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NEW DIAGRAM PROOFS OF THE HAUSDORFF-YOUNG THEOREM AND YOUNG'S INEQUALITY

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In this paper, the diagram proof of Riesz's Theorem proved by the author is used to give new diagram proofs for the classical Hausdorff-Young Theorem, and Young's Inequality, where Fourier transforms and convolutions are used respectively.

I. Introduction.[1] Very often, the same operator is investigated on several different function spaces. Thus, it is valuable to have theorems which give relationships between properties of the same operator considered in different function spaces. The well-known Marcel Riesz interpolation theorem [11] which was published in 1926 is a nontrivial example of such a theorem.

Since 1926, much work has been done in interpolation theory by A. P. Calderon in 1964, Lions-Peetre [9] in 1964, and M. Schechter in 1967.

More recently, V. Williams [16], in 1971, defined a generalized interpolation space, $X_{(T,C)}$, which generalizes each of the above-mentioned interpolation spaces. Also, a generalized interpolation theorem is proved in [16] which generalizes the Calderon, Lions-Peetre, and Schechter interpolation theorems.

The classical theorems of Riesz and Marcinkiewicz follow from interpolation theory, and there are many applications in differential equations, Banach algebras, and nonlinear, complex, and compact interpolation theories (see [13, 14, 3, 7, 8, and 5]).

We now give a definition:

DEFINITION. A *compatible triplet* $\{X_0, X_1, \mathscr{X}\}$ consists of two Banach spaces X_0 and X_1 which are continuously embedded in a Hausdorff topological vector space $\mathscr{X}$.

$s_x = s(P_0, E_0, X_0, P_1, E_1, X_1)$ denotes the Lions-Peetre [9] interpolation space which is also a generalized interpolation space [16].

In this paper, the diagram proof of Riesz's theorem proved by the author [6] is used to give new diagram proofs for the classical Hausdorff-Young theorem, and Young's inequality, where Fourier transforms and convolutions are used respectively.

II. Diagram proof of the Hausdorff-Young Theorem. As a corollary of the author's diagram proof [6] of Riesz's theorem, we

[1] Terms used in the introduction will be defined in the paper.

get the classical Hausdorff-Young theorem. The corollary is significant because we give a new diagram proof of this classical theorem.

First, we give our classical notation and state a classical lemma.

$L^P(R^1)$, $1 \leqq P \leqq \infty$, denotes functions defined on R^1 which have values in the complex numbers, C, with the usual norm, and m denotes the Lebesgue measure on R^1 divided by $\sqrt{2\pi}$.

We have

$$\int_{-\infty}^{\infty} f(x)dm(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)dx\ , \quad \text{where } dx \text{ refers to the ordinary Lebesgue measure.}$$

Define $\hat{f}$ by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt}dm(x)\ , \quad t \in R^1\ . \tag{4}$$

If $f \in L^1(R^1)$, the integral in Line (4) is well defined for every real t. The function $\hat{f}$, the *Fourier transform of* f, is denoted by $F(f)$, that is, F sends f to $\hat{f}$.

Since the Lebesgue measure of R^1 is infinite, $L^2(R^1)$ is not a subset of $L^1(R^1)$, and the definition of the Fourier transform is not directly applicable for every $f \in L^2(R^1)$. However, the definition does apply if $f \in L^1(R^1) \cap L^2(R^1)$, for in this case $\hat{f} \in L^2(R^1)$. In fact, $\|\hat{f}\|_{L^2(R^1)} = \|f\|_{L^2(R^1)}$. This isometry of $L^1(R^1) \cap L^2(R^1)$ into $L^2(R^1)$ extends to an isometry of $L^2(R^1)$ onto $L^2(R^1)$, and this extension defines the Fourier transform (sometimes called the *Plancherel transform*) of every $f \in L^2(R^1)$. The $L^2(R^1)$-theory has more symmetry than $L^1(R^1)$. In $L^2(R^1)$, f and $\hat{f}$ play exactly the same role.

We now state some classical results:

LEMMA. *The Fourier transform map,* $F\colon L^1(R^1) \to L^\infty(R^1)$ *where* $F(f) = \hat{f}$, *for* $f \in L^1(R^1)$ *is bounded, linear, and for every* $f \in L^1(R^1)$

$$\|F(f)\|_{L^\infty(R^1)} = \|\hat{f}\|_{L^\infty(R^1)} \leqq \|f\|_{L^1(R^1)}\ .$$

Therefore, $\|F\|_{B(L^1(R^1), L^\infty(R^1))} \leqq 1$.

