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POLYNOMIALS IN DENUMERABLE INDETERMINATES

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D. Knuth used the Robinson-Schensted "insertion into a tableau" algorithm to give a direct 1-to-1 correspondence between "generalized permutations" and ordered pairs of generalized Young tableaux having the same shape. Since a generalized permutation characterizes a power product of differential indeterminates, the work of D. Mead on the principal differential ideal generated by a Wronskian provided an independent proof of the existence of the Knuth bijection. This work led Mead to suggest that other interesting combinatorial results may be found by equating the cardinalities of different vector space bases for the same finite-dimensional subspace of a differential ring. In a previous paper the author showed how such combinatorial identities follow from the study of "strong bases" for certain ideals in a ring of polynomials in a denumerable set of indeterminates. The present paper completes that work by presenting an infinite number of such strong bases and thus greatly expands the ring theory and differential algebra having applications in the enumeration of tableaux.

1. The ideals I. Let $R = F[y_{ij}]$ denote the polynomial ring in the algebraically independent indeterminates y_{ij} $(i = 1, 2, \cdots, n;$ $j = 0, 1, 2, \cdots)$ over a field F. In applications to differential algebra, one lets $y_1, y_2, \cdots, y_n$ be n independent indeterminates and y_{ij} be the jth derivative of y_i. Then a principal differential ideal $[x]$ is the ideal $(x_0, x_1, x_2, \cdots)$ in which x_j is the jth derivative of x.

D. Mead's study in [9] of $[W_n]$, where W_n is the Wronskian of $y_1, y_2, \cdots, y_n$, gives a vector space basis for R consisting of determinantal products having a natural 1-to-1 correspondence with ordered pairs of Young tableaux of the same shape, having n or fewer rows. Let x_{nq} be the qth derivative $(y_1 y_2 \cdots y_n)^{(q)}$; it is shown below that ideals $(x_q, x_{q+1}, \cdots)$, related to $[x_{nq}]$, share combinatorial properties with $[W_n]$ when $q = n(n-1)/2$.

A combinatorial method for proving the existence of syzygies (i.e., the nonexistence of a strong basis) is also described. The structure of $(x_0, x_1, \cdots)$ is studied in a manner that gives the structures of all ideals generated by subsets of the x_j.

Let $\{x_j\} = x_0, x_1, \cdots$ be a finite or denumerable sequence in R, I be the ideal $(x_0, x_1, \cdots)$, and X be the set of all power products

$$\xi = (x_0)^{a_0}(x_1)^{a_1} \cdots (x_h)^{a_h}\ ; \qquad h,\ a_i \in \{0, 1, \cdots\}\ .$$

Let A be a linearly independent (over F) subset of R such that $L = \{\alpha\xi \mid \alpha \in A, \xi \in X\}$ generates the vector space R over F; then it is easily seen that the subspace I is generated by the subset C of all $\alpha\xi$ in L with $\xi \neq 1$. The set A is called an *α-set* for $\{x_j\}$ if L is a basis for the vector space R; if A is an α-set, C is a basis for I. If L is not a basis for R, the linear dependence relations of the elements of L are called syzygies. If $\{x_j\}$ has an α-set, the sequence is said to be *strong*. Below we describe a family of sequences $\{x_j\}$, develop a number of α-sets for each sequence, and for each α-set give an algorithm for determining membership in the ideal $(x_0, x_1, \cdots)$.

A power product (pp) π in the y_{ij} of degree $d = \deg \pi$ and weight $w = \operatorname{wgt} \pi$ is a product of d factors, each of which is one of the y_{ij}, with w the sum of the second subscripts j of these d factors. Below, $Q = (q_1, q_2, \cdots, q_n, q_{n+1})$ is an ordered $(n + 1)$-tuple of fixed nonnegative integers, $q = q_1 + q_2 + \cdots + q_{n+1}$, $T = \{t_1, t_2, \cdots\}$ is a subset (not necessarily proper) of $\{q, q + 1, \cdots\}$, and μ is a nonnegative real number. Whenever π is written as $\pi = \rho\eta$, ρ is the product of all the factors y_{ij} of π with $j < q_i$ and η is the product of the factors y_{ij} of π with $j \geqq q_i$.

The set of all pp in the y_{ij} is designated as P. The word *space* is used to denote a vector space over F; thus P is a space basis for the ring R.

For all t in T, let v_t be a linear combination with coefficients in F of the pp $\pi = \rho\eta$ with

$$\deg \pi + \operatorname{wgt} \pi \leqq n + t \quad \text{and} \quad \deg \eta + \mu \operatorname{wgt} \eta < n + \mu t$$

and let x_t be the sum of v_t and a linear combination with nonzero coefficients in F of all the products

$$y_{1j_1} y_{2j_2} \cdots y_{nj_n} \text{ with } j_i \geqq q_i \text{ for } 1 \leqq i \leqq n \text{ and } j_1 + \cdots + j_n = t .$$

Let $I = (x_{t_1}, x_{t_2}, \cdots)$ be the ideal in R generated by the x_t with t in T.

