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GENERALIZED THREE-MANIFOLDS WITH ZERO-DIMENSIONAL NONMANIFOLD SET

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This paper investigates the statement (GM): "If X is a compact generalized 3-manifold without boundary, and whose nonmanifold set is 0-dimensional, then X is the cell-like image of some closed 3-manifold." Some necessary and some sufficient conditions on X are given for (GM) to be true. The question of whether (GM) is true in general is shown to be inextricably tangled with the Poincaré Conjecture: (1) If the Poincaré Conjecture fails, then there is an acyclic, monotone union M of handlebodies whose one-point compactification $\hat{M}$ is a generalized 3-manifold, yet $\hat{M}$ is not the cell-like image of any compact 3-manifold. (2) If X is a compact generalized 3-manifold with zero-dimensional singular set S and no π_1-torsion in any sufficiently tight neighborhood of S, then (modulo the Poincaré Conjecture) X is the cell-like image of a compact 3-manifold.

1. Basic definitions and notation. In this paper a *generalized 3-manifold* (3-gm) X will be compact and without boundary, and so will be a compact, finite-dimensional absolute neighborhood retract (ANR) so that for every point $x \in X$

$$H_*(X, X - \{x\}) \cong H_*(S^3) .$$

We are using S^n to denote the unit sphere in Euclidean $(n+1)$-space $\boldsymbol{R}^{n+1}$. We will use B^n, Δ^n, $\boldsymbol{Z}$ and I to denote the unit n-ball, the standard n-simplex, the integers and the unit interval $[0, 1]$ respectively. All homologies will use $\boldsymbol{Z}$ for coefficients.

If X is a 3-gm, the *singular set* $S = S(X)$ of X will be the set of those points in X which have no neighborhood homeomorphic to $\boldsymbol{R}^3$. The set $X - S = M(X)$, called the *manifold set* of X, will be a noncompact 3-manifold without boundary if S is neither empty nor all of X. In this paper we will be concerned only with those 3-gms whose singular set is 0-dimensional. As of this writing, the status of (GM) for those 3-gms whose singular set has dimension greater than 0 is not known. For solutions to the analogous problems in dimensions five and higher, see [6] and [18].

A *cell-like* map is a surjection in which the inverse image of each point is cell-like. (All "maps" are continuous.) A continuum (compact, connected set) in an ANR is *cell-like* if it contracts in each neighborhood of itself. If X is a 3-gm, then we say that X *resolves*, or "admits a resolution," if there is a cell-like map from a

3-manifold onto X. A restatement of (GM) then says that a compact 3-gm X admits a resolution.

In this paper all spaces and maps of spaces will be, wherever possible, in the PL category. In particular, all compact manifolds will have a finite triangulation and all noncompact manifolds, including manifold sets of 3-gms, will have a locally finite triangulation. "Manifolds" are always connected. We will use ∂M and $\mathring{M}$ to denote the boundary and interior of M respectively. The relations "contained in the interior of," "homeomorphic to," "homotopy equivalent to" and "isomorphic to" will be written $\overset{\circ}{\subset}$, $\approx$, $\sim$ and $\cong$, respectively.

A *closed* manifold will be compact and without boundary, while an *open* manifold will be noncompact and without boundary. An *acyclic* manifold is one with the homology of a point. If a compact manifold A is embedded in a manifold B, it is *properly embedded* if $A \cap \partial B = \partial A$. A *surface* is a (connected) 2-manifold. The *genus* of a closed, orientable surface F is $1/2(2 - \chi F)$ where χF is the Euler characteristic of F.

A *handlebody of genus n* is a space homeomorphic to a regular neighborhood of a wedge of n simple closed curves in S^3. Note that the genus of a handlebody equals the genus of its boundary. A *complete set of cutting disks* for a handlebody H of genus n is a collection of n pairwise disjoint 2-disks, properly embedded in H, whose union does not separate H. The boundaries of a complete set of cutting disks for H comprise a *meridinal system* for H, and each curve in a meridinal system is a *meridinal curve*.

If $X \subset Y$ are topological spaces and J is a loop in Y, then we will say that J *shrinks in* Y mod X if there is a compact planar surface (disk-with-holes) that maps into Y so that the map on one boundary component gives J and the map takes all other boundary components into X. Lastly, a group is *perfect* if it is equal to its commutator subgroup.

2. Generalized three-manifolds: elementary properties and a criterion for resolution. In Theorem 1 we show that a 3-gm X with 0-dimensional singular set resolves if and only if the manifold set of X embeds in a compact 3-manifold. Thus, conditions on X that we show are sufficient to have X resolve (Theorems 3, 4 and 5.2) can also be interpreted as conditions under which an open 3-manifold embeds in a compact 3-manifold. These can be compared to [3; Theorem 2] and [16; Theorem 2.4].

Only Lemma 1 below is needed for the proof of Theorem 1. We have included Lemma 2 at this point because its proof is closely related to that of Lemma 1.

LEMMA 1. *Let X be a compact generalized 3-manifold and assume that $S(X)$ is 0-dimensional. Then the manifold set can be written as*

$$M(X) = \cup \{K_i | 1 \leqq i < \infty\} ,$$

where each K_i is a compact 3-manifold with nonempty boundary. If we let $N_i = (\overline{X - K_i})$, then the K_i can be chosen so that the following are true:

(a) *Each $K_i \subset \mathring{K}_{i+1}$ and each component of N_i has connected boundary;*

(b) *For all $i \geqq 2$ and $k \geqq 1$, $i_*: \pi_k(N_i) \to \pi_k(N_{i-1})$ is trivial;*

(c) *For all $i \geqq 2$ and for each component C of N_i,*

(c1) *$i_\#: H_1(\partial C) \to H_1(N_{i-1} - N_{i+1})$ is trivial;*

(c2) *$i_\#: H_1(C - S(C)) \to H_1(N_{i-1} - S(N_{i-1}))$ is trivial;*

(c3) *$C - S(C)$ is orientable;*

(c4) *$H_2(C) = 0$; and*

(c5) *$i_\#: H_1(C - S(C)) \to H_1(C)$ is an isomorphism.*

Proof. Since $M(X)$ is an open 3-manifold, it has a locally finite triangulation with a countable number of simplexes. We can obtain conclusion (a) by taking regular neighborhoods of submanifolds, simplexes and arcs. Since X is locally contractible, we can obtain conclusion (b) by taking a suitable subsequence and renumbering.

Let C be a component of N_i, $i \geqq 2$. Let x be a point in $S(N_{i-1})$. By the excision theorem and the definition of 3-gm,

$$i_\#: H_1(N_{i-1} - \{x\}) \longrightarrow H_1(N_{i-1})$$

is an isomorphism. Since every loop in ∂C shrinks in N_{i-1}, every loop in ∂C bounds a singular, compact, orientable surface in $N_{i-1} - \{x\}$. Since this surface is compact, it lies in $N_{i-1} - U_0(x)$ where $U_0(x)$ is a neighborhood of x in $\mathring{N}_{i-1} - \partial C$. Since $H_1(\partial C)$ is finitely generated, there is a neighborhood $U(x)$ of x in $\mathring{N}_{i-1} - \partial C$ so that

$$i_\#: H_1(\partial C) \longrightarrow H_1(N_{i-1} - U(x))$$

is trivial. Since $S(N_{i-1})$ is compact and totally disconnected, there exists a finite collection $\{U_1, \cdots, U_n\}$ of pairwise disjoint open sets in $\mathring{N}_{i-1} - \partial C$ so that

$$S(N_{i-1}) \subset \cup \{U_j | 1 \leqq j \leqq n\} ,$$

and so that

$$i_\#: H_1(\partial C) \longrightarrow H_1(N_{i-1} - U_j)$$

is trivial for all j, $1 \leqq j \leqq n$. We can also require that each $\partial \bar{U}_j$ is a compact 2-manifold in $\mathring{N}_{i-1}$ that is disjoint from $S(N_{i-1})$.

We will use the Mayer-Vietoris sequence to show that

$$i_{\#}\colon H_1(\partial C) \longrightarrow H_1(N_{i-1} - V)$$

is trivial, where $V = \cup \{U_j \mid 1 \leqq j \leqq n\}$. This will establish (c1) by taking a subsequence and renumbering. It suffices to show that

$$i_{\#}\colon H_1(\partial C) \longrightarrow H_1(N_{i-1} - U_1 - U_2)$$

is trivial. Let $A = N_{i-1} - U_1$ and let $B = N_{i-1} - U_2$. Then $A \cap B = N_{i-1} - U_1 - U_2$ and, since $U_1 \cap U_2 = \varnothing$, $A \cup B = N_{i-1}$. We can write

$$H_2(A) \oplus H_2(B) \xrightarrow{\alpha} H_2(N_{i-1}) \xrightarrow{\beta} H_1(A \cap B) \xrightarrow{\gamma} H_1(A) \oplus H_1(B) .$$

Let J be a loop in ∂C. We know $\gamma[J] = 0$. We would like to show $[J] = 0$ in $H_1(A \cap B)$. This will be the case if γ is one-to-one, which in turn will be true if $\operatorname{Im} \beta = 0$, which in turn will be true if α is onto. The spaces with which α is involved, A, B and N_{i-1}, are all subspaces of X that contract in X and that have 2-manifold boundary components. Such a space has the property that the inclusion of its boundary induces surjections on all homology groups. This can be shown by a direct geometric argument, or by using excision and (for example) the long exact sequences of the pairs $(A, \partial A)$ and $(X, X - \mathring{A})$. Thus $H_2(N_{i-1})$ is generated by 2-cycles in ∂N_{i-1}. But ∂A is the disjoint union of ∂N_{i-1} and ∂U_1. By choosing the appropriate elements of $H_2(A)$ and the zero element in $H_2(B)$ we can show that α is onto.

To show (c2), we note that each loop in $C - S(C)$ lies in $C - N_{i+k}$, for some $k \geqq 1$ that depends on the loop. Since the loop shrinks in N_{i-1}, it is homologous in $C - N_{i+k}$ to a sum of loops in $\partial C \cup \partial N_{i+k}$. We can now quote (c1) applied to ∂C and to the boundaries of the components of N_{i+k}.

