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TOPOLOGICAL PROOF OF THE G-SIGNATURE THEOREM FOR G FINITE

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The G-Signature Theorem was originally proved by Atiyah and Singer as a corollary of their general index theorem for elliptic operators. Subsequently Ossa gave a proof for G finite and the fix point set orientable. His methods are mainly topological. However, he uses the theory of elliptic operators to show the g-signature of a fix point free diffeomorphism of finite order is zero. Janich and Ossa gave a short completely topological proof of the theorem for involutions. In part one, we give a complete proof for semi-free actions and simultaneously a proof for general actions modulo the theorem for fix point free actions. In essence our argument here is similar to that of Ossa. However it is shorter and conceptually simpler. Also we derive the formula in a natural way as opposed to verifying it. In part two, we prove a theorem which we use in part one to prove the result for fix point free actions. **I wish to thank my advisor Professor E. Thomas for much help and encouragement.**

Part One. By considering the cyclic subgroup generated by a given element, we may restrict our attention to cyclic group actions. By a Z_d manifold (g, M) we will mean a smooth, oriented, compact manifold (without boundary unless otherwise stated or obvious) together with an orientation preserving diffeomorphism of order d. Frequently, we will omit the g in referring to Z_d manifolds M. We denote the disjoint union by $+$, the disjoint union of r copies of M by rM, and the disjoint difference (one reverses the orientation of the second manifold) by $-$. A Z_d manifold M bounds (resp. rationally bounds) if M (resp. $r \cdot M$) is the boundary of a Z_d manifold. Two Z_d manifolds are bordant (resp. rationally bordant) if their disjoint difference bounds (resp. rationally bounds). The collection of bordism classes of Z_d manifolds forms a graded ring $O_*(Z_d)$ in the usual manner.

Let (g, B) denote the d-fold branched cover of S^2 along d points together with the deck translation which rotates neighborhoods of the fixed points through an angle $2\pi/d$. Let (g, P_n) denote the action on n-dimensional complex projective space given by $g[z_0, \cdots, z_{n-1}, z_n] = [z_0, \cdots, z_{n-1}, \omega z_n]$. Here and throughout part one $\omega = \exp(2\pi i/d)$. The fix point set has two components: the point $[0, \cdots, 0, 1]$ where the action on the trivial normal bundle is multiplication by $\bar{\omega}$, and the P_{n-1} given by $z_n = 0$ where the action on the normal boundle is given by multiplication by ω.

Given a $\boldsymbol{Z}_d$ manifold (g, M), Atiyah and Singer [1] (p. 578) define a complex number, the g-signature Sign(g, M) (specify that g acts on $H^*(M, \boldsymbol{R})$ by g^{*-1} to get correct signs). The g-signature of a representative yields a ring homomorphism Sign: $O_*(Z_d) \rightarrowtail \boldsymbol{C}$ such that

$$(1) \qquad \operatorname{Sign}[g^j, B] = d\frac{\omega^j + 1}{\omega^j - 1} \qquad \text{for} \quad 0 < j < d$$

$$(2) \qquad \operatorname{Sign}[g^j, P_n] = \operatorname{Sign} P_n \qquad \text{for} \quad 0 \leqq j < d$$

$$(3) \qquad \operatorname{Sign}[g, M] = 0 \text{ if } g \text{ leaves no component of } M \text{ invariant}\,.$$

That Sign is well defined and a ring homomorphism are well known relatively immediate consequences of the definition. (1) is not too difficult a computation. See [4] (5.2). (2) is true since g is homotopic to the identity and therefore acts trivially on the cohomology. (3) follows because one can define a g-invariant positive definite inner product on the middle real cohomology of M that respects the direct sum decomposition into the cohomology of the connected components. Sign(g, M) is defined as a difference of two traces which in the above situation are easily seen to be zero. In part two, we show that some multiple of a given fix point free $\boldsymbol{Z}_d$ manifold is bordant to one where no connected component is invariant proving property:

$$(3') \qquad \operatorname{Sign}[g, M] = 0 \quad \text{if} \quad (g, M) \quad \text{is fix point free}\,.$$

