MANIFOLDS ADMITTING TAUT HYPERSPHERES

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The definition of taut submanifolds in Euclidean space is extended to submanifolds of an arbitrary complete Riemannian manifold. Manifolds containing a tautly embedded hypersphere are characterized up to homeomorphism. Also, a partial result in this direction is proved for manifolds containing a tautly embedded sphere of arbitrary codimension.

1. Taut submanifolds have received much attention in recent years [1], [3], [6], [7]. There the emphasis is on characterizing the taut submanifolds of a particular ambient space, usually Euclidean space, although there are studies involving hyperbolic space and complex projective space as well [4], [5]. In this paper the subject is approached from a different perspective: to characterize the ambient space given that it contains certain taut submanifolds. For example:

THEOREM 1. *A complete simply connected Riemannian manifold of dimension n that admits a taut embedding of S^{n-1} is either homeomorphic to S^n, diffeomorphic to R^n, or diffeomorphic to $S^{n-1} \times R$.*

In a Euclidean sphere or in a complete, simply-connected Riemannian manifold without conjugate points, every geodesic sphere is taut. The converse is also true.

THEOREM 2. *Suppose a complete Riemannian manifold has the property that about every point some small geodesic sphere is taut. Then the manifold is either simply connected without conjugate points or isometric to a Euclidean sphere.*

2. Let M be a complete Riemannian manifold and $N \subset M$ a proper submanifold. In particular, N is a closed submanifold with the subspace topology. For each $p \in M$, we define the function $L_p: N \to R$ by $L_p(x) = [d(x, p)]^2$ where $x \in N$ and d is the distance function on M. We say N is *taut* if for almost every $p \in M$ and almost every $r > 0$ the homeomorphism

$$i_*: H_*(L_p^{-1}([0, r])) \longrightarrow H_*(N)$$

induced by inclusion is injective, where the homology coefficients are in some field. Because of Lemma 2.8 on page 705 of [3], this definition coincides with the definition for taut submanifolds of

Euclidean space. It roughly states that on a dense set of $p \in M$, the function L_p has the least possible number of critical points.

L_p is a continuous function which is smooth at the points of N that are not in the cut locus $C(p)$ of p. At these points, one has a characterization of a critical point of L_p and its index in terms of the focal points of N [12].

PROPOSITION 1. *A point $x_0 \in N$, $x_0 \notin C(p)$ is a critical point of L_p if and only if the unique minimizing geodesic from p to x_0 is perpendicular to N at x_0. Furthermore, its index is the number of focal points of N between x_0 and p, counted with multiplicity, along this geodesic, and its nullity is the multiplicity of p as a focal point.*

Proof. The first statement follows from the first variation formula, and the second is essentially the finite dimensional approximation of the path space by broken geodesics, e.g., [10], page 160, where broken geodesics are unnecessary since we evade the cut locus of p. □

The nature of the critical points of L_p and $L_{p'}$ are similar if p and p' are close together.

PROPOSITION 2. (i) *Let $x_0 \in N$, $x_0 \notin C(p)$ and U be a neighborhood of x_0 in N. If x_0 is a nondegenerate critical point of L_p with index λ, then for all p' sufficiently near p, $L_{p'}$ has a nondegenerate critical point in U of index λ.*

(ii) *Let $K \subset N$ be a compact subset and U a relatively compact neighborhood of K in N. If L_p has a relative minimum on K, i.e., L_p is constant on K and $L_p(x) > L_p(K)$ for all $x \in U - K$, then for all p' sufficiently near p, $L_{p'}$ attains a relative minimum in U. A similar statement is true for relative maximums.*

Proof. (i) is a consequence of the fact that nearby geodesics have the same index. The proof of (ii) is analogous to Lemma 5.1 of [11]. □

In what follows, we will be concerned with taut embeddings of spheres S^k. In this case, the tautness condition and Proposition 2 imply that a relative minimum of L_p is an absolute minimum, that a relative maximum is an absolute maximum, and that the index of any nondegenerate critical point must be either 0 or k.

3. In this section we prove Theorem 1. We begin with the

following proposition.

