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TORSION DIVISORS ON ALGEBRAIC CURVES

GEORGE R. KEMPF

Let D be a divisor on a smooth complete algebraic curve C such that the multiple dD is rationally equivalent to zero for some positive integer d. We may write $D = D_0 - D_\infty$ where D_0 and D_∞ are distinct effective divisors.

The purpose of this note is to prove the

THEOREM. *Assume that C is a curve with general moduli in characteristic zero. For such a divisor D, the cohomology $H^1(C, \mathcal{O}_C(D_0 + D_\infty))$ must be zero (equivalently $|K - D_0 - D_\infty|$ is empty).*

This theorem answers a question which arose in Sevin Recillas' unpublished work which motivated this note. This paper contains simple computational techniques for handling infinitisimal deformations of curves. I hope that these techniques may be useful for solving similar problems.

The theorem is well-known for the case $d = 1$. For instance see Farkas' paper [2]. The recent work [1] contains further biliographic material and related material concerning deformations of mappings of curves into projective spaces.

1. First order deformations defined by principal parts.

Let $\mathscr{L}$ be an invertible sheaf on a smooth curve C. For any point c of C, $\text{Prin}_c(\mathscr{L}) \equiv \text{Rat}(\mathscr{L})/\mathscr{L}_c$ measures the principal (polar) part of the rational sections of $\mathscr{L}$. Let $\text{Prin}(\mathscr{L})$ be the space of principal parts, $\bigoplus \text{Prin}_c(\mathscr{L})$, where c runs through C. For a given principal part $p = (p_c)$ in Prin $(\mathscr{L})$, the support, supp(p), is the set of c in C with $p_c \neq 0$.

As in [3], we have a natural exact sequence

$$0 \longrightarrow \Gamma(C, \mathscr{L}) \longrightarrow \text{Rat}(\mathscr{L}) \longrightarrow \text{Prin}(\mathscr{L}) \longrightarrow H^1(C, \mathscr{L}) \longrightarrow 0 .$$

Let p_c be a principal part in $\text{Prin}_c(\mathscr{L})$ with a pole of order one. By duality the image of p_c in $H^1(C, \mathscr{L})$ is zero if and only if each section of $\Omega_C \otimes \mathscr{L}^{\otimes -1}$ vanishes at c. Consequently, if $H^1(C, \mathscr{L}) \neq 0$ (i.e., there is a nonzero such section), the cohomology class of p_c is nonzero for all but a finite number of points c.

Let $\Theta_C = \Omega_C^{\otimes -1}$ be the sheaf of regular vector fields on C. For any principal part X in $\text{Prin}(\Theta_C)$, we want to define a deformation C_X of C over $T = \text{Spec}(k[\delta])$, where $k[\delta]$ is the ring of dual numbers $k[t]/(t^2)$. First note that, if f is a rational function which is regular at each point of supp(X), then $X(f)$ makes sense in an obvious way as a principal part in $\mathrm{Prin}(\mathcal{O}_C)$. Secondly define the structure sheaf of C_X to be the subsheaf $\mathcal{A}_X$ of $\mathcal{O}_C \oplus \delta\mathfrak{Rat}(\mathcal{O}_C)$, whose sections $f_0+f_1\delta$ locally satisfy the conditions: f_0 is regular and $X(f_0)$ is the principal part of f_1 (symbolically $(1 - X\delta)(f_0 + f_1\delta)$ is regular).

Locally on C, X is the principal part of a rational vector field $\underline{X}$. For regular functions g_0 and g_1, the expression

$$\phi_{\underline{X}}(g_0 + \delta g_1) \equiv (1 + \underline{X}\delta)(g_0 + g_1\delta) = g_0 + (\underline{X}g_0 + g_1)\delta$$

defines an $k[\delta]$-algebra isomorphism between $\mathcal{A}_X$ and $\mathcal{O}_C \oplus \delta\mathcal{O}_C$, which preserves the augmentations of the two algebras onto $\mathcal{O}_C$. Therefore $C_X \to T$ is a (locally trivial) deformation of C over T, which has been given an obvious trivialization over the open subset C-$\mathrm{supp}(X)$.

EXAMPLE. If t is a parameter function at a point c and $t(c) = 0$, let X be the principal part of the vector field $(1/t)(d/dt)$ at c. This principal part is determined by c upto constant. In this case, C_X is called a (first-order) Schiffer variation of the curve C. (If one replaces $(1 + X\delta)$ by $\exp(Xt) \bmod t^n$, as $n \to \infty$ one gets the formal version of a Schiffer variation in the compact Riemann surface case.)

REMARK. The deformation $C_X \to T$ depends only on the cohomology class of X in $H^1(C, \Theta_C)$. This class is a form of the Kodaira-Spencer class of the deformation.

2. Obstructions to a particular deformation problem. Let D_0 and D_∞ be two effective divisors on our smooth curve C. Assume that D_0 and D_∞ have disjoint support. We are interested in the following special situation. For some positive integer d, dD_0 is linearly equivalent to dD_∞. In other words, there is a nonconstant morphism $f: C \to \boldsymbol{P}^1$ such that $f^{-1}(0) = dD_0$ and $f^{-1}(\infty) = dD_\infty$. We want to examine whether this special situation may be extended to a deformation of C.