If g acts freely on M, then $M \to M/\boldsymbol{Z}_d$ is a covering space and is classified by a map $M/\boldsymbol{Z}_d \to B\boldsymbol{Z}_d$ which in turn gives an element in $\Omega_*(B\boldsymbol{Z}_d)$. Since Ω_* (a point) $\otimes \boldsymbol{Q} \to \Omega_*(B\boldsymbol{Z}_d) \otimes \boldsymbol{Q}$ is an isomorphism, we easily see that some multiple of a given free $\boldsymbol{Z}_d$ manifold is bordant to one that leaves no component invariant, proving the weaker but easier:

$$(3'') \qquad \operatorname{Sign}[g, M] = 0 \quad \text{if} \quad (g, M) \quad \text{is free}\,.$$

The G-Signature Theorem expresses the g-signature of a $\boldsymbol{Z}_d$ manifold in terms of the characteristic classes of the fixed point set and its equivariant normal bundle. We will derive this expression for semi-free $\boldsymbol{Z}_d$ manifolds assuming 1, 2, and 3$''$ and for general $\boldsymbol{Z}_d$ manifolds assuming 1, 2, and 3$'$.

By a $\boldsymbol{Z}_d$ vector bundle (g, η), we mean a smooth real vector bundle η over a smooth manifold (without boundary unless otherwise stated) with total space E_η oriented as a manifold together with a bundle map g of order d, preserving some Riemannian metric as well as the orientation on E_η. We also require that g covers a diffeomorphism of the base and acts fix point free restricted to the sphere bundle. The associated disk bundle (g, D_η) of a $\boldsymbol{Z}_d$ vector bundle is a $\boldsymbol{Z}_d$ manifold with boundary. If g covers the identity on the base then (g, η) is called fixed. If g is free restricted to the sphere bundle, (g, η) is called semi-free. We frequently omit g in referring to $\boldsymbol{Z}_d$ vector bundles.

Fixed $\boldsymbol{Z}_d$ vector bundles η can be equivariantly decomposed $\eta = \bigoplus_{j=1}^{t} \eta_j$ where $\eta_j = \{x \in \eta \mid g^2(x) - 2(\text{Real } \omega^j)\ g(x) + x = 0\}$ where $0 < j < d/2 = t$ and $\eta_t = \{x \mid g(x) + x = 0\}$. For $j \neq t$, η_j can be given a complex structure by $J(x) = (1/(2 \text{ Imag } \omega^j))(g(x) - g^{-1}(x))$ with respect to which g acts by multiplication by ω^j. Note that η_t is even dimensional as g preserves orientation. Let $n_j = 1/2 \dim_R \eta_j$, then η is semi-free if j is relatively prime to d whenever $n_j \neq 0$. One also has notions of bordism for $\boldsymbol{Z}_d$ vector bundles: plain, fixed, oriented, or semi-free bordism accordingly as the cobounding $\boldsymbol{Z}_d$ vector bundle is.

We define a special submanifold of a $\boldsymbol{Z}_d$ manifold (g, M) to be an orbit of a component of the fixed point set of some iterate of g. The equivariant tubular neighborhood theorem as described by Conner and Floyd [2] (§22) asserts that an invariant tubular neighborhood of a special submanifold is diffeomorphic as a $\boldsymbol{Z}_d$ manifold with boundary to the disk bundle of a $\boldsymbol{Z}_d$ vector bundle (the equivariant normal bundle). If F is a component of M^g (the fixed point set of g), then the resulting normal bundle is a fixed $\boldsymbol{Z}_d$ vector bundle denoted η_F.