Conclusion (c3) holds since every loop in the 3-manifold $C - S(C)$ bounds in the 3-manifold $N_{i-1} - S(N_{i-1})$.

To show (c4), we again make use of the fact that the contractibility of C in X implies that $H_2(C)$ is generated by 2-cycles in ∂C. The boundary of C is a connected, orientable 2-manifold so that $H_2(\partial C)$ is generated by $[\partial C]$. We will show that $[\partial C] = 0$ in $H_2(C)$ by exhibiting the 3-chain that it bounds. Let $\{Q_1, \cdots, Q_m\}$ be the components of N_{i+2} in C. We know that $C - \cup \{\mathring{Q}_j \mid 1 \leqq j \leqq m\}$ is a compact, orientable 3-manifold with boundary. Form a compact, orientable 3-manifold C' by sewing handlebodies onto $C - \cup \{\mathring{Q}_j \mid 1 \leqq j \leqq m\}$ along the boundary components $\{\partial Q_1, \cdots, \partial Q_m\}$. We can define a map $f: C' \to C$ by first letting f be the identity on $C - \cup\{\mathring{Q}_j \mid 1 \leqq j \leqq m\}$ and then extending f to the handlebodies, skeleton by skeleton, using (b).

Lastly, to show that $i_\#$ in (c5) is one-to-one, let α be a loop in its kernel. Since α is contained in $C - S(C)$, it lies in $C - N_{i+k}$ for some $k \geqq 1$. Since α bounds in C, it bounds in $\mathring{C}$ and is homologous in $C - \mathring{N}_{i+k}$ to a sum of loops in ∂N_{i+k}. But now α is zero homologous in $C - S(C)$ by applying (c1) to ∂N_{i+k}. To show that $i_\#$ is onto, let β be a loop in C. Since each component of N_{i+1} has connected boundary, each arc of $\beta \cap N_{i+1}$ can be replaced by an arc that has the same endpoints and that lies in ∂N_{i+1}. In this way we can obtain a loop β' in $C - \mathring{N}_{i+1}$ so that $\beta - \beta'$ consists of loops in N_{i+1}. But now (b) says that β' is homotopic to β in C.

LEMMA 2. *Let X be a compact generalized 3-manifold whose singular set $S(X)$ is 0-dimensional. Let (N_i), $1 \leqq i < \infty$, be a nested sequence of neighborhoods of $S(X)$ as given in the conclusion of Lemma 1. Let C be a component of some N_i, $i \geqq 2$. Then there is a compact, orientable 3-manifold M, a finite collection of pairwise disjoint handlebodies $\{H_1, \cdots, H_m\}$ in $\mathring{M}$ and a map $f: M \to C$ with the following properties:*

(a) *The restriction of f to $M - \cup\{\mathring{H}_j \mid 1 \leqq j \leqq m\}$ is a homeomorphism onto $C - \mathring{N}_{i+3}$;*

(b) *The induced map f_* on π_1 is surjective;*

(c) *The induced map $f_\#$ on H_1 is an isomorphism;*

(d) *The kernel P of f_* is a perfect group.*

Proof. The proof will be an elaboration of the proof of Lemma 1 (c4). In that proof, the components of $N_{i+2} \cap C$ were replaced by handlebodies to form a compact, orientable 3-manifold. We will do almost exactly that here, but we will be more specific about the attaching maps for the handlebodies, and we will replace components of N_{i+3} instead of N_{i+2}.

Let W be a component of $C \cap N_{i+1}$ and let $Y = W - \mathring{N}_{i+3}$. Since loops in Y bound in X, we can use [19; Theorem 3.1], Lefschetz duality and the Universal Coefficient Theorem to embed Y in a closed, orientable 3-manifold Z with $H_1(Z) = 0$ (see [3; proof of Theorem 2]). Repeated applications of [19; Theorem 2.2] allow us to assume that $Z - \mathring{Y}$ is a union of handlebodies. One component of ∂Y is ∂W. One complementary domain of ∂W in Z contains Y. Denote the closure of this domain by W^*.

We have $\partial W^* = \partial W$ and W^* is simply Y with handlebodies sewn to all components of ∂Y except ∂W in such a way as to give the following property: Every loop α in $\partial Y - \partial W$ bounds in W^*. For, we know that α bounds a singular surface F in Z. Since every loop in ∂N_{i+2} bounds in $N_{i+1} - N_{i+3}$, and since ∂N_{i+2} separates ∂N_{i+3} from ∂N_{i+1}, we can cut F off on ∂N_{i+2} and replace F by a singular surface lying entirely in W^* and bounded by α.

Since $\partial W^* = \partial W$, we can remove $\mathring{W}$ from C and sew in W^*, using the identity map: $\partial W^* \to \partial W$ as the attaching map. If $\{W_1, \cdots, W_n\}$ are the components of $C \cap N_{i+1}$, we can repeat this process for each W_k, $1 \leqq k \leqq n$. This creates a compact, orientable 3-manifold M which has been obtained from C by removing from C the interiors of $\{Q_1, \cdots, Q_m\}$, components of $C \cap N_{i+3}$, and replacing them by handlebodies $\{H_1, \cdots, H_m\}$ to satisfy a certain homology condition. Namely, if H_j is a replacement handlebody, Q_j is the component of $C \cap N_{i+3}$ that H_j replaces and W_k is the component of N_{i+1} containing Q_j, then every loop in H_j bounds in W_k^*.

We can define a map f: $M \to C$ by first letting f be the identity on

$$M - \cup \{\mathring{H}_j \mid 1 \leqq j \leqq m\} = C - \cup \{\mathring{Q}_j \mid 1 \leqq j \leqq m\} .$$

The map can be extended to the handlebodies H_j as in the proof of Lemma 1 (c4) so that each H_j is carried by f into a component of $N_{i+1} \cap C$. Since each Q_j contracts in C and has connected boundary, every loop in C is homotopic to a loop in $C - \cup \{Q_j \mid 1 \leqq j \leqq m\}$. This says that f_* and $f_\#$ are surjections.

To show that $f_\#$ is one-to-one, let α be a loop in M so that $f\alpha$ bounds in C. Since loops can be homotoped off handlebodies, we can assume α lies in $M - \cup \{H_j \mid 1 \leqq j \leqq m\}$. Since $f\alpha$ bounds in C, $f\alpha$ is homologous in $C - \cup \{\mathring{Q}_j \mid 1 \leqq j \leqq m\}$ to a sum of loops on $\cup \{\partial Q_j \mid 1 \leqq j \leqq m\}$. Thus α is homologous in M to a sum of loop on $\cup \{\partial H_j \mid 1 \leqq j \leqq m\}$. But this means that α bounds in M.

To show that $\ker f_*$ is perfect, assume that β is a loop in $M - \cup \{H_j \mid 1 \leqq j \leqq m\}$ so that $f\beta$ shrinks in C. This says that $f\beta$ is one boundary component of a singular disk with holes in $C - \cup \{\mathring{Q}_j \mid 1 \leqq j \leqq m\}$ whose other boundary components lie in $\cup \{\partial Q_j \mid 1 \leqq j \leqq m\}$. Thus β is one boundary component of a singular disk with holes in $M - \cup \{\mathring{H}_j \mid 1 \leqq j \leqq m)$ whose other boundary components lie in $\cup \{\partial H_j \mid 1 \leqq j \leqq m\}$. But each of these other boundary components bounds in the W_k^* it lies in. Thus β bounds a singular surface E in M so that generating curves for $\pi_1(E)$ are mapped into $\cup \{W_k^* \mid 1 \leqq k \leqq n\}$. But each W_k^* is mapped into W_k by f and each W_k contracts in C. Thus β is a product of commutators of elements in $\ker f_*$. This says that $\ker f_*$ is perfect.

THEOREM 1. *Let X be a compact, generalized 3-manifold whose* *singular set S is 0-dimensional. Let M be the open 3-manifold $X - S$. Then, the following statements are mutually equivalent:*

(A) *X is the cell-like image of a compact (necessarily closed) 3-manifold;*

(B) *M embeds in a compact 3-manifold;*

(C) *For some compact $Y \subset M$, $M - Y$ embeds in $\boldsymbol{R}^3$.*

Proof. We consider first the implication from (A) to (B). If N is a closed 3-manifold and $f: N \to X$ is a cell-like surjection, then f is a closed mapping and

$$f| : N - f^{-1}(S) \longrightarrow X - S = M$$

is a cell-like, closed surjection between open 3-manifolds. Since the restriction of f to the inverse image of each open subset of X induces a π_1-isomorphism, for each $x \in M$ the cell-like continuum $f^{-1}(x)$ has arbitrarily tight open neighborhoods U such that each of U and $U - f^{-1}(x)$ is connected and simply connected. By Theorem 1 of [10], for each $x \in M$, $f^{-1}(x)$ has an arbitrarily tight compact, contractible 3-manifold neighborhood H for which

$$H - f^{-1}(x) \approx S^2 \times [0, 1) .$$

By H. Kneser's Finiteness Theorem (pages 252–255 of [7]), there are only finitely many $x \in M$ for which $f^{-1}(x)$ fails to be cellular in N.

Now, as in Cor. 2.1 of [10], we can identify to a point each of these noncellular point-inverses to obtain a closed 3-manifold K differing from N by finitely many surgeries, each of which replaces a fake 3-cell in N by a genuine 3-cell. There is still a cell-like surjection $g: K \to X$, but now $g^{-1}(x)$ is *cellular* whenever $x \notin S$. By [1] $K - g^{-1}(S) \approx M$, so that M embeds in the compact 3-manifold K. (In fact, g can be approximated by a cell-like mapping h so that $h^{-1}(x)$ is a single point whenever $x \notin S$. See [1] and Theorem 1 of [5].)

Now, to show "(B) $\Rightarrow$ (C)", we suppose that M embeds in a compact (without loss of generality, closed) 3-manifold K. By Lemma 1 (c3), for some compact $Y_1 \subset M$, $M - Y_1$ is orientable and thus lifts homeomorphically to the orientable double covering of K. By [8], for some compact $Y_2 \subset M - Y_1$, $M - (Y_1 \cup Y_2)$ embeds in $\boldsymbol{R}^3$. Hence the required Y is $Y_1 \cup Y_2$.

