-preserving autohomeomorphism $h_2\colon S^3 \to S^3$ taking $k_2$ onto $\rho_0 h_1(k_1)$ and leaving each point of $S^3 - \operatorname{Int} B_2$ fixed. The autohomeomorphism $h$ of $S^3$ given by

$$h(x) = \begin{cases} h_i(x)\,, & \text{if} \quad x \in \operatorname{Int} B_i(i = 1, 2) \\ x\,, & \text{otherwise}\,, \end{cases}$$

preserves the orientation of $S^3$ and takes $\rho_\alpha\varphi(K)$ onto a knot $h_1(k_1) \cup$

$\rho_0 h_1(k_1)$ that is in general position with respect to $S_0$ and that is strongly inverted by $\rho_0$. We set $K' = h\rho_\alpha\varphi(K)(=h_1(k_1) \cup \rho_0 h_1(k_1))$ and note that the knot $K'$, the involution $\rho_0$ with axis $A_0$, and the 2-sphere $S_0$ satisfy the hypothesis and conclusion of the lemma. The proof of the lemma now follows by taking $\rho = (h\rho_\alpha\varphi)^{-1}\rho_0(h\rho_\alpha\varphi)$, taking $A = A'(=\varphi^{-1}(A_0) = (h\rho_\alpha\varphi)^{-1}(A_0))$, and taking $S = (h\rho_\alpha\varphi)^{-1}(S_0)$.

## 2. Proofs.

*Proof of Theorem* 1. We shall assume that $K$ is not trivial, for otherwise, the theorem is evidently true.

(1) *Necessity.* We assume that $K$ is strongly invertible. Let $\rho$, and $A$, and $S$ denote the objects our Positioning Lemma guarantees, and let $K \cap A = \{x_1, x_2\}$. By the Positioning Lemma's proof, we can assume (without loss of generality) that $\rho$ is the 180°-rotation about $A(=(y\text{-axis}) \cup \{\infty\})$ and that $S = (yz\text{-plane}) \cup \{\infty\}$. Moreover, we can choose $\varepsilon(>0)$ and $V_\varepsilon$ exactly as in the lemma's proof. We have $K = k_1 \cup k_2$ (with $\rho(k_1) = k_2$) and $V_\varepsilon = B_1 \cup B_2$; moreover, $E_i \cap A (i = 1, 2)$ is a properly imbedded arc in $E_i$.

Let $C$ denote a cylindrical 3-cell with core $k$ and with two disks, $D_1$ and $D_2$, meeting in an arc and imbedded in $C$, as shown in Figure 1. Let $v$ be a (closed) arc in $\mathrm{Int}(E_2 \cap A)$ such that $x_2 \in \mathrm{Int}(v)$ (see Figure 2(a)). It is easy to find an arc $v_1 \subset \mathrm{Int}\, E_1$ such that $v_1 \cap A = \{x_1\} = v_1 \cap \rho(v_1) = A \cap \mathrm{Int}\, v_1$; note that $\rho(v_1) \subset \mathrm{Int}\, E_1$ (see Figure 2(b)).

Now, let $g: C \to B_1$ be a homeomorphism such that $g(E'_i) = E_i (i = 1, 2)$, such that $g(v'_1) = v_1$ and $g(v'_2) = \rho(v_1)$, such that $g(v') = v$, and such that $g(k) = k_1$. Then $[g(D_1 \cup D_2)] \cup [\rho g(D_1 \cup D_2)]$ is a singular disk $\Sigma$ with one clasping singularity, and the $\partial\Sigma$ is a double of $K$ with twisting number $\sigma$ (an integer depending on the homeomorphism $g: C \to B_1$) and with self-intersection number $\eta(=\pm 2)$. (By changing $g$ (to change $\sigma$) and by replacing $C$ with its mirror image (to change the sign of $\eta$), we can assume that $\partial\Sigma$ is any double of $K$ that we desire.) Evidently, $\partial\Sigma$ is strongly invertible (by the involution $\rho$). This completes the proof of the necessity.