LEMMA 1.1. *If η is a fixed (resp. and semi-free) $\boldsymbol{Z}_d$ vector bundle over a nonorientable base F then 2η is fixed (resp. and semi-free) bordant to $\tilde{\eta}$ the $\boldsymbol{Z}_d$ vector bundle pulled up to the orientable double cover $\tilde{F}$.*

Proof. We use a special case of the Dold construction as described by Hirzebruch and Janich [5]. $\tilde{F}$ can be constructed as follows. Pick a submanifold K of F Poincare dual to $w_1(F)$. Take two copies of F "cut" along K and identify them together using the antipodal map on the normal S^0 bundle of K in F. Let X be the double branched cover of $F \times [0, 1]$ along $K \times \{1/2\}$ obtained by slicing $F \times [0, 1]$ along $K \times [0, 1/2]$. Then X is a bordism between $2F$ and $\tilde{F}$. We can decompose η into η_j and then pull the η_j across $F \times [0, 1]$ and then up to X to get a bordism between 2η and $\tilde{\eta}$. We only need to check that the total space of the bundle over X is oriented. This is clear except over the branching locus as then we have an ordinary cover. However the branching locus is codimension two and if w_1 vanishes in the complement of a codimension two submanifold then it is zero. □

LEMMA 1.2. *Let η be a fixed Z_d vector bundle, then the sphere bundle rationally bounds a fixed point free Z_d manifold all of whose special submanifolds have higher dimension than the base. If η is semi-free, the cobounding manifold can be taken to be free.*

Proof. By Lemma 1.1 we can assume the base is oriented and therefore η_t can be given an $SO(2n_t)$ structure. The splitting of η gives us an element of $\Omega_* B(SO(2n_t) \times \prod_{j \neq t} U(n_j))$. See [2] §§4, 5, 6, and 44 for a definition of the homology theory $\Omega_*(\ \ )$ and an exposition of that part of the theory needed here. We think of integral elements of $\Omega_* B(SO(2n_t) \times \prod_{j \neq t} U(n_j)) \otimes \boldsymbol{Q}$ as rational bordism classes of fixed, oriented Z_d vector bundles with representation give implicitly by the sequence $\{n_j\}$. Thus the bordism is semi-free if the bundle is. The map $B(\prod T^{n_j}) \to B(SO(2n_t) \times \prod_{j \neq t} U(n_j))$ of the splitting principle induces an injection on rational cohomology and therefore a surjection on rational homology and thus on the E_2 term of the spectral sequence for the theory $\Omega_*(\ \ ) \otimes \boldsymbol{Q}$. Since both spaces only have rational homology in even dimensions $E_2 = E_\infty$, and the map induces an epimorphism on the filtration of $\Omega_*(\ \ ) \otimes \boldsymbol{Q}$, and so by induction using the five lemma on $\Omega_*(\ \ ) \otimes \boldsymbol{Q}$. In [2] (44.1), Conner and Floyd prove a Kunneth theorem which in particular gives an isomorphism $\bigotimes_{\Sigma n_j} \Omega_* BS^1 \to \Omega_*(BS^1)^{n_j} = \Omega_* B(\prod_{j=1}^{t} T^{n_j})$. Here the tensor product is over Ω_* and the map is given geometrically by the cross product of bundles. $\Omega_* BS^1$ is a free Ω_* module on generators given by the canonical bundle over P_n: ζ_n. This follows from [2] (18.1), however it is not difficult to see this directly (especially rationally). Given these facts one sees that one only needs to prove the lemma for a linear combination over Ω_* of cross products of canonical bundles. But given two bundles for which the lemma is true, one can use the rational null bordisms provided by the lemma to construct ambient Z_d manifolds such that some multiples of the given Z_d bundles occur as the normal bundles to the fixed point set. Forming the product of these Z_d manifolds and removing a tubular neighborhood of the fix point set, constructs a rational null bordism of the sphere bundle of the cross product of the required type. Also if the lemma is true for a bundle then it is true for any multiple of the bundle by a manifold, one simply multiplies the provided null bordism by the manifold. Thus we only have to prove the lemma for the canonical bundle over P_n. To do this take d^{n+1} copies of (g^j, P_{n+1}) and remove a tubular neighborhood of the d^{n+1} isolated fixed points. Also take $(-1)^n(g^j, B)^{n+1}$ and remove neighborhoods of the d^{n+1} fixed points. The resulting boundaries can be identified by an orientation reversing equivariant diffeomorphism to form a closed $\boldsymbol{Z}_d$ manifold for which the normal bundle to the fix point set is d^{n+1} copies of the canonical bundle. If we remove a tubular neighborhood of the fix point set we find our null bordism of the required type. □