Let $g: \widetilde{C} \to M$ be a deformation of $\widetilde{C}$. Thus g is a smooth morphism and C is a closed subscheme of $\widetilde{C}$, which has the form $g^{-1}(m)$ for some point m of M. Then we may ask if we can find a morphism $\widetilde{f}: \widetilde{C} \to \boldsymbol{P}^1$ such that $\widetilde{f}|_C = f$, $\widetilde{f}^{-1}(0)$ and $\widetilde{f}^{-1}(\infty)$ are defined and we have $\widetilde{f}^{-1}(0) = d\widetilde{D}_0$ and $\widetilde{f}^{-1}(\infty) = d\widetilde{D}_\infty$ for some effective relative Cartier divisors $\widetilde{D}_0$ and $\widetilde{D}_\infty$ on $\widetilde{C}$ with $\widetilde{D}_i|_C = D_i$ for $i = 0$ or ∞.

Next we will analyze this concept for first order deformations. Let $C_X \to T$ be the deformation defined in the first section for a given principal part X in $\mathrm{Prin}(\Theta_C)$. Here C and C_X are the same as topological spaces. Let C_i denote the open subscheme C-$\mathrm{supp}(D_j)$ of

C where $\{0, \infty\} = \{i, j\}$. Our original morphism f is given by a pair (f_0, f_∞) where each f_i is a regular function on C_i and $f_0 \cdot f_\infty = 1$ on $C_0 \cap C_\infty$. Similarly, our extension $\tilde{f}$ would be given by a pair $(f_0 + f_0'\delta, f_\infty + f_\infty'\delta)$, where $f_i + f_i'\delta$ is a section of $\mathscr{A}_X$ over C_i, such that

(*) $f_0 f_0' + f_\infty' f_\infty = 0$ on $C_0 \cap C_\infty$. As the $f_i + f_i'\delta$ are then reciprocals on $C_0 \cap C_\infty$, the conditions on the zeros and poles of $\tilde{f}$ are just that, near $\mathrm{supp}(D_i)$,

(**) there exist sections α and β of $\mathscr{A}_X$ with $f_i + f_i'\delta = \alpha\beta^d$ and α is a unit.

In the simplest case, the question of the existence of such an $\tilde{f}$ is easily interpreted.

PROPOSITION. *Assume that* $\mathrm{supp}(X)$ *is contained* $C_0 \cap C_\infty$ *and* $\mathrm{char}(k) \nmid d$. *Then the required extension* $\tilde{f}$ *exists if and only if the principal part* $(1/f)X(f)$ *interpreted in* $\mathrm{Prin}(\mathscr{O}_C(D_0 + D_\infty))$ *is cohomologous to zero.*

Proof. By the assumption on the $\mathrm{supp}(X)$, any section of $\mathscr{A}_X$ near $\mathrm{supp}(D_i)$ can be written as $\alpha = \alpha_0 + \alpha_1\delta$, where α_0 and α_1 are regular functions. Furthermore, this section $\alpha_0 + \alpha_1\delta$ is a unit near $\mathrm{supp}(D_i)$ if and only if α_0 is.

Claim. Given a rational function f_i' on C, then $f_i + f_i'\delta$ is a section of $\mathscr{A}_X$ over C_i-$\mathrm{supp}(X)$ and satisfies the condition (**) $\Leftrightarrow$ $\mathrm{div}(f_i'/f_i) + D_i \geqq 0$ on C_i-$\mathrm{supp}(X)$.

We will first see how this claim implies the proposition. Assume that $(f_1 + f_1'\delta, f_\infty + f_\infty'\delta)$ gives a required $\tilde{f}$. By the identity (*), we must have $f_0'/f_0 = -f_\infty'/f_\infty$. Thus the claim shows that the desired conditions on the zeros and poles of $\tilde{f}$ imply that f_i'/f_i is a regular section of $\mathscr{O}_C(D_0 + D_\infty)$ over C-$\mathrm{supp}(X)$. As $f_i + f_i'\delta$ is a section of $\mathscr{A}_X$ over C_i, $X(f_i)$ is the principal part of the rational function f_i' except at $\mathrm{supp}(D_0 + D_\infty)$. So f_0'/f_0 is a rational section of $\mathscr{O}_C(D_0 + D_\infty)$ with principal part $1/f_0 X(f_0)$. Therefore, if our $\tilde{f}$ exists, $(1/f)\ X(f)$ is cohomologous to zero in $\mathrm{Prin}(\mathscr{O}_C(D_0 + D_\infty))$.

Conversely, if $1/f X(f)$ is cohomologous to zero, we may have a rational section f_0'/f_0 of $\mathscr{O}_C(D_0 + D_\infty)$ with principal part $1/f_0 X(f_0)$. Define f_∞' by the identity (*). Reversing the above argument, one can readily check that the required extension $\tilde{f}$ is given by $(f_0 + f_0'\delta, f_\infty + f_\infty'\delta)$. Thus the claim implies the proposition.