Plancherel Theorem. One can associate to each $f \in L^2(R^1)$ a function $\hat{f} \in L^2(R^1)$ such that:

(a) If $f \in L^1(R^1) \cap L^2(R^1)$, then $\hat{f}$ is the previously defined Fourier transform of f.

(b) For every $f \in L^1(R^1)$, $\|\hat{f}\|_{L^2(R^1)} = \|f\|_{L^2(R^1)}$.

(c) The map $F\colon L^2(R^1) \to L^2(R^1)$, where $F(f) = \hat{f}$, for each $f \in L^2(R^1)$ is an isomorphism of $L^2(R^1)$ onto $L^2(R^1)$. In particular, F is bounded, linear, and by Part (b),

$$\|F\|_{B(L^2(R^1), L^2(R^1))} = 1\ .$$

Note: Since $L^1(R^1) \cap L^2(R^1)$ is dense in $L^2(R^1)$, Parts (a) and (b) determine the map F uniquely.

Hausdorff-Young Theorem. With the above notation, let $1 \leqq P \leqq 2$, let $1 \leqq q \leqq \infty$ be such that $1/P + 1/q = 1$; if $f \in L^P(R^1)$ then

$$\|\hat{f}\|_{L^q(R^1)} \leqq \|f\|_{L^P(R^1)} .$$

We now state our main theorem:

THEOREM. *If $1 < P < 2$, the Hausdorff-Young Theorem follows from the diagram proof Riesz's Theorem.*

Proof. First, we consider a commutative diagram (see below).

By the author's work [4], for any fixed $0 < s < 1$, there exist E_0 and E_1 in R^1 such that $E_0 \cdot E_1 < 0$ and $s = E_0/(E_0 - E_1)$.

Let $P_0 = 1$, $P_1 = 2$, $q_0 = \infty$, $q_1 = 2$, so $P_i \leqq q_i$, $i = 0, 1$. Let P and q satisfy

$$\frac{1}{P} = \frac{1-s}{P_0} + \frac{s}{P_1}, \qquad \frac{1}{q} = \frac{1-s}{q_0} + \frac{s}{q_1} .$$

From previous work [4], $\{L^1(R^1), L^2(R^1), L^1_{\text{loc}}(R^1)\}$ and $\{L^\infty(R^1), L^2(R^1), L^1_{\text{loc}}(R^1)\}$ are compatible triplets; therefore the sums in the diagram make sense.

From above work, all spaces in the diagram are Banach. Let $L = F$ be the Fourier transform map as defined above.

By the lemma and Plancherel's theorem,

(9) $L \in B(L^1(R^1), L^\infty(R^1))$ and $L \in B(L^2(R^1), L^2(R^1))$, with respective norms N_0 and N_1, where $N_0 \leqq 1$, $N_1 = 1$.

$$\begin{tikzcd}
L^1(R^1) + L^2(R^1) \arrow[loop left] \arrow[r, "L = F"] & L^\infty(R^1) + L^2(R^1) \arrow[loop right] \\
\mathscr{W}(1, s, L^1(R^1), 2, s-1, L^2(R^1)) \arrow[u, "T_1"'] \arrow[r, "L'"] & \mathscr{W}(1, s, L^\infty(R^1), 2, s-1, L^2(R^1)) \arrow[u, "T_2"]
\end{tikzcd}$$

$$\begin{array}{c}
s(1, s, L^1(R^1), 2, s-1, L^2(R^1)) \\
\| \ [9] \\
S(1, s, L^1(R^1), 2, s-1, L^2(R^1)) \\
\| \ [9] \\
L^P(R^1)
\end{array}
\qquad
\begin{array}{c}
s(1, s, L^\infty(R^1), 2, s-1, L^2(R^1)) \\
\| \ [9] \\
S(1, s, L^\infty(R^1), 2, s-1, L^2(R^1)) \\
[9] \\
S(\infty, s, L^\infty(R^1), 2, s-1, L^2(R^1)) \\
\| \ [9] \\
L^q(R^1)
\end{array}$$

From previous work [4], L is a bounded linear map from $L^1(R^1) + L^2(R^1)$ to $L^\infty(R^1) + L^2(R^1)$.

L', T_1, and T_2 are defined as follows:

$$L'(f) = L \circ f\,, \qquad T_1(f) = \sum_{-\infty}^{\infty} f_n\,, \qquad T_2(g) = \sum_{-\infty}^{\infty} g_n\,,$$

where $f \in \mathscr{W}(1, s, L^1(R^1), 2, s-1, L^2(R^1))$ and $g \in \mathscr{W}(1, s, L^\infty(R^1), 2, s-1, L^2(R^1))$, that is,

$$f\colon z = \{0, \pm 1, \pm 2, \cdots\} \longrightarrow L^1(R^1) + L^2(R^1)\,,$$
$$g\colon z \longrightarrow L^\infty(R^1) + L^2(R^1)\,.$$

L', T_1, and T_2 are bounded linear maps. By the definition of L', we have $LT_1 = T_2L'$.