Given a fixed $\boldsymbol{Z}_d$ vector bundle η, we define an invariant $\sigma(\eta) \in \boldsymbol{Q}$ as follows. By Lemma 1.2, there is a integer r and a $\boldsymbol{Z}_d$ manifold X whose boundary is r copies of the sphere bundle of η. If η is semi-free, we insist that X is free. Define

$$\sigma(\eta) = (1/r)\,\mathrm{Sign}\left(g,\, rD_\eta \bigcup_\partial X\right) .$$

THEOREM 1.3. *If (g, M) is semi-free or if we accept $3'$, then*

$$\mathrm{Sign}\,(g, M) = \sum_{F \subset M^g} \sigma(\eta_F) .$$

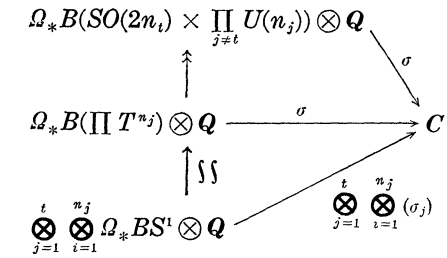
Here the summation is over the components F of M^g. It follows that σ is well-defined for semi-free bundles and in general assuming $3'$.

Proof. For each F, let $r_F \in \boldsymbol{Z}$ and X_F denote any choices as above for calculating $\sigma(\eta_F)$. Let

$$Y = \Big(\prod_F (r_F)\Big)M - \sum_F\Big(\prod_{F' \neq F}(r_{F'})\Big)\Big(r_F D_{\eta_F} \bigcup_\partial X_F\Big) .$$

Then $\mathrm{Sign}(g, Y) = (\prod_F (r_F))(\mathrm{Sign}(g, M) - \sum_F \sigma(\eta_F))$. If we take $Y \times I$ and attach copies of $D_{\eta_F} \times I$ appropriately, we may form a $\boldsymbol{Z}_d$ manifold whose boundary is $Y +$ some free or fixed point free $\boldsymbol{Z}_d$ manifold as the case may be. Thus $\mathrm{Sign}(g, Y) = 0$, and the main equality is established. The proof is independent of the choices of r_F and X_F for calculating $\sigma(\eta_F)$. Every fixed $\boldsymbol{Z}_d$ bundle can occur as the equivariant normal bundle of a component of a fixed point set (by (1.2)). It follows that σ is well defined. □

We now derive a formula for σ. Note that σ is an invariant of semi-free fixed bordism (or simply fixed bordism assuming $3'$). To see this attach to $(rD_\eta \bigcup_\partial X) \times I$ r copies of the disk bundle associated to a bordism of η. In view of Lemma 1.1, we can concentrate on bundles with oriented base. We have the following commutative diagram (specify $n_j = 0$ if j is not prime to d, if one only accepts $3''$):



where σ is defined as σ of a representative bundle. It is easy to see that σ is an Ω_* module homomorphism where $\boldsymbol{C}$ is made an Ω_* module via the signature homomorphism. σ_j is the map $\Omega_* BS^1 \otimes \boldsymbol{Q} \to \boldsymbol{C}$ that is obtained by taking σ of a representative bundle with action given by multiplication by ω^j. The lower triangle commutes because σ is multiplicative for cross products. To see this, realize η and η' rationally as the normal bundles of the fixed point set for two Z_d manifolds and take their product as in the proof of Lemma 1.2.

We wish to determine σ_j. To place the formula in context, we consider first a more general situation. If K is a multiplicative sequence with characteristic series $\sqrt{s}/R(\sqrt{s})$ and $\alpha\colon \Omega_* BS^1 \to \boldsymbol{C}$ is a Ω_* module homomorphism where $\boldsymbol{C}$ has been made an Ω_* module via the K-genus and $f(t)$ is the power series $R'(t)\sum_{n=0}^{\infty}\alpha[\zeta_n]R^n(t)$ then $\alpha[\zeta_{F,x}] = \{f(x)K(F)\}[F]$ where $\zeta_{F,x}$ is the line bundle over F with first Chern class x. This is easily seen to be correct on $[\zeta_n]$.