For the claim "(C) $\Rightarrow$ (A)", we suppose that for some compact 3-manifold $Y \subset M$ there is an embedding

$$h: M - \mathring{Y} \longrightarrow S^3 ,$$

where h extends to an embedding of a bicollar neighborhood of ∂Y. Let N be the closed 3-manifold formed by glueing Y onto the image of h, via $h|_{\partial Y}$. Identifying to a point each compact component of $(S^3 - \text{Image } h)$ yields a surjection

$$f: N \longrightarrow X' \approx X .$$

As in part (c2) of Lemma 1, each point-inverse of f is acyclic over $\boldsymbol{Z}$ in the sense of [13]. Since X' is locally contractible, Theorem 3 of [13] implies that each point-inverse of f has property $1-UV$. This, combined with their acyclicity, proves that each is cell-like. Hence, f is cell-like.

3. Connections with the Poincaré Conjecture. If the Poincaré Conjecture is false, then there is a compact, contractible 3-manifold that is not homeomorphic to a 3-cell. Such an object is called a *fake 3-cell* and must have a 2-sphere as its boundary. If a fake 3-cell exists, then (as is well known) a counterexample to (GM) with one singular point can be constructed by taking an infinite, disjoint, null sequence of 3-cells in the 3-sphere that converge to one point, and replacing each by a fake 3-cell. In [3; Theorem 3] it is shown that the existence of fake 3-cell implies the existence of a counterexample to (GM) with one singular point and whose manifold set is *irreducible* (i.e., every 2-sphere bounds a 3-cell).

Assuming the existence of a fake 3-cell, Theorem 2 below builds a counterexample to (GM) with one singular point and whose manifold set is an acyclic, monotone union of handlebodies. Thus, simplifying assumptions on the manifold set of a 3-gm do not seem to avoid the Poincaré conjecture. We do not know whether an example can be constructed with the properties given in Theorem 2 and with a uniform bound on the genera of the handlebodies. The manifold set of the example in [3] is a monotone union of compact 3-manifolds each bounded by a torus.

THEOREM 2. *If fake 3-spheres exist, then for some acyclic,*[1] *monotone union M of handlebodies, the one-point compactification $\hat{M}$ is a generalized 3-manifold, yet $\hat{M}$ is not the cell-like image of any compact 3-manifold.*

Proof. Let Σ be a closed, simply connected 3-manifold not homeomorphic to S^3. Since Σ is closed and orientable, there are handlebodies K_1, K_2 with $K_1 \cup K_2 = \Sigma$ and

$$\partial K_1 = K_1 \cap K_2 = \partial K_2 .$$

[1] Using [**15**; Theorem 3], M can be constructed to be contractible.

(For example, take K_1 to be a regular neighborhood of the 1-skeleton of a triangulation of Σ, and let $K_2 = \Sigma - \mathring{K}_1$.) Since $\pi_1(\Sigma) = \{1\}$, Lemma 1 of [15] yields a homeomorphism h of Σ onto Σ that is isotopic to the identity, and for which the inclusion

$$h(K_1) \overset{\circ}{\subset} K_1$$

is homotopic to a constant.

Hence, the continuum

$$X = \bigcap_{n=1}^{\infty} h^n(K_1)$$

is cell-like and is defined as a nested intersection of handlebodies $h^n(K_1)$, where the superscript denotes iteration of h. Since some self homeomorphism of Σ takes K_1 into $\Sigma - K_1$ (and hence X into $\Sigma - X$), two copies of the open set

$$U = \Sigma - X$$

cover Σ. Further U is acyclic, by cell-likeness of X and duality. By Corollary 1 of [14], U fails to embed in S^3, even though it is the monotone union of handlebodies $h^n(K_2)$. (We remark also that the one-point compactification $\hat{U}$ of U is topologically Σ with the cell-like set X identified to a point. Hence, $\hat{U}$ is a generalized 3-manifold. M will be a kind of "connected sum" of infinitely many copies of U.)

We need some technicalities before defining M. Let

$$V = (\partial\Delta^2 \times \Delta^2) \cup (\Delta^2 \times \{q\}) ,$$

where $q \in \partial\Delta^2$. (V is a solid torus with 2-disk attached along a longitudinal simple closed curve.) Wedge an interval J onto V at a point of $\partial\Delta^2 \times (\partial\Delta^2 - \{q\})$, and denote by P the resulting polyhedron. Let g be a PL embedding of P into Σ so that $g^{-1}(X)$ consists of the other endpoint of J.

Let

$$p \colon T \longrightarrow \Sigma - g(\partial\Delta^2 \times \mathring{\Delta}^2)$$

be the universal covering of our homotopy solid torus. Put

$$Y = X \cup g[(\partial\Delta^2 \times \Delta^2) \cup J]$$

and define

$$M = p^{-1}(\Sigma - Y) .$$

Since X is a nested intersection of handlebodies, the inclusion

$$\Sigma - Y \longrightarrow \Sigma - g(\partial\Delta^2 \times \mathring{\Delta}^2)$$

induces a surjection on fundamental groups. Hence, M is connected. (M is the covering of $\Sigma - Y$ corresponding to the commutator subgroup of its fundamental group.)

Attach a 3-cell Q to the two-point compactification of T along its 2-sphere "boundary" to obtain a compact generalized 3-manifold T^* with the homotopy type of S^3. Then $\hat{M}$ is obtained from T^* by identifying to a point the cell-like set

$$Q \cup p^{-1}(X \cup g(J)) .$$

It follows that M is acyclic, and that $\hat{M}$ is a generalized 3-manifold.

We verify next that M is a monotone union of handlebodies. Since any covering space of an open handlebody is a monotone union of handlebodies, it suffices to show that $\Sigma - Y$ is a monotone union of handlebodies. First, note that cutting $\Sigma - Y$ along $g(\mathring{\Delta}^2 \times \{q\})$ yields

$$\Sigma - X - g(P) \approx \Sigma - X ,$$

which as a monotone union of handlebodies, is irreducible. Hence, $\Sigma - Y$ is irreducible. Now let Z be a given compact 3-polyhedron in $\Sigma - Y$, with each component of Z a compact 3-manifold. As in §2 of [11], apply *simple moves* to Z in $\Sigma - Y$ to obtain a *simple* Z_0. (That is, cut Z along properly embedded 2-disks, fill in spherical holes, add 2-handles, and run tubes to join up with 3-cell components, until no more of these "moves" can be done nontrivially. See [11] for details.)

We claim that Z_0 is a 3-cell. If not, then Lemma D of [11] implies that each component of ∂Z_0 is incompressible in $\Sigma - Y$ and none is a 2-sphere. Then, using irreducibility of $\Sigma - Y$ and standard "cut and paste", Z_0 can be isotoped to miss $g(\mathring{\Delta}^2 \times \{q\})$. Thus, the interior of some handlebody in $\Sigma - X - g(P)$ contains the isotoped Z_0. This is a contradiction, since no handlebody contains an incompressible closed surface. Hence, Z_0 is a 3-cell. Theorem 1 of [11] then gives a handlebody in $\Sigma - Y$ containing Z.

Now suppose that $\hat{M}$ were the cell-like image of a compact 3-manifold. Then by Theorem 1, there is a compact $C \subset M$ and an embedding of $M - C$ into $\boldsymbol{R}^3$. But only finitely many components of

$$p^{-1}(\Sigma - X - g(P))$$

can intersect the compact set C. Thus, most such components embed in $\boldsymbol{R}^3$. Since p embeds each such component, each is homeomorphic to $\Sigma - X$. This contradicts the fact that $\Sigma - X$ fails to embed in $\boldsymbol{R}^3$. The proof is complete.

ADDENDUM 1. *Any fake* 3-*sphere* Σ *contains a continuum* Y

(with the "shape" of a circle) such that the covering space M of $\Sigma - Y$ corresponding to the commutator subgroup of $\pi_1(M - Y)$, has the properties claimed in Theorem 2.

ADDENDUM 2. *If M is the open 3-manifold constructed in the proof of Theorem 2, then the fundamental group of each connected, open subset of $\hat{M}$ is torsion-free.*

Proof. It suffices to show that if $U \subset T^*$ is connected, open, and either contains the cell-like set

$$Q \cup p^{-1}(X \cup g(J)) ,$$

or is disjoint from it, then $\pi_1(U)$ is torsion-free. In the latter case, $U \cap Q = \varnothing$ and hence each compact subset of U embeds in the connected sum of finitely many copies of Σ, which is again a homotopy 3-sphere. It is well-known (Theorem 31.2 of [17]) that the fundamental group of no open, connected set in a homotopy 3-sphere can have torsion in its fundamental group. In the former case, some neighborhood R of Q in U is simply connected and has frontier a bicollared 2-sphere. Hence,

$$\pi_1(U) \cong \pi_1(U - \bar{R}) ,$$

where $U - \bar{R}$ is an open set disjoint from Q, so that the previous argument applies.

Theorem 3 presents a condition on a 3-gm X with 0-dimensional singular set sufficient for X to resolve. The condition asks that each compact subset of the manifold set of X should embed in a 3-manifold in a manner that reflects the local simple connectivity of X. (There are also some "irreducibility" requirements.) This is used in Theorem 4 which, assuming the Poincaré conjecture, establishes another condition sufficient for X to resolve. (Namely, there should be a "full" set of disjoint surfaces with nice properties, properly embedded in but not separating a tight neighborhood of the singular set.) Corollaries 1 and 2 show what positive results can be obtained from those portions of the Poincaré Conjecture that are already known.

