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A NON-COMPACT MINIMAX THEOREM

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This paper contains an extension of Ky Fan's theorem on sets with convex sections (for the case two sets are involved) by relaxing the compactness condition. It is then applied to obtain a generalization of Sion's minimax theorem in which neither underlying set is assumed to be compact.

Ky Fan gave his theorem on sets with convex sections for a family of n sets ($n \geqq 2$) and its various interesting consequences in [1, 2]. It was recently extended in [4] by removing the compactness condition on the underlying sets. Our first result is the following Theorem 1, which is also an extension of the Fan's theorem for the case $n = 2$ in the same direction, but under a much weaker condition. The proof of Theorem 1 relies on the minimax inequality of Fan in its geometric formulation ([3], Theorem 2). It says that if Z is a nonempty compact convex set in a Hausdorff topological vector space and if S is a subset of $Z \times Z$ such that the set $\{z \in Z: (x, z) \in S\}$ is open in Z for each $x \in Z$ and the set $\{x \in Z: (x, z) \in S\}$ is nonempty and convex for each $z \in Z$, then there exists $x_0 \in Z$ such that $(x_0, x_0) \in S$.

THEOREM 1. *Let X, Y be nonempty convex sets, each in a Hausdorff topological vector space, and let A, B be subsets of $X \times Y$ such that*

(a) *For each $x \in X$, the set $A(x) = \{y \in Y, (x, y) \in A\}$ is open in Y, and the set $B(x) = \{y \in Y: (x, y) \in B\}$ is nonempty and convex;*

(b) *For each $y \in Y$, the set $B(y) = \{x \in X: (x, y) \in B\}$ is open in X, and the set $A(y) = \{x \in X: (x, y) \in A\}$ is nonempty and convex.*

If there exists a nonempty compact convex subset K of X such that

(c) *The set $\{y \in Y: (x, y) \notin A$ for all $x \in K\}$ is compact in Y, then $A \cap B \neq \emptyset$.*

Proof. By (b) we have $Y \subset \bigcup_{x \in X} A(x)$. The condition (c) implies that $Y \backslash \bigcup_{x \in K} A(x)$ is compact. Since each $A(x)$ is open, there exists a finite subset $\{x_1, x_2, \cdots, x_m\}$ of X such that $Y \backslash \bigcup_{x \in K} A(x) \subset \bigcup_{i=1}^{m} A(x_i)$. Let Z be the convex hull of $K \cup \{x_1, x_2, \cdots, x_m\}$, then Z is compact convex in X and $Y \subset \bigcup_{x \in Z} A(x)$. By (a) we have $Z \subset X \subset \bigcup_{x \in Y} B(y)$. Since each $B(y)$ is open, there exists a finite subset $\{y_1, y_2, \cdots, y_n\}$ of Y such that $Z \subset \bigcup_{j=1}^{n} B(y_j)$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be a continuous partition of unity subordinated to the cover $\{B(y_j): 1 \leqq j \leqq n\}$ of Z, that is, $\alpha_1, \alpha_2, \cdots, \alpha_n$ are nonnegative real-valued continuous functions on Z

such that α_j vanishes on $Z\backslash B(y_j)$ for each $1 \leqq j \leqq n$ and $\sum_{j=1}^{n} \alpha_j(x) = 1$ for $x \in Z$. We define a map $p: Z \to Y$ by $p(x) = \sum_{j=1}^{n} \alpha_j(x) y_j$, then p is continuous. Moreover, for each $x \in Z$ and $1 \leqq j \leqq n$, if $\alpha_j(x) \neq 0$, then $x \in B(y_j)$ and so $(x, y_j) \in B$. By (a) we have $(x, p(x)) \in B$ for all $x \in Z$. Now let

$$S = \{(x, z) \in Z \times Z : (x, p(z)) \in A\} .$$

By the construction of Z, clearly S satisfies the hypothesis of the geometric form of the minimax inequality stated above. Consequently there exists $x_0 \in Z$ such that $(x_0, x_0) \in S$, that is, $(x_0, p(x_0)) \in A$. But we also have $(x_0, p(x_0)) \in B$ and so the result follows.

The condition (c) in Theorem 1 is obviously fulfilled if either X or Y is compact. In this case, Theorem 1 reduces to the Fan's theorem on sets with convex sections involving two sets. Among various applications given in [1, 2], Fan used his theorem to derive in a direct and simple way the Sion's minimax theorem [5]. In a similar fashion, we shall use Theorem 1 to obtain a noncompact minimax theorem that generalizes the Sion's result. Let X be a convex set in a vector space and let f be a real-valued function defined on X. We recall that f is called *quasi-concave* if for any real number t the set $\{x \in X, f(x) > t\}$ is convex; f is called *quasi-convex* if $-f$ is quasi-concave.

THEOREM 2. *Let X, Y be nonempty convex sets, each in a Hausdorff topological vector space, and let f be a real-valued function defined on $X \times Y$ such that*

(a) *For each $x \in X$, $f(x, y)$ is lower semi-continuous and quasi-convex on Y;*

(b) *For each $y \in Y$, $f(x, y)$ is upper semi-continuous and quasi-concave on X.*

If there exists a nonempty compact convex set K in X and a compact set H in Y such that

$$\inf_{y \in Y} \operatorname{suf}_{x \in X} f(x, y) \leqq \inf_{y \notin H} \max_{x \in K} f(x, y) , \tag{1}$$

then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y) . \tag{2}$$

Proof. We always have that the left-hand side of (2) is no less than the right-hand side. To prove (2) we can assume that the left-hand side of (2) is not equal to $-\infty$. We choose any real number t such that $\inf \sup_{y \in Y\, x \in X} f(x, y) > t$ and let

$$A = \{(x, y) \in X \times Y : f(x, y) > t\} ;$$
$$B = \{(x, y) \in X \times Y : f(x, y) < t\} .$$

Obviously $A \cap B = \varnothing$. On the other hand, the condition (1) implies that the set $\{y \in Y: (x, y) \notin A \text{ for all } x \in K\}$ is compact. Clearly the sets A, B also verify other conditions in the hypothesis of Theorem 1 with the exception that the sets $B(x) \neq \varnothing$ for all $x \in X$. We must have $B(x_0) = \varnothing$ for at least one point $x_0 \in X$, that is, $(x_0, y) \notin B$ for all $y \in Y$. This shows that $\sup\inf_{x \in X\, y \in Y} f(x, y) \geqq t$ and therefore completes the proof.

By a similar argument we can show that Theorem 1 remains valid if the condition (c) is replaced by the contion that: *There exists a nonempty compact convex subset H of Y such that the set* $\{x \in X: (x, y) \notin B$ for all $y \in H\}$ *is compact in* X. Consequently the minimax identity (2) also holds if (1) is replaced by the condition that: *There exists a nonempty compact convex set H in Y and a compace set K in X such that*

$$\sup_{x \in K} \inf_{y \in H} f(x, y) \leqq \sup_{x \in X} \inf_{y \in Y} f(x, y) .$$

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