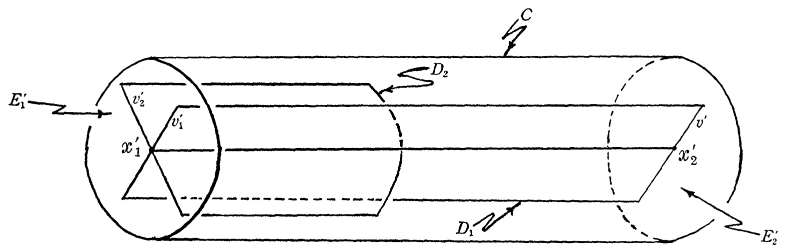


FIGURE 1

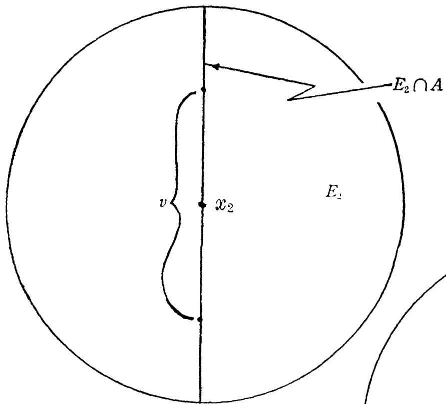


FIGURE 2(a)

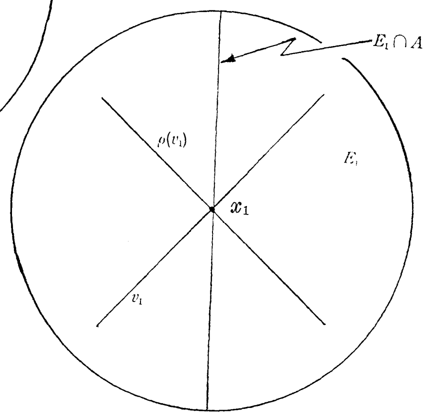


FIGURE 2(b)

(2) *Sufficiency*. We assume that some double, $D_K$, of $K$ is strongly invertible. Replace $K$ by $D_K$ in the Positioning Lemma; we can assume that $\rho$ is the standard rotation (of period 2) about $A(=(y\text{-axis}) \cup \{\infty\})$, that $S=(yz\text{-plane}) \cup \{\infty\}$, and that $D_K \cap A = \{x_1, x_2\}$.

Let $V^*$ denote a (closed) regular neighborhood of a clasping disk whose boundary is $D_K$; note that $K$ is equivalent to a core of $V^*$ [6; p. 238]. Now $K$ is a unique maximal companion $D_K$ [6; p. 242]; that is, any companion of $D_K$, other than $K$, is also a companion of $K$. Hence, the torus $\rho(\partial V^*)$ is ambient isotopic to $\partial V^*$ in $S^3 - D_K$. So, by [9; Theorem 1, p. 223], the $\partial V^*$ is ambient isotopic (in $S^3 - D_K$) to a torus $T$ in general position with respect to $A$, and either $\rho(T) \cap T = \emptyset$ or $\rho(T) = T$. If $\rho(T) \cap T = \emptyset$, then $T$ and $\rho(T)$ are parallel. Because $\rho^2(T) = T$ and because each of $\rho(T)$ and $T$ separates $S^3 - D_K$, it easily follows that $\rho$ moves fixed points of itself, which is absurd. Thus, $\rho(T) = T$.

Now $T$ splits $S^3$ into a solid torus $V$ (containing $D_K$ in its interior) and a $K$-knot manifold. If $A \cap T = \emptyset$, then $A \subset \operatorname{Int} V$, because $A \cap D_K \neq \emptyset$. Because $K$ is knotted and $A$ is unknotted, $A$ belongs to a polyhedral 3-cell $\subset \operatorname{Int} V$; otherwise, $A$ would have a companion, which it does not [6]. Applying Tollefson's lemma [8; Lemma 1, p. 141], we can find a 2-sphere $S' \subset \operatorname{Int}(V - A)$ such that $S'$ bounds no 3-cell in $V - A$ and such that either $\rho(S') \cap S' = \emptyset$ or $\rho(S') = S'$. As with the tori $T$ and $\rho(T)$ in the preceding paragraph, we cannot have $\rho(S') \cap S' = \varnothing$. If $\rho(S') = S'$, then take the 3-cell $B^3(\subset S^3)$ that does not contain $A$ and that $S'$ bounds (in $S^3$), and consider the homeomorphism $\rho | B: B \to B$. By the Brouwer fixed-point theorem, $\rho | B$ has a fixed point, and so $\rho$ has a fixed point not on $A$ (which it does not). Hence, $A \cap T \neq \varnothing$.

Because $T$ is in general position with respect to $A$, the cardinality $b$ of $A \cap T$ is finite. Let $T_0$ denote the orbit space of $\rho | T$. The projection $p: T \to T_0$ is a branched covering, and the two Euler characteristics, $\chi(T)$ and $\chi(T_0)$, are related by the Riemann-Hurwitz branch-point formula,

$$\chi(T) = 2\chi(T_0) - b\ ;$$

see [1; p, 93]. But $\chi(T) = 0$ and $b > 0$. Hence, $\chi(T_0) = 2$, and so $T_0$ is a 2-sphere and $b = 4$. (Because the orbit space of $\rho$ is $S^3$ and because $S^3$ contains no projective planes, we cannot have $\chi(T_0) = 1$.)