In our case $K = L$ and $R(t) = \tan h(t)$. Define $f_j(t)$ as above so that $\sigma_j[\eta_{F,x}] = \{f_j(x)L(F)\}[F]$. Theorem 1.3 applied to (g^j, P_{n+1}) and (g^{d-j}, β) yields $\sigma_j[\zeta_n] = \operatorname{Sign}(P_{n+1}) + (-1)^n((\omega^j+1)/(\omega^j-1))^{n+1} = \operatorname{Sign}(P_{n+1}) + (-1)^n[\coth(a)]^{n+1}$ where a is the complex number $(j\pi/d)i$. Thus

$$f_j(t) = \operatorname{sech}^2(t)\left[\frac{\tanh(t)}{1-\tanh^2(t)} + \frac{1}{\tanh(a)+\tanh(t)}\right].$$

Some hyperbolic trigonometric identities yield:

$$f_j(t) = \coth(t+a) = \frac{\omega^j e^{2t}+1}{\omega^j e^{2t}-1}. \tag{1.4}$$

Note also that

$$f_t(x) = \tanh(x) = e(\eta_{F,x})L^{-1}(\eta_{F,x}).$$

Our commutative diagram and the σ_j determine σ. In fact if we write $c(\eta_j) = \prod_{k=1}^{n_j}(1+x_{k,j})$ formally for $j \neq t$ then the formula

$$\sigma(\eta) = \left\{L^{-1}(\eta_t)e(\eta_t)\Bigl(\prod_{j\neq t}\prod_{k=1}^{n_j} f_j(x_{k,j})\Bigr)L(F)\right\}[F] \tag{1.5}$$

makes the diagram commute and therefore is correct. Finally if F is nonorientable, so is η_t. F has a fundamental class in the homology with coefficients twisted by the orientation system and $e(\eta_t)$ lies in the cohomology similarly twisted. The above formula makes sense and yields half the value it would were we to pull η up to the oriented double cover and calculate σ of that bundle. So by Lemma 1.1 the formula is correct for nonorientable base as well. Theorem 1.3 together with the formulas (1.4) and (1.5) constitute the content of the G-Signature Theorem.

Part Two. In this part we prove the following theorem.

THEOREM 2.1. *If (g, M) is a fixed point free Z_d manifold then some multiple of (g, M) is bordant to a Z_d manifold where g leaves no component invariant.*

We define the Z_s prolongation of a Z_r manifold (h, F) to be the Z_{rs} manifold $(g, Z_s \times F)$ where $g(a, f) = (a + 1, f)$ for $0 \leqq a < s$ and $f \in F$ and $g(s, f) = (0, h(f))$. One has a similar notion of a Z_s prolongation of a Z_r vector bundle. We need the following lemma. We delay its proof showing first how 2.1 follows from it.

LEMMA 2.2. *Let (g, η) be a Z_{rs} vector bundle such that g acts freely of order s restricted to the base and (g^s, η) is a fixed Z_r vector bundle, then some multiple of η is bordant through such bundles to a Z_s prolongation of a fixed Z_r vector bundle.*

Proof of 2.1. The proof is by induction on the maximum codimension k of the special submanifolds of (g, M). If k is zero then g is free of some order on the orbit of any component and we already noted in the proof of 3″ that the conclusion of this theorem holds for each such orbit. Suppose the theorem holds for $k < j$, and consider the case $k = j$. The special submanifolds F_i of codimension j are disjoint as otherwise their intersections would consist of special submanifolds of higher codimension. Moreover g restricted to F_i acts freely of some order s_i, as otherwise there would be special submanifolds of higher codimension. Thus the equivariant normal bundle η_i of F_i satisfies the hypothesis of Lemma 2.2. Let $r_i s_i = d$. Let $p_i \eta_i$ be bordant through λ_i to the Z_{s_i} prolongation of the fixed Z_{r_i} vector bundle ψ_i. Lemma 1.2 provides N_i a Z_{r_i} manifold with normal bundle to the fixed point set $q_i \psi_i$, and all other special submanifolds with lower codimension. Let $r = \prod_i p_i q_i$, L_i be the Z_{s_i} prolongation of N_i, and $X = rM - \sum_i (r/p_i q_i) L_i$. X is a Z_d manifold for which the tubular neighborhood to the collection of special submanifolds of maximal codimension is provided with a null bordism, namely $Y = \sum_i (r/p_i)D_{\lambda_i}$. $X \times I + Y$ with appropriate identifications is a bordism between rM and a Z_d manifold all of whose special submanifold have codimension less than k. By the inductive hypothesis, some multiple of this manifold is bordant to a manifold where no component is invariant. Putting appropriate multiples of these two bordisms together, we complete the inductive step. □