By Riesz's theorem

$$L \in B(L^P(R^1), L^q(R^1))\,, \quad \text{and by Line (9)}\,,$$

$$\|L\| \leqq \max\{N_0, N_1\} \leqq \max\{1, 1\} = 1. \quad \text{Therefore,}$$

$$\|L\|_{B(L^P(R^1), L^q(R^1))} \leqq 1\,.$$

By the definition of P and q,

$$\text{(12)} \qquad \frac{1}{P} = \frac{1-s}{P_0} + \frac{s}{P_1} = \frac{1-s}{1} + \frac{s}{2} = 1 - s + \frac{s}{2} = 1 - \frac{s}{2}\,, \quad \text{and}$$
$$\frac{1}{q} = \frac{1-s}{q_0} + \frac{s}{q_1} = 0 + \frac{s}{2} = \frac{s}{2}\,.$$

Therefore, $1/P + 1/q = 1$.

From above, $s = E_0/(E_0 - E_1) \in (0, 1)$.

Note: If $s = 0$, then $P = 1$; if $s = 1$ then $P = 2$. $s \in (0, 1)$ implies $0 < s/2 < 1/2$, which implies $1 > 1 - s/2 > 1/2$, which implies $1 > 1/P > 1/2$, which implies $1 < P < 2$.

So, P, as defined in Line (12), will always be such that $1 < P < 2$.

From above,

$$L = F \in B(L^P(R^1), L^q(R^1))\,,$$

therefore, for every $f \in L^P(R^1)$,

$$\|L(f)\|_{L^q(R^1)} = \|F(f)\|_{L^q(R^1)} \leqq 1 \cdot \|f\|_{L^P(R^1)} = \|f\|_{L^P(R^1)}\,.$$

By definition,

$$F(f) = \hat{f}\,,$$

therefore, for every $f \in L^P(R^1)$, we have

$$\|\hat{f}\|_{L^q(R^1)} \leqq \|f\|_{L^P(R^1)}\,,$$

so the Hausdorff-Young theorem holds if $1 < P < 2$.

III. A diagram proof of Young's Inequality. Next, we show that as a consequence of the diagram proof of Riesz's theorem, we get an inequality of W. H. Young.

Some classical results are:

DEFINITION. Let f and g be two m-measurable functions on R^1 such that $f(x-y)$. $g(y)$ is m-integrable on R^1, $x, y \in R^1$; the *convolution*, $f*g$, of f and g at point x is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dm(y) , \quad x \in R^1 .$$

Young's Inequality. Suppose that

$$1 \leqq P \leqq \infty , \quad 1 \leqq q \leqq \infty , \quad \frac{1}{r} = \frac{1}{P} + \frac{1}{q} - 1 \geqq 0 .$$

Let $f \in L^P(R^1)$, $g \in L^q(R^1)$. Then

$$\|f*g\|_{L^r(R^1)} \leqq \|f\|_{L^P(R^1)} \cdot \quad \|g\|_{L^q(R^1)} .$$

Our main theorem for this section is:

THEOREM. *Young's Inequality above follows from the diagram proof of Riesz's Theorem if* $1 < P, q < \infty$, *and* $1/r = 1/P + 1/q - 1 > 0$.

Proof. Let $1 < P$, $q < \infty$, be such that $1/P + 1/q - 1 > 0$, let $1/r = 1/P + 1/q - 1$, let P' be such that $1/P + 1/P' = 1$.

For $q_0 = 1$, $q_1 = P'$, $r_0 = P$, $r_1 = \infty$, we show that there is an s such that $0 < s < 1$, and these equations hold:

$$\frac{1}{q} = \frac{1-s}{q_0} + \frac{s}{q_1} = \frac{1-s}{1} + \frac{s}{P'} ,$$

and

$$\frac{1}{r} = \frac{1-s}{r_0} + \frac{s}{r_1} = \frac{1-s}{P} + 0 = \frac{1-s}{P} .$$

Now,

$$\frac{1}{q} = \frac{1-s}{1} + \frac{s}{P'} = \frac{P'(1-s)+s}{P'} .$$

Thus,

$$\frac{P'}{q} = P'(1-s) + s ,$$

$$P' - P's + s = \frac{P'}{q} ,$$

and

$$s(1-P') = \frac{P'}{q} - P' .$$

Thus, $s = (P'/q - P')/(1 - P')$, $P' \neq 1$ since $1 < P < \infty$. Thus, $s = (1/q - 1)/(1/P' - 1) > 0$, since $q > 1$, $1 < P < \infty$, and $P' > 1$.