THEOREM 3. *Let X be a compact, generalized 3-manifold whose singular set S is 0-dimensional. Let M be the open 3-manifold $X - S$. Suppose that for some expanding sequence*

$$K_0 \overset{\circ}{\subset} K_1 \overset{\circ}{\subset} K_2 \overset{\circ}{\subset} \cdots$$

of compact 3-manifolds exhausting M the following is true: For *each* $n \geqq 1$, *the inclusion* $X - K_n \to X - K_{n-1}$ *is homotopic to a constant; for each* $n \geqq 2$ *there is an embedding*

$$h_n : K_n \longrightarrow N_n ,$$

where N_n *is a closed 3-manifold that is separated by each component of* $h_n(\partial K_n)$, *where for* $n \geqq 3$ *each component of* $N_n - h_n(K_{n-2})$ *is irreducible (For* $n = 2$, *we assume only that each component of* $N_2 - h_2(K_1)$ *is irreducible), and the inclusion*

$$N_n - h_n(K_n) \longrightarrow N_n - h_n(K_{n-1})$$

componentwise induces the trivial π_1*-homomorphism. Then for each* $n \geqq 2$, *some homeomorphism* $\Phi_n : N_n \to N_{n+1}$ *extends*

$$\varphi_n = h_{n+1} \cdot h_n^{-1} | : h_n(K_{n-1}) \longrightarrow h_{n+1}(K_{n-1}) .$$

Further, M *embeds in* N_2, *and for some* n, $M - K_n$ *embeds in* $\boldsymbol{R}^3$.

Proof. We collect some facts for later reference. First, since each loop in $X - K_n$ contracts in $X - K_{n-1} (n \geqq 1)$, we conclude that each component of $M - K_1$ is orientable. In fact, for $n \geqq 2$, each component of $N_n - h_n(K_1)$ is orientable since by the above this is true of $h_n(K_n - K_1)$, since each component of $h_n(\partial K_n)$ separates N_n, and since each loop in $N_n - h_n(K_n)$ is contractible in $N_n - h_n(K_{n-1})$. Second, each component of $h_n(\partial K_{n-i})$ separates N_n, for $0 \leqq i \leqq n - 1$. For such a component F (considered as a 2-cycle over $\boldsymbol{Z}$) is homologous in $h_n(K_n - K_{n-i-1})$ to some 2-cycle in $h_n(\partial K_n)$, and hence is homologous to zero in $N_n - h_n(K_1)$.

We now show that for $n \geqq 2$ the inclusion

$$N_n - h_n(K_{n-1}) \longrightarrow N_n - h_n(K_{n-2})$$

componentwise induces zero on π_1. For, since each component of $h_n(\partial K_n)$ separates N_n, a given loop in $N_n - h_n(K_{n-1})$ is homotopic to a product of finitely many loops each conjugate to a loop in $h_n(K_n - K_{n-1})$ or to a loop in $N_n - h_n(K_n)$. Each loop of the latter type is contractible in $N_n - h_n(K_{n-1})$ by explicit hypothesis. Each loop f of the former type contracts in $N_n - h_n(K_{n-2})$ as follows: $f' = h_n^{-1}(f)$ is a loop in $K_n - K_{n-1}$ and thus f' contracts in $X - K_{n-2}$; use this to homotope f' in $K_n - K_{n-2}$ to a product of conjugates of loops in ∂K_n; apply h_n to this last statement to homotope f in $h_n(K_n - K_{n-2})$ to a product of conjugates of loops in $h_n(\partial K_n)$ (which then contract in $N_n - h_n(K_{n-1})$).

Now, we show how to extend φ_n to a homeomorphism $\Phi_n : N_n \to N_{n+1} (n \geqq 2)$. Let C be a component of $N_n - h_n(\mathring{K}_{n-1})$. Then $\partial C = C \cap h_n(K_{n-1})$ is connected and C is irreducible. It suffices to extend

$\varphi_n|_{\partial C}$ to a homeomorphism of C onto the (irreducible) component C^* of $N_{n+1} - h_{n+1}(\mathring{K}_{n-1})$ with $\partial C^* = \varphi_n(\partial C)$. By Lemma 2 of [12], it is enough to show that $\varphi_n|_{\partial C}$ extends to a mapping $C \to C^*$ that induces a π_1-monomorphism. Thus, we fix basepoints in ∂C and ∂C^* and decide how to map a given loop f in C to one in C^*. (The map is thus defined between the 1-skeleta. The extension to the 2-skeleta follows from the fact that the definition is invariant for the homotopy class of f. Asphericity completes the job.)

Now, as in our "first fact" above, f is homotopic in C to a loop f' in $h_n(K_n - \mathring{K}_{n-1})$. We define

$$\Phi'_n|_C(f) = h_{n+1}h_n^{-1}(f') .$$

This definition is independent of choice of f', and is well-defined on the homotopy class of f, by our proof's second paragraph. To show that $\Phi'_n|_C$ induces a π_1-monomorphism, consider a loop f' in $h_n(K_n - \mathring{K}_{n-1})$ for which $h_{n+1} \circ h_n^{-1}(f') = f''$ is a contractible loop in C^*. Then f'' is homotopic in $h_{n+1}(K_n - K_{n-1})$ to a product of conjugates of loops in $h_{n+1}(\partial K_n)$. Applying $h_n \circ h_{n+1}^{-1}$ to this last statement shows that f' is homotopic in $h_n(K_n - K_{n-1})$ to a product of conjugates of loops in $h_n(\partial K_n)$. Since each loop in $h_n(\partial K_n)$ contracts in $N_n - h_n(K_{n-1})$, f' contracts in C as desired. Hence, the required homeomorphism Φ_n exists for $n \geqq 2$.

Finally, we define an embedding $F: M \to N_2$ by

$$F|_{K_n} = \Phi_2^{-1} \cdots \circ \Phi_n^{-1} \circ \Phi_{n+1}^{-1} \circ h_{n+2}|_{K_n} \quad (n \geqq 2) .$$

Our last conclusion is immediate from Theorem 1.

THEOREM 4. *Let X be a compact, generalized 3-manifold whose singular set S is 0-dimensional. Let M be the open 3-manifold $X - S$. Suppose that for some expanding sequence*

$$K_0 \mathring{\subset} K_1 \mathring{\subset} K_2 \mathring{\subset} \cdots$$

of compact 3-manifolds exhausting M the following is true: For each $n \geqq 1$, the inclusion $X - K_n \hookrightarrow X - K_{n-1}$ is homotopic to a constant; for each $n \geqq 1$ and each component C of $X - \mathring{K}_n$, ∂C is connected and there exist disjoint orientable, compact surfaces

$$S_1, \cdots, S_{g(C)}\left(g(C) = genus\, \partial C = 1 - \frac{1}{2}\cdot\chi(\partial C)\right)$$

in $C - S$ such that $C - \cup S_i$ and each $\partial S_j = S_j \cap \partial C$ are connected. Then if the Poincaré Conjecture is true, M embeds in a compact 3-manifold and hence X is the cell-like image of a closed 3-manifold.

Proof. (Unless specified otherwise, the index i is always understood to range over the values $1, 2, \cdots, g(C)$. The correct choice of C should be clear from the context.) As in the proof of Theorem 3, each component of $M - K_1$ is orientable. By taking a subsequence of the K_n's, if necessary, and reindexing, we assume that (for $n \geqq 0$) the union of all the S_i's associated with the components of $X - \mathring{K}_n$ is contained in $\mathring{K}_{n+1}$.

For each $n \geqq 2$ and each component C of $X - \mathring{K}_n$, let $T_n(C)$ be an associated handlebody disjoint from X with $\partial T_n(C) \approx \partial C$. (If $C' \neq C$, we choose $T_n(C')$ disjoint from $T_n(C)$.) For fixed $n \geqq 2$ and component C of $X - \mathring{K}_n$, attach $T_n(C)$ to K along its corresponding ∂C via a homeomorphism that induces a bijection between the union of the boundaries of a complete set of cutting disks for $T_n(C)$ and the components of $\cup \partial S_i$. (Note that $\partial C - \cup \partial S_i$ is connected.) Denote the resulting closed 3-manifold by N_n and let h_n be the inclusion of K_n into N_n.

We must verify the hypotheses of Theorem 3. Towards this goal, we construct some mappings with degree one. Let C be a component of $X - \mathring{K}_n (n \geqq 2)$. Then, some mapping

$$f_{n,C} \colon C \longrightarrow T_n(C)$$

extends the identity $\partial C \to \partial T_n(C)$, and hence induces a π_1-surjection. To define $f_{n,C}$, we first send each S_i onto the cutting disk of $T_n(C)$ with the same boundary by squashing an appropriate finite graph in $\mathring{S}_i$ to a point. Next, we extend the map to take a bicollar neighborhood of each S_i to a bicollar neighborhood of the corresponding cutting disk in a level-preserving way. Finally, we extend to all of C by mapping C minus its bicollar neighborhoods into $T_n(C)$ minus its bicollar neighborhoods (a 3-cell) by the Tietze Extension Theorem.

Now let $T_n(C_0)$ be a component of $N_n - h_n(\mathring{K}_n)$, and let B be the component of $N_n - h_n(\mathring{K}_{n-1})$ containing $T_n(C_0)$. Denote by B' the component of $X - \mathring{K}_{n-1}$ containing C_0. Then, by piecing together all the $f_{n,C}$ for which $C \subset B'$, we obtain $\varphi \colon B' \to B$ that extends the identity on $B' \cap K_n$, and induces a consistent diagram

$$\begin{tikzcd} C_0 \arrow[r, hook] \arrow[d, "\varphi|"] & B' \arrow[d, "\varphi"] \\ T_n(C_0) \arrow[r, hook] & B . \end{tikzcd}$$

The top inclusion is homotopic to a constant and the vertical arrows induce π_1-surjections. Hence, $T_n(C_0) \hookrightarrow B$ induces zero on π_1, as required.

It remains only to show that each component of $N_2 - h_2(K_1)$ is irreducible, and that for $n \geqq 3$ each component of $N_n - h_n(K_{n-2})$ is irreducible. First, using the result of the previous paragraph and the hypothesis that $X - K_j \hookrightarrow X - K_{j-1}$ is homotopic to a constant for each $j \geqq 1$, one easily verifies that for fixed $n \geqq 1$, each inclusion

$$N_n - h_n(\mathring{K}_{n-k}) \hookrightarrow N_n - h_n(\mathring{K}_{n-k-1}) , \quad k = 0, 1, \cdots, n-1 ,$$

componentwise induces zero on π_1. (Use induction on k.)