2. Ordering of power products. Associated with the η of a fixed pp $\pi = \rho\eta$ is a function $j(i, k)$ such that $\eta = \eta_1\eta_2 \cdots \eta_n$ with either $\eta_i = 1$ or

$$\begin{aligned} &\eta_i = y_{ij(i,1)} y_{ij(i,2)} \cdots y_{ij(i,d_i)} , \\ &q_i \leqq j(i, k) \leqq j(i, k + 1) \quad \text{for} \quad 1 \leqq k < d_i = \deg \eta_i . \end{aligned}$$

(If $\eta_h = 1$, $j(i, k)$ is not defined for $i = h$.) This is next used to define nonnegative integers g_i, a function $M[i, k]$, and a sequence $\sigma(\pi) = s_0, s_1, \cdots$. Then $\sigma(\pi)$ will be used in a partial ordering of the *pp* which is the key tool for the study of the structure of the ideal I.

Let $g_1 = d_1$ and $M[1, k] = n(k-1)+1$ for $1 \leqq k \leqq g_1$. Now assume that $i > 1$, that g_{i-1} is defined, and that $M[i-1, k]$ is defined for $1 \leqq k \leqq g_{i-1}$. Let g_i be the largest positive integer m with $m + j(i, m) - q_i \leqq g_{i-1}$ if such an m exists and let $g_i = 0$ otherwise. Also let

$$(1) \qquad M[i, k] = M[i-1, k + j(i, k) - q_i] + 1 \quad \text{for} \quad 1 \leqq k \leqq g_i .$$

For those i with $g_i > 0$, this defines $M[i, k]$ for $1 \leqq k \leqq g_i$. If $m = M[i, k]$ for such an i and k, let $s_m = j(i, k)$; if m is a positive integer not in the image set of M, let $s_m = \infty$. Also let $s_0 = \deg \eta + \mu \operatorname{wgt} \eta$. Since M is easily shown to be injective, the sequence $\sigma(\pi) = s_0, s_1, \cdots$ is now well defined. [$\sigma(\rho\eta)$ depends only on η.]

Let $\sigma(\pi) = s_0, s_1, \cdots$ and $\sigma(\pi') = s'_0, s'_1, \cdots$. If there is an integer m such that $s_m < s'_m$ and $s_k = s'_k$ for $k < m$, then π is said to be *stronger than* π' ($\pi \gg \pi'$) *at* m. The stronger than relation is transitive but is not a complete linear ordering.

3. The set A. An i-tuple

$$(2) \qquad s_{cn+1}, s_{cn+2}, \cdots, s_{cn+i}$$

in $\sigma(\pi)$ for which each of these i terms is finite is an *i-run* for π and the sum of the i terms is the *weight* of the i-run. It can be shown that the weight of the i-run (2) is a nondecreasing function of c in (2). If the weight of an n-run for π is in the given set T, the associated product

$$(3) \qquad b = y_{1s_{cn+1}} y_{2s_{cn+2}} \cdots y_{ns_{cn+n}}$$

is called a *β-factor* of π. The set A is now defined to consist of all π having no β-factors and the set C to consist of all

$$\gamma = \alpha\xi , \qquad \alpha \in A , \qquad \xi = x_{t_1} x_{t_2} \cdots x_{t_e} , \qquad e \geqq 1 , \qquad t_k \in T .$$

In §5, C and $L = A \cup C$ will be shown to be space bases for I and R, respectively. When $q = 0$, $T = \{0, 1, \cdots\}$, and each $v_t = 0$, A and C can be shown to be the same as the sets of α-terms and β-terms respectively, defined in [5], using the machinery in [7] and induction on n.

4. The bijection θ. Next we define a mapping θ from P to L and, as in Levi's work in [7], show that θ is a bijection and then show that L and C are space bases for R and I, respectively.

Let π have the b of (3) as a β-factor and let $\sigma(\pi) = s_0, s_1, \cdots$. It is easily seen that

$$(4) \qquad \sigma(\pi/b) = s_0, \cdots, s_{cn}, s_{(c+1)n+1}, \cdots$$

i.e., that $\sigma(\pi/b)$ is $\sigma(\pi)$ with the n-run corresponding to b deleted. This implies that π can be written as $\alpha b_1 b_2 \cdots b_r$, with the b_k all the β-factors of π and $\alpha \in A$; then θ is defined by

$$(5) \qquad \theta(\pi) = \theta(\alpha b_1 b_2 \cdots b_r) = \alpha x_{t_1} x_{t_2} \cdots x_{t_r}, \quad \text{where} \quad t_k = \operatorname{wgt} b_k .$$

If π has no β-factors, π is an α in A and $\theta(\pi) = \theta(\alpha) = \alpha$.