PROPOSITION 3. *Let M be a complete Riemannian manifold and N a taut connected hypersurface which separates M. Let γ be a geodesic starting at $x_0 \in N$ perpendicularly to N. If there exists a cut point of x_0 along γ, then the first focal point of N along γ occurs between x_0 and the cut point.*

Proof. Since N separates M, $M - N$ consists of two connected components. Furthermore, being taut, the relative minimums of L_p are absolute minimums. Let p_0 be the cut point of x_0 along γ and suppose the first focal point of N along γ does not occur between x_0 and p_0. Then, by Proposition 1, for any p along γ between x_0 and p_0, L_p has a nondegenerate relative minimum at x_0, there being no focal points between x_0 and p. This is an absolute minimum. Thus for all $x \in N$, $d(x, p) \geqq d(x_0, p)$ and by taking limits as p approaches p_0, $d(x, p_0) \geqq d(x_0, p_0)$ for all $x \in N$. This shows that the segment of γ between x_0 and p_0 is contained in one of the components of $M - N$. (If it wasn't we could find a point of N closer to p_0 than x_0.) Extend γ on the "other side" of N and take a point q on γ close enough to N so that there is no focal point between x_0 and q, and q lies on the opposite side of N from p_0. As before, L_q attains its absolute minimum at x_0. Thus $d(x, q) \geqq d(x_0, q)$ for all $x \in N$. Now, the segment of γ between q and p_0 is not minimizing, since it contains the cut point x_0 of p_0. Hence, by completeness of M, there is a minimizing segment σ connecting q to p_0. Since N separates M, σ intersects N at some point x. Hence $d(p_0, q) = d(p_0, x) + d(x, q) \geqq d(p_0, x_0) + d(x_0, q) = $ length of the segment of γ, which contradicts the nonminimality of γ. Therefore we have shown the first focal point of N occurs between x_0 and p_0. $\square$

REMARK. By [13], if M is simply connected, every properly embedded, connected hypersurface of M separates M. Thus the hypothesis that N separates M can be dropped from Proposition 3 if M is simply connected.

In view of this remark, Theorem 1 is a consequence of the following Theorem 1'.

THEOREM 1'. *Let M be a complete Riemannian manifold of dimension n that admits a taut embedding of S^{n-1} which separates M, then M is either homeomorphic to S^n, diffeomorphic to $\mathbf{R}^n$, or diffeomorphic to $S^{n-1} \times \mathbf{R}$.*

Proof. Let N be the taut embedding of S^{n-1} separating M.

Thus by Proposition 3, we can conclude that the first focal points occur prior to the cut points. The results of § 2 allow us to conclude that a first focal point must be of multiplicity $n - 1$. (If not, by choosing a point p slightly beyond the first focal point on the geodesic, the function L_p would have a nondegenerate critical point of index λ with $0 < \lambda < n - 1$.) If p_0 is the first focal point of N along γ starting perpendicularly at $x_0 \in N$, then L_{p_0} is constant on N. For taking p between x_0 and p_0, L_p has a minimum at x_0, and p with p_0 between x_0 and p, L_p has a maximum at x_0. Since these are absolute extrema, on taking limits as p approaches p_0 we find L_{p_0} is constant. Since the multiplicity of the first focal points is constant, the first focal locus is an $n - 1$ dimensional submanifold of the normal bundle of N [9]. (A generalization of Warner's regular conjugate locus [14].) Furthermore, we have essentially shown that the distance to the first focal point is constant on each side of N. (We are in a situation analogous to the regular spherical conjugate loci of [15].) Hence, the components of the first focal locus in the normal bundle get identified each to a single point, since the focal points are of maximal multiplicity. Now the first focal locus has at most one component on each side of N. Hence this leaves three possibilities, either there are two focal points of N, one on either side; there is only one focal point; or there are no focal points. Now, two geodesics starting perpendicularly to N cannot meet until at least one of them has reached the first focal point, otherwise if p is the point of intersection, L_p has two distinct minima in contradiction to the tautness of N. This says that the exponential map of the normal bundle of N into M is one-one on the set of normal vectors shorter than the distance to the focal point. Thus if there are no focal points the exponential map gives a diffeomorphism of the normal bundle of N to M, hence M is diffeomorphic to $S^{n-1} \times \mathbf{R}$. If there is only one focal point, p_0, then N is a geodesic sphere about p_0 and since the focal points of a geodesic sphere about a point correspond to the conjugate points of that point, p_0 has no conjugate points. Furthermore, since the exponential map on the normal bundle of N is one-one up to the focal points, the exponential map at p_0 is one-one. Hence, M is diffeomorphic to $T_{p_0}M \cong \mathbf{R}^n$ under the exponential map at p_0. Lastly, if there are two focal points, let p_0 be one of them. Then, N is a geodesic sphere about p_0, and, by the same reasoning as in the previous case, the first conjugate points to p_0 occur at constant distance and no pair of geodesics emitted from p_0 meet before this conjugate point which is of multiplicity $n - 1$. We can conclude that M is a Blaschke manifold at p_0, and is homeomorphic to S^n. (See Theorem 5.43 of [2] and Theorem 3.1 of [15].)   □