In the following, let either $(n, k) = (2, 1)$, or $k = 2$ and $n \geqq 3$. Suppose that a 2-sphere F in $N_n - h_n(\mathring{K}_{n-k})$ is given. Then by the above, F separates the component of $N_n - h_n(\mathring{K}_{n-k})$ containing it. Since each such component has connected boundary, F bounds a compact 3-submanifold L of $N_n - h_n(\mathring{K}_{n-k})$. Since ∂L is simply connected,

$$L \hookrightarrow N_n - h_n(\mathring{K}_{n-k-1})$$

induces a π_1-monomorphism. Hence L is simply connected. Since ∂L is connected, L is (modulo Poincaré Conjecture) a 3-cell. Thus, each component of $N_n - h_n(\mathring{K}_{n-k})$ is irreducible. The proof is completed by applying Theorem 3.

COROLLARY 1. *Assume all the hypotheses of Theorem* 4. *In addition, suppose that each K_n is a handlebody of genus k or less.* (*Hence, S is a single point.*) *Assume that each homotopy* 3*-sphere of genus k or less is a genuine* 3*-sphere. Then, for some embedding $F: M \to S^3$, X is homeomorphic to the quotient space of S^3 obtained by crushing to a point the cell-like set $S^3 - F(M)$.*

Proof. We note, in particular, that a byproduct of the recently proven Smith Conjecture (no reference available) is that our corollary applies for $k = 2$.

We simply work through the proof of Theorem 4, observing that (even without the full strength of the Poincaré Conjecture) $N_n \approx S^3$ for each $n \geqq 2$. For, the union of K_n and the surfaces associated with $X - \mathring{K}_n$ embeds in S^3. Thus we can construct (in the manner used to obtain the $f_{n,C}$'s in Theorem 4) a mapping $f: S^3 \to N_n$ with $f|_{f^{-1}(K_n)}$ a homeomorphism. Such a mapping clearly induces a π_1-surjection. Hence, N_n is a homotopy 3-sphere of Heegaard genus $\leqq k$. Thus, $N_n \approx S^3$ and the manifolds considered in the last paragraph of the proof of Theorem 4 are irreducible, as required.

The proofs of Theorems 3 and 4 show that M embeds in $N_2 \approx S^3$. The proof of "(C) $\Rightarrow$ (A)" in Theorem 1 (with $Y = \varnothing$) shows that X is the cell-like image of S^3 in the manner claimed.

COROLLARY 2. *Let X be a compact, generalized 3-manifold whose singular set S is a single point. Let M be the open 3-manifold $X - S$. Suppose that for some expanding sequence*

$$K_0 \overset{\circ}{\subset} K_1 \overset{\circ}{\subset} K_2 \overset{\circ}{\subset} \cdots$$

of compact 3-manifolds exhausting M, each ∂K_n is a torus and K_n embeds in S^3. Then for some embedding $F: M \to S^3$, X is homeomorphic to the quotient space of S^3 obtained by crushing to a point the cell-like set $S^3 - F(M)$.

Proof. Our hypotheses guarantee that each K_n is topologically either a *solid torus* ($\approx S^1 \times \Delta^2$) or a *knot space* ($S^3$ minus the interior of a solid torus with knotted core). We assume without loss of generality that each inclusion $X - K_n \to X - K_{n-1} (n \geqq 1)$ is homotopic to a constant. As in Lemma 1 (c1) we also arrange that each inclusion

$$i_n: \partial K_n \longrightarrow K_{n+1} - \mathring{K}_{n-1} \quad (n \geqq 1)$$

induces zero on H_1 (first homology).

Using the exact homology sequence of the pair $(X - \mathring{K}_{n-1}, X - \mathring{K}_n)$ and excision, we have:

$$H_1(X - \mathring{K}_{n-1}) \cong H_1(K_n - \mathring{K}_{n-1}, \partial K_n) .$$

Let e_n embed K_n in S^3. Since i_n induces zero on H_1 and since

$$e_{n+1}(\partial K_n) \hookrightarrow S^3 - e_{n+1}(\mathring{K}_n)$$

induces a surjection on H_1, it follows that

$$S^3 - e_{n+1}(\mathring{K}_n) \hookrightarrow S^3 - e_{n+1}(\mathring{K}_{n-1})$$

induces zero on H_1. Hence, for $n \geqq 1$,

$$H_1(X - \mathring{K}_{n-1}) \cong H_1(S^3 - e_{n+1}(\mathring{K}_{n-1}), S^3 - e_{n+1}(\mathring{K}_n)) \cong H_1(S^3 - e_{n+1}(\mathring{K}_{n-1})) ,$$

and this last group is $\boldsymbol{Z}$. Since $H_1(\partial K_n) \cong \boldsymbol{Z} \times \boldsymbol{Z}$ and since $\partial K_n \hookrightarrow X - \mathring{K}_n$ induces a surjection on H_1 for $n \geqq 1$, there is, for each $n \geqq 1$, a simple closed curve in ∂K_n which, when considered as an integral 1-cycle, is homologous to zero in $X - \mathring{K}_n$ (and so, by Lemma 1, in $M - \mathring{K}_n$), but not in ∂K_n.

By the previous paragraph, there is for each $n \geqq 1$ an orientable compact surface F_n properly embedded in $M - \mathring{K}_n$ such that $M - \mathring{K}_n - F_n$ and ∂F_n are connected. Returning to the proof of Theorem 4, we need only show that each $N_n \approx S^3$. Let g_n embed $K_n \cup F_n$ in S^3. Then, as in Corollary 1, it follows that N_n is a homotopy 3-sphere. If K_n is a solid torus, then N_n has Heegaard genus $\leqq 1$, and thus is S^3. If K_n is a knot space, then $S^3 - g_n(\mathring{K}_n)$ must be a solid torus. Further, $g_n(\partial F_n)$ is then homologous (and hence homotopic) to zero in $S^3 - g_n(\mathring{K}_n)$. Hence, the union of K_n and a 2-disk attached along ∂F_n embeds in S^3. It thus follows that $N_n \approx S^3$ in this case also. The proof is now completed as in Corollary 1.

4. The torsion-free case: resolution modulo Poincaré Conjecture. Theorem 4 of §3 is used to establish Theorem 5.2. The latter assumes the Poincaré conjecture to show that a 3-gm X with 0-dimensional singular set S resolves if S does not have arbitrarily small neighborhoods whose fundamental groups have torsion. This condition is shown to be necessary by Theorem 5.1. We note that [3; Theorem 1] shows that this "no torsion" assumption is not needed if S has arbitrarily small neighborhoods bounded by tori.

LEMMA 3. *Let X be a compact generalized 3-manifold whose singular set $S(X)$ is 0-dimensional. Let N_i, $1 \leqq i < \infty$, be a nested sequence of neighborhoods of $S(X)$ as given in the conclusion of Lemma 1. Let C be a component of some N_i, $i \geqq 2$. Assume that every closed neighborhood N' of $S(C)$ in C with 2-manifold boundary has no torsion in $\pi_1(N')$. Then there exists a closed neighborhood E of $S(C)$, not necessarily connected, with $C \cap N_{i+1} \subset \mathring{E} \subset \mathring{C}$ and so that for each component E' of E, $\partial E'$ is a connected 2-manifold, and there exist pairwise disjoint, compact, orientable surfaces $\{F_1, \cdots, F_{g(E')}\}$, $g(E') =$ genus $(\partial E')$, embedded in $E' - S(E')$ such that $E' - \cup \{F_j \mid 1 \leqq j \leqq g(E')\}$ and each $\partial F_j = F_j \cap \partial E'$ are connected.*

Proof. In order to motivate the proof and to establish an outline, we will state and prove a simpler theorem that deals entirely with 3-manifolds. Portions of the proof will be referred to in the proof of Lemma 3. (A different proof of this result is given by the second author in [9; Lemma 1].)

LEMMA 3′ (*McMillan*). *Let M be a compact, orientable, irreducible 3-manifold with nonempty, connected boundary. Assume that every closed, orientable 2-manifold embedded in $\mathring{M}$ separates M. Let N be a compact, connected 3-manifold embedded in $\mathring{M}$. Assume that $i_*\colon \pi_1(\partial N) \to \pi_1(M)$ is trivial. Then N is contained in a handlebody in $\mathring{M}$.*

Proof of Lemma 3′. Since every loop in ∂N shrinks in M, ∂N can be completely compressed in $\mathring{M}$ until what is left of the boundary is a union of 2-spheres. The sequence of compressions that accomplishes this can be thought of as a sequence of modifications to N. Let $N(j)$ denote the result of the first j of these modifications. If a compression of $\partial N(j)$ is to take place along a disk contained in $N(j)$, we may say that the compression *removes a 1-handle* from $N(j)$. If a compression of $\partial N(j)$ is to take place along a disk contained in the closure of the complement of $N(j)$ in M, we may say that the compression *adds a 2-handle* to $N(j)$. Removing a 1-handle from $N(j)$ is accomplished by removing from $N(j)$ the "half open" 3-cell $C_{j+1} = [B^2 \times (0, 1)]$ which is embedded in $N(j)$ so that

$$C_{j+1} \cap \partial N(j) = [\partial B^2 \times (0, 1)] .$$

Adding a 2-handle to $N(j)$ is accomplished by adding to $N(j)$ the 3-cell $C_{j+1} = \{B^2 \times [0, 1]\}$ which is embedded in M so that

$$C_{j+1} \cap N(j) = \{\partial B^2 \times [0, 1]\} .$$

In either case we will refer to C_{j+1} as the 1-handle or 2-handle.

It is possible that a 2-handle may pass through space in M occupied by a previously removed 1-handle. In this case we require that the boundary of the 2-handle be in general position with respect to the boundary of the 1-handle. We make no special requirements when a 1-handle consists partly of portions of an $N(j)$ that are contained in some previously added 2-handle. We always require that the annulus $\partial B^2 \times (0, 1)$ removed from $\partial N(j)$ by the $(j + 1)st$ compression be disjoint from all previous 1-handles or 2-handles involved in the first j compressions.