Proof of 2.2. This proof is based partially on an idea of Conner the Floyd [3] 3.1. Consider the fixed Z_r vector bundle (g^s, η) and write as before $\eta = \bigoplus_{j=1}^t \eta_j$ where now $t = r/2$, $\omega = e^{2\pi i/r}$ and $n_j = 1/2 \dim_R \eta_j$. Let F be the base and $\pi: F \to F/Z_s = B$ the projection. Let $n = \dim F$ and $k \gg n$. Embed B in S^{n+k} with normal bundle ν. Let $S(p, q)$ be $SO(p+q) \cap (O(p) \times O(q)) \subset O(p+q)$. Let H be $S(k, 2n_t) \times \prod_{j \neq t} U(n_j)$ and G be $Z_{rs} \times H/\Delta$ where Δ is the cyclic group of order r generated by $(s, I_k - I_{2nt}, \omega I_{n_1}, \omega^2 I_{n_2}, \cdots, \omega^j I_{n_j}, \cdots)$. H is normal in G with quotient Z_s. Since $w_1\pi^*(\nu) = w_1(F) = w_1(\eta_t)$, $\pi^*\nu \oplus \eta$ can be given an H structure. $\pi^*\nu$ has a natural Z_{rs} action of order s covering the action on F, and η comes with a Z_{rs} action. Thus the associated principal H bundle with total space S has a Z_{rs} action. (i.e., think of elements of S as admissible maps, then Z_{rs} acts by composition of mappings.) Let $Z_{rs} \times H$ act on S by $s(a, h) = a^{-1}sh$. Note $s(a, h) = s$ if and only if (a, h) is in Δ. The quotient space is B, thus S is a principal G bundle over B. If we take the associated vector bundle to its $O(k)$ "extension", we recover ν.

Since G is a subgroup of some $O(m)$, if we let E be $EO(m)$ restricted to the Grassmann of m planes in $R^N (N \gg n)$, then $Z_s = G/H \to E/H \to E/G$ will approximate $G/H \to BH \xrightarrow{p} BG$ (from the point of view of an n-complex) by an s-fold covering projection of smooth closed manifolds. We write BH for E/H etc. Let γ be the vector bundle associated to the $O(k)$ "extension" of the G bundle over BG. We need that $T(p)_\#: \pi_{n+k}(T(p)^*\gamma) \to \pi_{n+k}(T\gamma)$ is rationally surjective. To see this, note that the rational Hurewicz homomorphism from $\pi_{n+k}(\ ) \otimes Q$ to $H_{n+k}(\ , Q)$ is an isomorphism on both Thom spaces as they are $k - 1 \gg n$ connected: see [7] 18.3. Using excision, we need only show the map on the disk bundle rel the sphere bundle induces a surjection in rational homology. However, as this is a finite covering map, one has a right inverse on the relative rational chain level. The result follows.