4. In order to prove Theorem 2, suppose that for every $p \in M$, some geodesic sphere $S(p, r)$ about p of radius r less than the injectivity radius at p is taut. Certainly, every such geodesic sphere separates M, and p is a focal point of $S(p, r)$. This leaves two possibilities. Either M is diffeomorphic to R^n and every $p \in M$ has no conjugate points, or M is homeomorphic to S^n and is a Blaschke manifold at every point. In the second case, Berger's theorem in Appendix D of [2], states that M is isometric to a Euclidean sphere. This proves Theorem 2.

5. In this section we sketch a partial result for manifolds admitting taut spheres of arbitrary codimension. The following was proved in [8] with an extra assumption of simple connectivity.

THEOREM 3. *Let M be a compact Riemannian manifold admitting a taut embedding of S^k. Suppose that the first focal point on every geodesic starting at $x_0 \in S^k$ perpendicularly to S^k occurs between x_0 and its cut point. Then M is homeomorphic to a sphere.*

Proof. As in § 3, we show that the multiplicity of a first focal point is k, that for every first focal point p_0, L_{p_0} is constant on S^k, and that a pair of geodesics starting perpendicularly to S^k do not meet until at least one of them has reached the first focal point. If we consider the first focal locus X of S^k in the normal bundle, it can be shown that X is a hypersurface of the normal bundle and that the exponential map restricted to X is a submersion into M [9]. Since M is compact, every geodesic has cut points, and we may conclude that X is compact. Hence, using the submersion, $\exp(X) = K$ is a compact submanifold of M of dimension $n - k - 1$. Futhermore, one can prove that M is expressible as the union of the normal disk bundle of S^k and the normal disk bundle of K pasted together by a diffeomorphism of their boundaries. These common boundaries may be identified with X in such a way that the exponential map restricted to X is the projection of the normal sphere bundle onto K. The map $(\pi, \exp) : X \to S^k \times K$, where π is the restriction to X of the projection of the normal bundle of S^k onto S^k, is a diffeomorphism because, first of all, it is clearly an immersion of manifolds of the same dimension and compactness implies it is onto, and secondly, if it was not one to one, one could find two minimizing geodesics from a point $x_0 \in S^k$ to a focal point $p_0 \in K$, contradicting the condition that the first focal point occurs before the cut point. Similarly, for any $x_0 \in S^k$, $\exp : \pi^{-1}(x_0) \to K$ is a diffeomorphism. Therefore, K is diffeomorphic to S^{n-k-1} since $\pi^{-1}(x_0)$ is diffeomorphic to S^{n-k-1}, and X is diffeomorphic to $S^k \times S^{n-k-1}$. By

the mapping cylinder construction, we see that the normal disk bundle of S^k is homeomorphic to $S^k \times D^{n-k}$, that the normal disk bundle of K is homeomorphic to $D^{k+1} \times S^{n-k-1}$, and that the pasting map on the boundaries corresponds to the identity map. However, $S^k \times D^{n-k}$ and $D^{k+1} \times S^{n-k-1}$ pasted together via the identity map on the boundaries is the sphere S^n. $\square$

REMARK. The condition on the focal points is not generally satisfied by taut spheres when the codimension is greater than one. A easy example is the following. Let $M = S^m \times S^n$. Let S^k be a great sphere in S^m with $k < m$, and $p \in S^n$. The submanifold $N = S^k \times \{p\}$ is taut. However, along the geodesic $\gamma = (x_0, \gamma_0)$, where γ_0 is a geodesic in S^n starting at p and $x_0 \in S^k$, the first focal point of N and the cut point to (x_0, p) coincide.

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