In order to reconstruct N from the fully compressed state, it is necessary to remove the 2-handles and add the 1-handles. If instead the 2-handles are left in place and the 1-handles are restored (even though they intersect the 2-handles), a submanifold N' of M is obtained that contains N. Since N is connected, N' is connected.

If a complementary domain of N' in M had two differect components of $\partial N'$ in its closure, then neither of those boundary components would separate M. Since everything in sight is orientable, this would violate a hypothesis. Thus the complementary domain of N' in M containing ∂M contains only one component F of $\partial N'$ in its closure. Let N^* be the closure of the complementary domain of F in M containing N'. We will show that N^* is a handlebody.

Our aim will be to find a collection of pairwise disjoint simple closed curves $\{J_1, \cdots, J_n\}$ on ∂N^* so that each J_k shrinks in N^* and so that each component of $\partial N^* - \cup \{J_k \mid 1 \leqq k \leqq n\}$ is planar. It is then easy to show, using the irreducibility of M and the presence of ∂M, that N^* is a handlebody.

The boundary of N^* is made of pieces of three possible origins:

Portions of that part of ∂N untouched during the entire compression process; portions contained in annuli of the form $\partial B^2 \times (0, 1)$ that were removed and later restored as part of the boundaries of 1-handles; portions contained in disks of the form $B^2 \times \partial[0, 1]$ that were added when 2-handles were added. Our collection of simple closed curves on ∂N^* will draw from two sources. First, our collection will include all boundary components of annuli of the form $\partial B^2 \times [0, 1]$ along which 2-handles hit the appropriate $\partial N(j)$, and all boundary components of closures of annuli of the form $\partial B^2 \times (0, 1)$ along which 1-handles hit the appropriate $\partial N(j)$. Secondly, our collection will contain all simple closed curves in ∂N^* arising from intersections of those parts in the boundaries of 2-handles of the form $B^2 \times \partial[0, 1]$ with those parts of the boundaries of 1-handles of the form $\partial B^2 \times (0, 1)$. By our general position assumption, these intersections are unions of simple closed curves.

We now have our collection $\{J_1, \cdots, J_n\}$. Each J_k shrinks in N^* since it is contained in a 3-cell in N^*. There are enough curves in $\{J_1, \cdots, J_n\}$ so that no component of $\partial N^* - \cup\{J_k \mid 1 \leqq k \leqq n\}$ intersects two or more of the sources of ∂N^* as described in the last paragraph. Since ∂N was completely compressed to spheres by the addition and subtraction of handles, the components of $\partial N^* - \cup \{J_k \mid 1 \leqq k \leqq n\}$ coming from the uncompressed part of ∂N are all planar. The portions from the boundaries of 1-handles and 2-handles are subsets of annuli and disks and therefore are planar. This completes the proof.

Proof of Lemma 3 (continued). All of the action of this proof will take place inside C. For this reason and to simplify notation, we will limit our scope and renumber the neighborhoods of $S(X)$. We will regard C as the only component of some neighborhood called N_0 and what used to be $C \cap N_{i+1}, C \cap N_{i+2}, \cdots$ will now be called $N_1, N_2, \cdots$. This notation which will be maintained throughout the rest of the proof, has the secondary advantage that it frees the letter i for general use.

In order to imitate the proof of Lemma 3′, we must define what we mean by compression. Ordinarily, compression removes an annulus from a surface and adds two disks. Our compression will remove an annulus and will add two surfaces with connected boundary but of unknown genus. Thus our handles, these surfaces cross an interval, will have bumps. We will call them *jagged* 1-handles and *jagged* 2-handles. Conditions will be placed on them to make them look algebraically like real 1 and 2-handles. We will also place strict requirements on how they may intersect. Because of this we will have to give partial definitions of first jagged 1-handles and then jagged 2-handles before we can finish listing all their properties.

Lastly, we will require that new jagged handles stay away from the high genus surfaces introduced by old jagged handles. This is the source of the surfaces G_i in the following description.

Let N be a closed neighborhood of $S(C)$ in C containing N_2 so that each component of ∂N is a 2-manifold. We do not require that N be connected. Let $\{G_1, \cdots, G_n\}$ be a pairwise disjoint collection of compact surfaces in ∂N so that each ∂G_i is connected. Assume that

$$F = \partial N - \cup \{\mathring{G}_i \mid 1 \leqq i \leqq m\}$$

has nonzero genus. Let J be a simple closed curve in F so that $[J]$ is not in the normal closure of the elements $[\partial G_i]$ in $\pi_1(F)$. Assume that J bounds a surface S in N. We will put three requirements on S, one to be described now, and two others to be described after the definition of a jagged 2-handle. We require of S:

(i) S is compact, orientable and properly embedded in $N - N_2$ with $\partial S = J$; (ii) and (iii) later. A *jagged* 1-*handle* in N with *core* S is a closed regular neighborhood U of S in N of the form $S \times [-1, 1]$ where

$$S = S \times \{0\}\ , \quad \text{and} \quad U \cap \partial N = J \times [-1, 1]$$

is an annular regular neighborhood of J in ∂N. We can compress N along S by removing $S \times (-1, 1)$ from N.

Note that this removes the open annulus $J \times (-1, 1)$ from ∂N and adds the surfaces $S \times \{\pm 1\}$. We can remove several jagged 1-handles from N in sequence if, after each compression, we add to the collection $\{G_i\}$ the surfaces $S_j \times \{\pm 1\}$ where each S_j is the core of a jagged 1-handle.

Now let N be obtained from N_1 by removing a sequence of jagged 1-handles. Let $\{G_i \mid 1 \leqq i \leqq 2m\}$ be exactly the collection of surfaces $\{S_j \times \{\pm 1\} \mid 1 \leqq j \leqq m\}$, where the S_j are cores of the jagged 1-handles. Let F and J be defined as above. Assume that J bounds a surface S with the following properties:

(iv) S is compact, orientable and properly embedded in $C - \mathring{N}$ with $\partial S = J$;

(v) For all j, $1 \leqq j \leqq m$, each component of $S \cap (S_j \times [-1, 1])$ is of the form $S_j \times \{t\}$, $t \in (-1, 1)$; and

(vi) $S - \cup \{\mathring{S}_j \times [-1, 1] \mid 1 \leqq j \leqq m\}$ is a disk with holes.

Then a *jagged* 2-*handle with core* S attached to N along J is a closed regular neighborhood U of S in $C - \mathring{N}$ of the form $S \times [-1, 1]$ where $S = S \times \{0\}$ and $U \cap \partial N = J \times [-1, 1]$ is an annular regular neighborhood of J in ∂N and so that each component of $U \cap (S_j \times [-1, 1])$ is of the form $S_j \times [a, b]$, $-1 < a < b < 1$ for all j, $1 \leqq j \leqq m$. We can compress N along S by adding $S \times [-1, 1]$ to N.

Note that this removes the open annulus $J \times (-1, 1)$ from ∂N and adds the surfaces $S \times \{\pm 1\}$. From now on we will look at neighborhoods of $S(C)$ obtained from N_1 by a sequence of compressions that consist of removing jagged 1-handles and adding jagged 2-handles. If $N(k)$ represents the result of applying to N_1 a sequence of k compressions, then the collection of surfaces $\{G_i \mid 1 \leqq i \leqq 2k\}$ will consist of the introduced surfaces $\{S_j \times \{\pm 1\} \mid 1 \leqq j \leqq k\}$, where the S_j are the cores of the jagged handles used in the compressions. The surfaces $F(k)$ will be defined as

$$F(k) = \partial N(k) - \cup \{\mathring{G}_i \mid 1 \leqq i \leqq 2k\} .$$

We can now state the two remaining requirements of cores of jagged 1-handles. Let $N(k)$, $\{G_i\}$ and $F(k)$ be as described in the last paragraph. Let J, S and $U = S \times [-1, 1]$ be as in the definition of jagged 1-handle. Let

$$\{S_{j_i} \mid 1 \leqq j_1 < \cdots < j_r \leqq k\}$$

be the cores of all the jagged 2-handles added to get $N(k)$. We ask that S satisfy the following additional requirements:

(ii) Each component of $S \cap (S_{j_i} \times [-1, 1])$, $1 \leqq i \leqq r$, is of the form $S_{j_i} \times \{t\}$, $t \in (-1, 1)$; and

(iii) every loop in $S - \cup \{S_{j_i} \times [-1, 1] \mid 1 \leqq i \leqq r\}$ must shrink in N_1 mod the loops ∂S_{j_i}, $1 \leqq i \leqq r$.

We can now define a complexity of each $N(k)$, denoted $KN(k)$, to be the sum of the squares of the genera of the components of $F(k)$. That $KN(k) = 0$ if and only if $F(k)$ is a union of planar surfaces, that $KN(k) \geqq 0$ for all k, and that $KN(k) < KN(k-1)$ for all $k > 1$ are all standard observations.

At this point it is not clear that even one compression can be performed on N_1. Most of the effort of this proof will go into showing that in fact a sequence of compressions can be performed on N_1 at the end of which the complexity will be zero. Toward that end we will prove the following:

CLAIM. Let $N(k)$ be obtained from N_1 by a sequence of k compressions, $k \geqq 0$. Assume that $KN(k)$ is greater than zero. Then there exists a simple closed curve J in $F(k)$ with $[J]$ not in the normal closure in $\pi_1 F(k)$ of $[\partial S_j \times \{\pm 1\}]$, $1 \leqq j \leqq k$, if $k > 0$, and there exists a surface S_{k+1} bounded by J along which $N(k)$ can be compressed.

Proof of Claim. We outline the proof.

(I) Find a loop L in F that satisfies the algebraic requirements mentioned above for J and that shrinks in a complementary domain of $\partial N(k)$ in C mod the surfaces $\{S_j \times \{\pm 1\} \mid 1 \leqq j \leqq k\}$.

(II) Modify the complementary domain of $\partial N(k)$ in C mentioned in (I) and show that L shrinks in this new domain without the aid of the surfaces $\{S_j \times \{\pm 1\} \mid 1 \leqq j \leqq k\}$.

(III) Use Dehn's lemma and the loop theorem, or use Lemma 2 and Theorems 1 and 3 of [4] to obtain the J and S_{k+1} of the claim.