Pick a map of principal G bundles $S \to EG$ and consider the induced map $\nu \to \gamma$. The Thom construction then gives a map $f: S^{n+k} \to T\gamma$ representing $[f]$ in $\pi_{n+k}(T\gamma)$. By the above, for some $m \in Z$, $m[f] = [g]$ where $g: S^{n+k} \xrightarrow{h} T(p^*\gamma) \to T(\gamma)$. If we embed mB

in $\#^m S^{n+k} = S^{n+k}$ and use the same bundle map $\nu \to \gamma$, the above construction leads to a map (say) $mf\colon S^{n+k} \to T(\gamma)$ representing $m[f]$. We can insure that h is smooth away from $h^{-1}(\infty)$ and transverse to BH. Therefore g will be smooth away from $g^{-1}(\infty)$ and transverse to BG. We can pick a homotopy F between mf and g that is smooth away from $F^{-1}(\infty)$ and transverse to BG as well as being exactly mf and g on a collar of the boundary. Let $B' = F^{-1}(BG)$ and $\bar{B} = g^{-1}(BG)$ so $\partial B' = mB + \bar{B}$. There is a principal G bundle over B' with total space S' such that the associated vector bundle to the $O(k)$ "extension" is given an isomorphism to the normal bundle ν' of B' in $S^{n+k} \times I$. Moreover, restricted to mB we have m isomorphic copies of our original G bundle over B, and over $\bar{B}$ we have a G bundle which is an extension of an H bundle. Thus the associated bundle over $\bar{B}$ with fiber $G/H = Z_s$ has a section and thus is trivial as it is also principal.

We now use this bordism of G bundles to construct the required bordism of Z_{rs} vector bundles. Consider the array

$$\begin{array}{ccccc}
S' & \longrightarrow & S'/SO(2n_t) \times \prod_{j \neq t} U(n_j) & \longrightarrow & S'/Z_{rs} \times SO(2n_t) \times \prod_{j \neq t} U(n_j)/\Delta \\
\downarrow & \searrow^{d} & \downarrow b & & \downarrow a \\
S'/SO(k) & \xrightarrow{c} & S'/H = F' & \xrightarrow{\pi} & S'/G = B'
\end{array}$$

of quotients of S'. a is the projection of the $O(k)$ "extension". π is the projection of the associated principal $G/H = Z_s$ bundle and b is a pulled up to F'. c is the projection of the associated principal $H/SO(k) = O(2n^t) \times \prod_{j \neq t} U(n_j)$ bundle with associated vector bundle η'. The total spaces of the vector bundles associated to a and b are oriented. d gives an orientation to η' Whitney sum the bundle associated to b. It follows that the total space of η' is orientable. Let Z_{rs} generated by g act on $S'/SO(k)$ via $Z_{rs} \xrightarrow{(-1,0)} Z_{rs} \times H \to G$. Since Δ acts trivially, g^s which fixes F' will act in the fibers by multiplication by $(-1, \omega^j)$. Moreover g covers a map free of order s on F'. Let g act on η' in such a way that it induces the above action on $S'/SO(k)$. In this way one gets a Z_{rs} vector bundle (g, η') with boundary of the type in the lemma. One can check that one recovers m isomorphic copies of (g, η) if one restricts to $\pi^{-1}(mB)$. Let $(g, \bar{\eta})$ be the restriction to $\pi^{-1}(\bar{B})$. Since $\pi^{-1}(\bar{B}) \to \bar{B}$ is a trivial principal Z_s bundle, one sees that $(g, \bar{\eta})$ is isomorphic to the Z_s prolongation of $(g^s, \bar{\eta})$ restricted to a single sheet. □

REFERENCES

1. M. F. Atiyah and I. M. Singer, *The index of elliptic operators: III*, Ann. of Math., **87** (1968), 546–604.

2. P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Ergebnisse Series vol. 33, Springer-Verlag, 1964.
3. ———, *Maps of odd period*, Ann. of Math., **84** (1966), 132–156.
4. P. M. Gilmer, *Configurations of surfaces in 4-manifolds*, Trans. Amer. Math. Soc., **264** (1981), 353–380.
5. F. Hirzebruch and K. Janich, *Involutions and singularities*, Proc. Bombay Colloq. on Alg. Geom., (1968), 219–240.
6. K. Janich and E. Ossa, *On the signature of an involution*, Topology, **8** (1969), 27–30.
7. J. W. Milnor and J. A. Stasheff, *Characteristic Classes*, Annals of Math. Studies, Princeton University Press, 1974.
8. E. Ossa, *Aquivariante Cobordismus Theorie*, Diplomarbeit, Bonn, 1967.

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