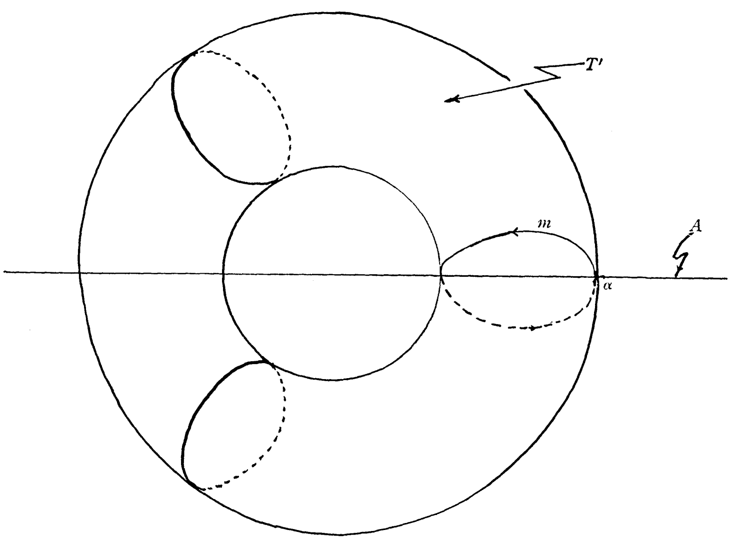


FIGURE 3

Now let $T'$ denote the torus $(r - 2)^2 + z^2 = 1$ (see Figure 3), let $m$ denote the curve $\{(r, z) | \theta = 0 \text{ and } (r - 2)^2 + z^2 = 1\}$ (which we shall take as one of the two components of $T' \cap S$), and let $K_{(\bar{p},\bar{q})}((\bar{p}, \bar{q}) = 1)$ denote the torus knot $\{(r, z) | r = 2 + \cos(\bar{p}\theta/\bar{q}),\ z = \sin(\bar{p}\theta/\bar{q})\}$ on $T'$ (cf. [2; p. 92]). To fix the $(r, \theta, z)$-coordinate system on $T'$, let the point $\alpha$ shown in Figure 3 have $(r, \theta, z)$-coordinates $(3, 0, 0)$. Note that $\rho(T') = T'$ and that $\rho(m) = m^{-1}$ (after we have oriented $m$). If $T'_0$ denotes the orbit space of $\rho | T'$, then the projection $p': T' \to T'_0$ is a branched covering. As with $p: T \to T_0$, the covering $p'$ has four branch points, and $T'_0$ is a 2-sphere.

According to [1; Theorem 3.4, p. 94], the coverings $p$ and $p'$ are equivalent; that is, there exist homeomorphisms $\psi\colon T \to T'$ and $\gamma\colon S^2 \to S^2$ such that $p'\psi = \gamma\rho$. It follows easily that $\varphi$ preserves covering fibers.

Thus, if $\{x, \rho(x)\}$ is a fiber of $p$, then $\{\psi(x), \psi\rho(x)\}$ is a fiber of $p'$, and so $(\rho\,|\,T')\psi(\rho\,|\,T)(x) = \psi(x)$; that is, $\psi = (\rho\,|\,T')\psi(\rho\,|\,T)$. Because $\rho^2 = id.$, we have $(\rho\,|\,T')\psi = \psi(\rho\,|\,T)$. Notice that $\rho(K_{(\bar{p},\bar{q})}) = K^{-1}_{(\bar{p}\,\bar{q})}$; thus, for any $(\bar{p}, \bar{q})$-torus knot, there exists a representative, $K_{(\bar{p}\,\bar{q})}$, of it on $T'$ that $\rho$ inverts (and, hence, *strongly* inverts).

If $\lambda$ is an (oriented) longitude of $K$ on $T$, then $\psi(\lambda)$ is isotopic on $T'$ to $m$ or to one of the torus knots $K_{(\bar{p}_1,\bar{q}_1)}$, for some pair $(\bar{p}_1, \bar{q}_1)$. Thus, either $\psi^{-1}(m)$ or $\psi^{-1}(K_{(\bar{p}_1,\bar{q}_1)})$ is a longitude of $T$ meeting the axis $A$ of $p$ in exactly two points, because $\psi$ maps branch points of $p$ to branch points of $p'$. Because $(\rho\,|\,T')\psi = \psi(\rho\,|\,T)$, we have either $\rho(\psi^{-1}(m)) = \psi^{-1}(\rho\,|\,T')(m) = \psi^{-1}(m^{-1}) = [\psi^{-1}(m)]^{-1}$ or, similarly, $\rho(\psi^{-1}(K_{(\bar{p}_1,\bar{q}_1)})) = [\psi^{-1}(K_{(\bar{p}_1,\bar{q}_1)})]^{-1}$. Therefore, $\rho$ strongly inverts a longitude of $K$, and it follows that $K$ itself is strongly invertible.