We are actually going to introduce two modifications of complements of $\partial N(k)$ in C. One of them will be an exact counterpart of the space N' used in the proof of Lemma 3′. However in that proof N' did not appear until all compressions were done. In the current proof N' will have a counterpart for each k and these will be used in several places in the proof. They will be defined in the next paragraph. The other modification to a complementary domain of $\partial N(k)$ in C will have a more temporary use and will be defined later, when needed.

Let

$$N'(k) = N_1 \cup [\cup \{S_{j_i} \times [-1, 1] \mid 1 \leqq i \leqq r\}]$$

where the S_{j_i} are the cores of the jagged 2-handles as mentioned above. In words, $N'(k)$ is N_1 to which the jagged 2-handles have been added without removing the jagged 1-handles. Observe that from property (vi) of jagged 2-handles, $N'(k)$ may be obtained from N_1 by adding spaces of the form $E_{j_i} \times [-1, 1]$, $1 \leqq i \leqq r$, where each E_{j_i} is a disk with holes and, for each i,

$$(E_{j_i} \times [-1, 1]) \cap N_1 = (\partial E_{j_i} \times [-1, 1])\ .$$

A useful property of the space $N'(k)$ is that every loop in each of the surfaces S_j, $1 \leqq j \leqq k$, shrinks in $N'(k)$. For, since N_1 is contained in $N'(k)$, property (iii) of the definition of a jagged 1-handle tells us that every loop in that part of the core S_j of jagged 1-handle that is not in a jagged 2-handle of lower index must shrink in $N'(k)$ mod the boundaries of cores of jagged 2-handles of index lower than j. Property (vi) of the definition of jagged 2-handle gives a similar statement with the roles of jagged 1-handles and jagged 2-handles reversed. Property (ii) assures us that those parts of cores of jagged 1-handles in jagged 2-handles of lower index are simply parallel copies of cores of jagged 2-handles of lower index. Property (v) repeats the same with role reversal. The statement at the beginning of this paragraph now follows by induction.

We now attack step I of the outline. Since each component of N_1 contracts in C, every loop in ∂N_1 shrinks in C. Since every loop in an S_j shrinks in $N'(k)$, it follows that every loop in $\partial N(k)$ shrinks in C.

Let

$$\{G_i \mid 1 \leqq i \leqq 2k\}$$

be the collection of surfaces

$$\{S_j \times \{\pm 1\} \mid 1 \leqq j \leqq k\} .$$

Let H be the monotone decomposition of C whose only nondegenerate elements are the surfaces $\{G_i \mid 1 \leqq i \leqq 2k\}$. Let q be the projection map $q: C \to C/H$. Since each surface G_i has only one boundary component, $q\partial N(k)$ is a union of surfaces. By Lemma 1 (c4), each component of $\partial N(k)$ separates C. Thus each component of $q\partial N(k)$ separates C/H. Since every loop in $\partial N(k)$ shrinks in C, every loop in $q\partial N(k)$ shrinks in C/H.

If it were true that the inclusion of each component of $q\partial N(k)$ into the closures of the various components of $C/H - q\partial N(k)$ induced monomorphisms on the fundamental groups, then $\pi_1(C/H)$ could be obtained by a sequence of free products with amalgamations where the groups involved in the products as factors would be the fundamental groups of the components of $C/H - q\partial N(k)$ and the amalgamating subgroups would be the fundamental groups of the components of $q\partial N(k)$. This would imply that the inclusion of each component of $q\partial N(k)$ into C/H would induce a monomorphism on the fundamental groups. This contradicts the fact that every loop in $q\partial N(k)$ shrinks in C/H.

Since each component of $q\partial N(k)$ is a surface, and since there are only a finite number of points of the form qG_i, any loop in $q\partial N(k)$ can be homotoped in $q\partial N(k)$ so that it misses all points of the form qG_i. We can thus find a loop on a component of $q\partial N(k)$ that misses the qG_i, that shrinks in the closure of a component of C/H-$q\partial N(k)$, but does not shrink on $q\partial N(k)$. Since q is one-to-one off the surfaces G_i, we can call this loop qL. This reflects the fact that there is a loop L in $F(k)$ which, when composed with q, gives the loop that we have called qL.

Since each of the surfaces G_i has a product regular neighborhood in $C - S(C)$, and, again, since q has an inverse off the surfaces G_i, we know that L shrinks in the closure of a component of $C - \partial N(k)$ mod some of the surfaces G_i. This is established by cutting off the disk bounded by qL on some level of the product neighborhoods of the G_i as carried over by q. We also know that $[L]$ is not in the normal closure of the classes $[\partial G_i]$ in π_1 of the component of $F(k)$ containing L. This follows from the fact that qL does not shrink on $q\partial N(k)$. This completes step I.

There are now two possibilities, either L shrinks in a component of $N(k)$ mod some of the surfaces G_i, or L shrinks in a component of $C - \mathring{N}(k)$ mod the G_i. We will describe in detail the arguments for steps II and III only for the first possibility. The second possibility is very much simpler, and at the end of the proof of the claim we will indicate how the argument that we give should be modified to handle this case.

Assume that L shrinks in $N(k)$ mod the surfaces G_i. We need only work in the component of $N(k)$ that contains L, so we will assume that $N(k)$ is connected for the rest of the proof of the claim. Since each $F(j)$, $1 \leqq j \leqq k$, is a subset of ∂N_1, L lies in ∂N_1. It need not be true that L shrinks in $N(k)$ or N_1. It is the case that L shrinks in $N'(k)$. We will also describe a new space created from N_1 in which L shrinks. This space is unfortunately not a subset of C in general, but it does have the advantage of being geometrically simpler than $N'(k)$.

Each curve ∂S_j, $1 \leqq j \leqq k$, lies on ∂N_1. As before let

$$\{S_{j_1}, \cdots, S_{j_r}\}, 1 \leqq j_1 < \cdots < j_r \leqq k ,$$

be the cores of all the jagged 2-handles. Let $N(k, 2)$ be the space obtained from N_1 by adding 2-handles of the form $(B^2 \times [-1, 1])_{j_i}$, $1 \leqq i \leqq r$, to N_1 where each

$$(B^2 \times [-1, 1])_{j_i} \cap N_1 = (\partial B^2 \times [-1, 1])_{j_i}$$

is the annular region $(\partial S_{j_i} \times [-1, 1])$ on ∂N_1 along which $N(j_i - 1)$ and the jagged 2-handle $(S_{j_i} \times [-1, 1])$ intersect. Contained in $N(k, 2)$ is a family of surfaces that correspond naturally to the cores of jagged 1-handles. Let

$$\{S_{q_1}, \cdots, S_{q_s}\} , \quad 1 \leqq q_1 < \cdots < q_s \leqq k ,$$

be the cores of the jagged 1-handles. Let $(S_{j_i} \times [-1, 1])$ be a jagged 2-handle that one of the surfaces S_{q_u} intersects. Corresponding to each component $(S_{j_i} \times \{t\})$ of $(S_{j_i} \times [-1, 1]) \cap S_{q_u}$, introduce a disk $(B^2_{j_i} \times \{t\})$ in $N(k, 2)$ and attach it to the surface $S_{q_u} \cap N_1$. If this is done for all jagged 2-handles that S_{q_u} intersects, we obtain a surface in $N(k, 2)$ that we will call T_{q_u}. This can be done for the cores of all the jagged 1-handles. We will have occasion to reverse this operation later.

We will now show:

(a) The loop L shrinks in $N(k, 2)$;

(b) every loop in a surface T_{q_u} shrinks in $N(k, 2)$; and

(c) $\pi_1 N(k, 2)$ is naturally isomorphic to the subgroup of $\pi_1 N'(k)$ that is generated by all loops in N_1.

Statement (a) will complete step II. The other two statements will be used to finish step III.

Statement (a) is equivalent to the statement that L shrinks in N_1 mod the curves $\{\partial S_{j_i} \mid 1 \leqq i \leqq r\}$. Thus, statement (a) will follow inductively when we demonstrate that the following two statements are true. First, if S_k is the core of a jagged 2-handle, then L shrinks in $N'(k-1)$ mod ∂S_k and the surfaces $\{S_j \times \{\pm 1\} \mid 1 \leqq j \leqq k-1\}$. Second, if S_k is the core of a jagged 1-handle, then L shrinks in $N'(k-1)$ mod the surfaces $\{S_j \times \{\pm 1\} \mid 1 \leqq j \leqq k-1\}$. The second assertion is clear since, in that case, $N'(k-1)$ and $N'(k)$ are identical and we have already shown that every loop in S_k shrinks in $N'(k)$. To see the first assertion, let A be the disk with holes that demonstrates that L shrinks in $N(k)$ mod the G_i. Put A in general position with respect to ∂N_1 and remove the jagged 2-handle $(\mathring{S}_k \times [-1, 1])$. This punches more holes in A and all the new boundary components lie in the annulus $\partial S_k \times [-1, 1]$.

To show (b), we note that every loop in T_{q_u} is homotopic to a loop in $T_{q_u} \cap N_1$. By property (iii), every loop in

$$T_{q_u} \cap N_1 = S_{q_u} - \cup \{\mathring{S}_{j_i} \times [-1, 1] \mid 1 \leqq i \leqq r\}$$

shrinks in N_1 mod boundaries of cores of jagged 2-handles of index lower than q_u. But all such curves are trivial in $N(k, 2)$.

We start the argument for (c) by observing that N_1 is a subset of both $N(k, 2)$ and $N'(k)$. Let the inclusion maps be

$$g\colon N_1 \longrightarrow N(k, 2) \quad \text{and} \quad h\colon N_1 \longrightarrow N'(k) .$$

The map g_* is a surjection on π_1. It suffices to show that $\ker g_* = \ker h_*$. A loop in $\ker g_*$ shrinks in N_1 mod the curves $\{\partial S_{j_i} \mid 1 \leqq i \leqq r\}$. A loop in $\ker h_*$ shrinks in N_1 mod the curves $\{\partial S_j \mid 1 \leqq j \leqq k\}$ (see the observation at the end of the paragraph in which $N'(k)$ was defined). We are done if every curve in $\{\partial S_j \mid 1 \leqq j \leqq k\}$ shrinks in $N(k, 2)$, and if every curve in $\{\partial S_{j_i} \mid 1 \leqq i \leqq r\}$ shrinks in $N'(k)$. The first condition holds because of statement (b) and the definition of $N(k, 2)$ while the second condition was proven immediately after $N'(k)$ was defined.