To prove the claim, assume that $f_i + f_i'\delta$ satisfies condition (**). Then near $\mathrm{supp}(D_i)$ we have $f_i + f_i'\delta = (\alpha_0 + \alpha_1\delta)(\beta_0 + \beta_1\delta)^d = \alpha_0\beta_0^d +$

$(\alpha_1\beta_0^d + d\beta_0^{d-1}\beta_1\alpha_0)\delta$, where the α's and β's are regular and α_0 is a unit. So $\beta_0 f_i'/f_i = (\alpha_1\beta_0 + d\beta_1\alpha_0)/\alpha_0$ is regular. As $D_i = (\beta_0 = 0)$, this means that $\operatorname{div}(f_i'/f_i) + D_i \geqq 0$ along D_i. Conversely, assume that this is ture. By the assumption on f, we have $f_i = \alpha_0\beta_0^d$, where α_0 is unit and $D_i = (\beta_0 = 0)$ near D_i. Take $\alpha_1 = 0$ and $\beta_1 = 1/d(\beta_0 f_i'/f_i)$. Thus β_1 is regular and condition (**) is satisfied for $\alpha = \alpha_0 + \alpha_1\delta$ and $\beta = \beta_0 + \beta_1\delta$ near D_i.

Away from D_i on C_i-supp(X), $\operatorname{div}(f_i'/f_i) + D_i \geqq 0 \Leftrightarrow f_i'$ is regular because f_i is a unit there. On the other hand, away from supp(X), $f_i + f_i'\delta$ is a section of $\mathscr{A}_X \Leftrightarrow f_i'$ is regular. This completes the proof of the claim. □

Immediately we get

COROLLARY. *Assume that $H^1(C, \mathscr{O}_C(D_0 + D_\infty))$ is not zero and $\operatorname{char}(k) \nmid \deg(f)$. Then, for all but a finite number of points c of C, the required extension $\tilde{f}$ does not exist if X is the Schiffer variation $(1/t)(d/dt)$ at the point c.*

Proof. The rational differential df is not zero. In characteristic zero, this is true because f is not constant. In characteristic p, it follows because p does not divide the degree of f.

Thus for all but a finite number of points c of C, $X(f) = (1/t)(df/dt)$ is a principal part at c of order one. As $H^1(C, \mathscr{O}_C(D_0 + D_\infty)) \neq 0$, by the previous remark, we know that almost of these principal parts are not zero. Hence our corollary follows from the proposition. □

Now we will be done as soon as we finish the

Proof of the theorem. Let $g: \tilde{C} \to M$ be a family of smooth complete curves where M is an irreducible variety. We will also assume that, for each point m of M, g gives a universal deformation of the fiber $g^{-1}(m)$.

For some positive integer e, let $\operatorname{Div} \to M$ be the variety of effective relative divisors on $\tilde{C}/M$ of degree e. Let X be the subset $\{(D_0, D_\infty) \mid dD_0 \sim dD_\infty\}$ of $\operatorname{Div} \times_M \operatorname{Div}$.

By the see-saw principle, X is a closed subset. Consider the conditions $D_0 \neq D_\infty$ and $H^1(C, \mathscr{O}_C(D_0 + D_\infty)) \neq 0$. The first condition defines an open subset of X and the second condition defines a closed subset of X by upper-semicontinuity. Let Y be the subvariety of X defined by these two conditions. We have a tautological morphism $\sigma: Y \to M$. The precise statement of the theorem is that σ omits an open dense subset of M.

Assume otherwise. Then by Sard's theorem ($\operatorname{char} k = 0$) we may find a locally closed smooth subvariety M' of Y, which is etale over

M. Replacing M by M' and $\tilde{C}$ by $\tilde{C} \times_M M'$, we may assume that we have effective relative divisors $\tilde{D}_0$ and $\tilde{D}_\infty$ on C, whose fibers over M satisfy the above two conditions. We may find effective relative divisors $\bar{E}$, $\bar{D}_0$ and $\bar{D}_\infty$ on $\tilde{C}$ such that $\bar{D}_0$ and $\bar{D}_\infty$ have no common component and $\bar{E} + \bar{D}_i = \tilde{D}_i$. Further replacing M by an open subset we may assume that $\bar{D}_0$ and $\bar{D}_\infty$ are disjoint.

By construction we have a rational function $\tilde{f}$ on $\tilde{C}$ with zero divisor $d\bar{D}_0$ and polar divisor $d\bar{D}_\infty$. By the first condition $\bar{D}_0$ and $\bar{D}_\infty$ are relative divisors of positive degree and consequently $\tilde{f}$ is not constant. We may now specialize our attention to a particular fiber C of g. By the corollary and the universality of the deformation, $H^1(C, \mathcal{O}_{\tilde{C}}(\bar{D}_0+\bar{D}_\infty)|_C) = 0$. Hence its quotient $H^1(C, \mathcal{O}_{\tilde{C}}(\tilde{D}_0+\tilde{D}_\infty)|_C) = 0$. As this violates the second condition, the theorem follows by contradiction. □

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THE JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218