*Proof of Theorem* 2. We need only note that, in the proof of Theorem 1, the sufficiency portion depends on the uniqueness of the maximal companion $K$ of $D_K$ and not on the knot type of $D_K$.

**3. A counterexample.** The noninvertible knot $\mathscr{K}$ in [11; Figure 3, p. 1275] is a counterexample to the converse of Theorem 2. Because the knots $3_1$ and $5_1$ (of the Alexander-Briggs table) are simple, one can apply Schubert's theorem [6; p. 216] to show that $\mathscr{K}$ has exactly one maximal companion, which is a trefoil knot and, hence, strongly invertible; details of the application are routine, and we shall omit them.

**4. A conjecture.** A link $L$ in $S^3$ is *strongly invertible*, if there exists an orientation-preserving PL involution of $S^3$ that inverts each component of $L$. In [5, Theorem 1, p. 231], Montesinos proved that any 3-manifold derived from surgery on a strongly invertible link is a 2-fold cyclic covering space of $S^3$ branched over a link and, conversely, that one can produce any particular 2-fold branched cyclic covering space of $S^3$ by surgery on a suitable, strongly invertible link. I do not know whether nontrivial surgery on a *knot that is not strongly invertible* will produce a 2-fold branched cyclic covering space of $S^3$. It is, however, a different story for links. Here are some examples.

F. González-Acuña and J. Montesinos gave the first such examples (unpublished). Assign any rational coefficient to the component $K_1$ of the unsplittable and noninvertible Borromean rings, $K_1 \cup K_2 \cup K_3$

[5]. Take nonzero integers, $a$ and $b$, and assign the coefficient $1/a$ to $K_2$ and the coefficient $1/b$ to $K_3$. We now have a surgical description of a closed, connected, orientable 3-manifold, $M$. By applying an appropriate twist across a disk spanning each of $K_2$ and $K_3$, we can replace our original surgical description on $M$ by one involving only a knot, $K$, which (with a little adjusting) is easily seen to be strongly invertible. Hence, $M$ is a 2-fold branched cyclic covering space of $S^3$. Some of the various knots that $K$ might be are $8_3$, $10_3$, and any twist knot.

For the second group of examples, let $K_1$ denote a double of a noninvertible knot and let $K_2$ denote a trivial knot in $S^3 - K_1$ placed near the "critical" part of $K_1$ so that exactly one (suitable) twist, $t$, across a disk spanning $K_2$, will unknot $K_1$. Now assign any rational coefficient to $K_1$ and assign either $+1$ or $-1$ to $K_2$ so that the coefficient of $K_2$ becomes $\infty$ after the twist $t$. The link $K_1 \cup K_2 (= L)$ is invertible, but not strongly invertible. Furthermore, with the two coefficients attached, $L$ provides a surgical description of a manifold $N$. After twisting by $t$ about a disk spanning $K_2$, we can replace our first surgical description of $N$ by one involving only a trivial knot. Hence, $N$ is a 2-fold branched cyclic covering space of $S^3$; in fact, $N$ is a lens space.

CONJECTURE. *No manifold obtained from nontrivial surgery on a double of a noninvertible knot is a 2-fold branched cyclic covering space of* $S^3$.

We conclude with two remarks, added in October, 1980, just before the paper went to press.

REMARK 1. Let $K$ be a knot nontrivially imbedded in the interior of an unknotted solid torus $V$ in $S^3$, and suppose that one can invert $K$ inside $V$ (without disturbing $S^3 - \text{Int}(V)$). Let $W$ be a solid torus in $S^3$ whose core is *not* strongly invertible, and let $f: V \to W$ be a faithful homeomorphism. With only minor technical restrictions on $K$, we can conclude that $f(K)$ is invertible but not strongly (see Theorem 2 of [12]). One can easily construct examples (each with genus $> 1$) that are not double knots (see [12]).

REMARK 2. Richard Hartley has independently constructed counterexamples to Montesinos's conjecture (that every invertible knot is strongly invertible); see Hartley's paper [*Knots and involutions*, Math. Zeit., 171 (1980), 175–185].

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