We can now start on step III in earnest. We concentrate on the fact that L shrinks in $N(k, 2)$. We would like to replace L by a simple closed curve. However $N(k, 2)$ is not a 3-manifold and the loop theorem and Dehn's lemma are not available. We will use Lemma 2 and the results of [4] to complete the proof of the claim. Since $N(k, 2)$ contains N_1, and N_2 contracts in N_1, we know N_2 contracts in $N(k, 2)$. This allows us to conclude from Lemma 2 that there exists a compact, orientable 3-manifold M formed by replacing components of N_4 by a collection of handlebodies $\{H_j\}$, and there exists a map $f: M \to N(k, 2)$ so that f carries $M - \cup\{H_j\}$ homeomorphically onto $N(k, 2) - \mathring{N}_4$, f_* is a surjection on π_1, and P, the kernel of f_*, is perfect.

The map f when restricted to $M - \cup\{\mathring{H}_j\}$ has an inverse which we will denote by $\bar{f}$. We know that $\bar{f}L$ is a loop in ∂M with $[\bar{f}L]$ in P. If we let K be the normal subgroup of $\pi_1(\partial M)$ generated by the curves in

$$\{\bar{f}\partial G_i \mid 1 \leqq i \leqq 2k\}\ ,$$

then our choice of L says that $[\bar{f}L]$ is not in K. By the hypotheses of Lemma 3, $\pi_1 N'(k)$ is torsion free. Then, using statement (c) above and the fact that f_* is surjective, we can say that $\pi_1 N(k, 2)$ and $\pi_1(M)/P$ are torsion free. This allows us to use [4, Theorem 1] to conclude that there is a simple closed curve J' in a regular neighborhood of $\bar{f}L$ with $[J']$ in P but in K. Thus, $J = fJ'$ is a simple closed curve in $F(k)$ that shrinks in $N(k, 2)$, and $[J]$ is not in the normal closure in $\pi_1\partial N(k, 2)$ of the curves $\{\partial G_i \mid 1 \leqq i \leqq 2k\}$. Since $\partial N(k, 2)$ contains $F(k)$, we have that $[J]$ is not in the normal closure of $\pi_1 F(k)$ of the curves $\{\partial G_i \mid 1 \leqq i \leqq 2k\}$. The curve J will be the boundary of the core of new jagged 1-handle.

Since all of the cores of the existing jagged 1-handles miss N_2, all of the surfaces in $\{T_{q_u} \mid 1 \leqq u \leqq s\}$ have homeomorphic copies in $M - \cup\{\mathring{H}_j\}$. Also since every loop in a T_{q_u} shrinks in $N(k, 2)$, every loop in $\bar{f}T_{q_u}$ represents an element in P in $\pi_1(M)$. We also observe that since N_2 contracts in $N(k, 2)$, every loop in $\bar{f}\partial N_2$ represents an element in P. We can now apply [4, Theorem 3] to conclude that J bounds a compact, orientable surface T' so that T' is properly embedded in $M - \bar{f}\partial N_2$, so that every loop in T' represents an element of P, and so that T' is disjoint from all of the surfaces $\bar{f}T_{q_u}$.

The surface $T = fT'$ in $N(k, 2) - N_2$ may not intersect the 2-handles $(B^2 \times [-1, 1])_{j_i}$ nicely. However, by pushing out radially from each $(\{0\} \times [-1, 1])$, we can isotop T to a surface T_{k+1} for which each component of the intersection of T_{k+1} with a 2-handle is of the form $B^2 \times \{t\}$, $t \in (-1, 1)$. This can be done so as not to disturb the surfaces T_{q_u}.

We can now apply to T_{k+1}, the inverse of the operation that turned the surfaces S_{q_u} into the surfaces T_{q_u}. This gives a surface S_{k+1} in $N(k)$ that satisfies properties (i) and (ii). Every loop in T' represents an element of P. Thus, every loop in T and in T_{k+1} shrinks in $N(k, 2)$. This says that S_{k+1} also satisfies property (iii).

This completes the proof of the claim in the case that L shrinks in $N(k)$ mod the G_i. If L shrinks in $C - \mathring{N}(k)$ mod the G_i, then a space similar to $N(k, 2)$ is defined. It is gotten by sewing 2-handles along the ∂S_{q_u}, boundaries of cores of jagged 1-handles. Surfaces T_{j_i} are also defined but these are all disks. The new space is a 3-manifold and the claim follows using Dehn's lemma and the loop theorem. This finishes the claim.

Proof of Lemma 3 (continued). We can now let k be an integer so that $KN(k) = 0$, i.e., each component of $F(k)$ is planar. The space $N'(k)$ now corresponds to the space N' in the proof of Lemma 3'. Note that because of the intersections of jagged 1-handles and jagged 2-handles, $N'(k)$ may have fewer components than either N_1 or $N(k)$. However it cannot have more.

The following steps are identical to corresponding steps in Lemma 3'. From Lemma 1 (c4), we know that every closed, orientable surface in $C - S(C)$ separates. Let V be a component of $N'(k)$. One boundary component of V has a complementary domain whose closure E' contains V but does not contain ∂C. Other components of $N'(k)$ may be swallowed up by E', but this does not matter. Since all loops in cores of jagged handles shrink in $N'(k)$, a system of pairwise disjoint simple closed curves $\{K_i' \mid 1 \leqq i \leqq m\}$ exists on $\partial E'$ so that each $[K_i'] = 1$ in $\pi_1(E')$ and so that every component of

$$\partial E' - \cup \{K_i' \mid 1 \leqq i \leqq m\}$$

is planar.

Since E' is not a 3-manifold we need an extra argument. Consider $\partial E'$ in limbo and attach 2-handles along disjoint annuli $\{K_i' \times [-1, 1] \mid 1 \leqq i \leqq m\}$. A complex is formed which embeds in a handlebody with $\partial E'$ as boundary. Let

$$\{K_i \mid 1 \leqq i \leqq \text{genus}(\partial E')\}$$

be a complete system of meridinal curves for this handlebody. In $\pi_1(\partial E')$, these curves are in the normal closure of the curves $\{K_i' \mid 1 \leqq i \leqq m\}$. Thus these new curves shrink in E'. We can now use Lemma 2 again and use [4, Theorem 3] repeated genus$(\partial E')$ times to obtain the surfaces required in E' in the conclusion of Lemma 3. There are a finite number of components of $N'(k)$ not contained in E', and the process of the last two paragraphs can be repeated. The component E' may be swallowed up by another component during this process, but this is also no matter. This completes the proof of Lemma 3.

THEOREM 5. *Let X be a compact generalized* 3-*manifold whose* *singular set S is 0-dimensional. Consider the statements:*

(a) *X is the cell-like image of a compact 3-manifold; and*

(b) *there is a neighborhood U of S so that if N is a neighborhood of S in U, then $\pi_1(N)$ has no torsion.*

Then (a) *implies* (b), *and* (b) *together with the Poincaré Conjecture implies* (a).

Proof. The forward direction follows from Theorem 5.1 and the proof of Theorem 1. The reverse direction is given as Theorem 5.2 which uses a hypothesis that is minutely weaker than statement (b).

THEOREM 5.1. *Let f be a cell-like mapping of the closed 3-manifold N onto the Hausdorff space X. Suppose that the closure S of the set*

$$\{x \in X \mid f^{-1}(x) \text{ contains more than one point}\}$$

is zero-dimensional. Then, the fundamental group of each component of each sufficiently tight neighborhood of S in X is torsion-free.

Proof. Since each component of $f^{-1}(S)$ is cell-like, it follows from Theorem 3 of [11] that each component of some compact neighborhood U of $f^{-1}(S)$ in N is a *homotopy handlebody* (i.e., a homotopy 3-cell with orientable 1-handles attached to its boundary). In particular, the fundamental group of each component of U is free. By Theorem 31.2 of [17], each open connected subset of U has torsion-free fundamental group. Further $f(U)$ is an open neighborhood of S in X. Hence, if $V \subset f(U)$ is connected and open in X, with $V \cap S$ both open and closed in S, then the restriction of f to $f^{-1}(V) \subset U$ induces an isomorphism between $\pi_1(f^{-1}(V))$ and $\pi_1(V)$. Hence, $\pi_1(V)$ has no elements of finite order.

THEOREM 5.2. *Let X be a compact generalized 3-manifold whose singular set S is 0-dimensional. Assume there is a neighborhood U of S so that if N is a closed neighborhood of S bounded by 2-manifolds and $N \subset U$, then $\pi_1(N)$ is torsion free. Then, modulo the Poincaré Conjecture, X is the cell-like image of a compact 3-manifold.*

Proof. This follows from Lemma 3 and Theorem 4. We use this space to make the following remarks. If every compact generalized 3-manifold X with 0-dimensional singular set S admitted a resolution, then there would be a "loop theorem" for closed neighborhoods of S with 2-manifold boundary. It was our hope that an independent proof of a "loop theorem" would lead to the existence of resolutions.

The "loop theorem" is obtained for neighborhoods whose fundamental groups have no torsion from Lemma 2 and the results of [4]. The torsion free requirement comes entirely from [4]. Lemma 3 shows how a "loop theorem" can be used to verify the hypotheses of Theorem 4. If a "loop theorem" without a torsion free restriction could be found, Lemma 3 would then be valid with its torsion free hypothesis removed. (See Thickstun's announcement in Bull. (New Series) Amer. Math. Soc., 4 (1981), 192–194, especially his last Corollary.)

We also note that if the manifold set of X embeds in a compact 3-manifold, then the results of [2] say that it embeds in a closed 3-manifold so that its complement is a nested intersection of unions of handlebodies. The proof of Theorem 1 then says that there are arbitrarily small neighborhoods of the singular set which are cell-like images of unions of handlebodies. The neighborhoods constructed by Lemma 3 are such neighborhoods.

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