Now, $1/q - 1 > -1/P$, $1/(-1/P) < 0$; thus, $(1/q - 1)/(-1/P) < (-1/P)/(-1/P) = 1$, $-1/P = 1/P' - 1$, thus, $0 < s = (1/q - 1)/(-1/P) < 1$.

We show this same $0 < s < 1$ works for r, where

$$\frac{1}{r} = \frac{1-s}{r_0} + \frac{s}{r_1} = \frac{1-s}{P} + 0 = \frac{1-s}{P}\,.$$

Also, $1/P + 1/q - 1 = (1 - s)/P$

$$\begin{aligned}\frac{1}{r} = \frac{1}{P} + \frac{1}{q} - 1 &= \frac{1}{P} + \left(\frac{1-s}{1} + \frac{s}{P'}\right) - 1 \\ &= \frac{1}{P} + \left[1 - s + s\left(1 - \frac{1}{P}\right) - 1\right] \\ &= \frac{1}{P} + \left[1 - s + s - \frac{s}{P} - 1\right] = \frac{1-s}{P}\,.\end{aligned}$$

As shown on the preceding page, for $0 < s < 1$ fixed, there exist reals E_0 and E_1 such that $E_0 \cdot E_1 < 0$ and $s = E_0/(E_0 - E_1)$.

Let $f \in L^P(R^1)$ be fixed, once chosen. Let $g \in L^1(R^1)$, and $g \in L^{P'}(R^1)$, define L by $L(g) = f * g$.

From classical work,

$$L \in B(L^1(R^1), L^P(R^1))\,, \qquad L \in B(L^{P'}(R^1), L^\infty(R^1))\,,$$

where

$$\| L \|_{B(L^1(R^1), L^P(R^1))} \leqq \| f \|_{L^P(R^1)}\,,$$

and

$$\| L \|_{B(L^{P'}(R^1), L^\infty(R^1))} \leqq \| f \|_{L^P(R^1)}\,.$$

Under the above conditions, we now consider a commutative diagram:

$$\begin{tikzcd}
L^1(R^1) + L^{P'}(R^1) \arrow[r, "L"] & L^P(R^1) + L^\infty(R^1) \\
\mathscr{W}(1, s, L^1(R^1), P', s-1, L^{P'}(R^1)) \arrow[u, "T_1"] \arrow[r, "L'"] & \mathscr{W}(1, s, L^P(R^1), P', s-1, L^\infty(R^1)) \arrow[u, "T_2"']
\end{tikzcd}$$

$$s(1, s, L^1(R^1), P', s-1, L^{P'}(R^1)) \overset{[9]}{=} S(1, s, L^1(R^1), P', s-1, L^{P'}(R^1)) \overset{[9]}{=} L^q(R^1)$$

$$s(1, s, L^P(R^1), P', s-1, L^\infty(R^1)) \overset{[9]}{=} S(1, s, L^P(R^1), P', s-1, L^\infty(R^1)) \overset{[9]}{=} S(P, s, L^P(R^1), \infty, s-1, L^\infty(R^1)) \overset{[9]}{=} L^r(R^1)$$

As above,

$$\{L^1(R^1),\ L^{p'}(R^1),\ L^1_{\mathrm{loc}}(R^1)\}$$

and

$$\{L^p(R^1),\ L^\infty(R^1),\ L^1_{\mathrm{loc}}(R^1)\}$$

are compatible triplets. Therefore, the sums in the diagram make sense.

Also, all spaces are Banach in diagram, and L defines a linear map from $L^1(R^1) + L^{p'}(R^1)$ to $L^p(R^1) + L^\infty(R^1)$.

The bounded linear maps L', T_1, and T_2 are defined as above, and $LT_1 = T_2L'$.

From Riesz's theorem,

$$L \in B(L^q(R^1),\ L^r(R^1)) ,$$

with

$$\| L \| \leqq \max\{\| f \|_{L^p(R^1)},\ \| f \|_{L^p(R^1)}\} = \| f \|_{L^p(R^1)} .$$

Therefore, for any $g \in L^q(R^1)$, we have

$$\| L(g) \|_{L^r(R^1)} \leqq \| f \|_{L^p(R^1)} \cdot \| g \|_{L^q(R^1)} , \quad \text{where } f \text{ is fixed.}$$

By definition of L, $L(g) = f * g$, therefore, for any $g \in L^q(R^1)$, we have

$$\| f * g \|_{L^r(R^1)} \leqq \| f \|_{L^p(R^1)} \cdot \| g \|_{L^q(R^1)} ,$$

and therefore Young's inequality holds.

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