Examination of the sequence $\sigma(\pi) = s_0, s_1, \cdots$ shows that θ is injective. Since the terms $\alpha x_{t_1} x_{t_2} \cdots x_{t_r}$ in (5) are easily seen to be in one-to-one correspondence with the $3n$-section partitions dealt with in [3], Theorem 1 of that paper shows that θ is a bijection.

5. The space bases C and L.

LEMMA. *If π has a β-factor $b = y_{1j_1} \cdots y_{nj_n}$ of weight t, then*

$$\pi = f_0 \pi_0 x_t + f_1 \pi_1 + f_2 \pi_2 + \cdots + f_s \pi_s$$

where $f_h \in F$, $\pi_h \gg \pi$, and $\deg \pi_h + \operatorname{wgt} \pi_h \leqq \deg \pi + \operatorname{wgt} \pi$ for $0 \leqq h \leqq s$. Also, $\deg \pi_0 = \deg \pi - n$ and $\operatorname{wgt} \pi_0 = \operatorname{wgt} \pi - t$.

Proof. By definition of x_t,

$$(6) \qquad x_t - v_t = e_0 b + e_1 b_1 + \cdots + e_r b_r ,$$

where each e_h is a nonzero element of F and for $1 \leqq h \leqq r$,

(7) $b_h = y_{1k_1} \cdots y_{nk_n}$, with $k_1 + \cdots + k_n = t = j_1 + \cdots + j_n$ and $k_i \neq j_i$ for some i.

Solving (6) for b and letting $\pi_0 = \pi/b$ and $\pi_h = \pi_0 b_h$ for $1 \leqq h \leqq r$ yields

$$\pi = \pi_0 b = -f_0 \pi_0 v_t + f_0 \pi_0 x_t + f_1 \pi_1 + \cdots + f_r \pi_r .$$

By definition of v_t, one can write

$$-f_0 \pi_0 x_t = f_{r+1} \pi_{r+1} + \cdots + f_s \pi_s$$

where, for $r < h \leqq s$, one has $f_h \in F$, and $\pi_h \gg \pi$ at 0.

For $1 \leqq h \leqq r$, $\pi_h = (\pi b_h)/b$, with b_h as in (7) and so $\deg \pi_h = \deg \pi$, $\operatorname{wgt} \pi_h = \operatorname{wgt} \pi$. From (7), it follows that $k_i < j_i$ for some i. Let the β-factor b of π be as in (3). If $k_1 < j_i$, it can be seen that $\pi_h \gg \pi$ at some m with $m \leqq cn + 1$, and if $k_i < j_i$, $i > 1$, then

$\pi_h \gg \pi$ at some m with $m \leqq cn$. Since $\pi_0 = \pi/b$, we have $\deg \pi_0 = \deg \pi - n$, $\operatorname{wgt} \pi_0 = \operatorname{wgt} \pi - t$, and $\pi_0 \gg \pi$ at 0.

THEOREM 1. *L and C are space bases for R and I, respectively.*

Proof. Let B be the complement of A in P. For every nonnegative integer s, let $P(s)$ consist of all π in P with $\deg \pi + \operatorname{wgt} \pi \leqq s$. Let $R(s)$ be the subspace of R generated by $P(s)$. Let $A(s)$, $B(s)$, $C(s)$, $I(s)$, and $L(s)$ be the intersections with $R(s)$ of A, B, C, I, and L, respectively. Note that $R(s)$ has finite dimension.

The space $R(s)$ is generated by its elements of the form $\pi\xi$, with π in $P(s)$ and ξ a *pp* in the x_t with t in T, since the elements of this form with $\xi = 1$ generate $R(s)$. Since $B(s)$ is finite and the "stronger than" relation is transitive, the lemma implies that $R(s)$ is generated by its elements $\alpha\xi$ with α in $A(s)$ and ξ a *pp* in the x_t with t in T, i.e., $L(s)$ generates $R(s)$.

Since θ with its domain restricted to $B(s)$ is a bijection onto $C(s)$, $L(s) = A(s) \cup C(s)$ has the same finite number of elements as $P(s) = A(s) \cup B(s)$. Since $P(s)$ is a basis for the space $R(s)$, this means that the set $L(s)$ of generators for $R(s)$ is also a basis for $R(s)$. Then it follows that L is a space basis for R.

The space I is generated by the $\pi\xi$ with π in P and ξ a *pp* of positive degree in the x_t with t in T; then C generates I since the π in B can be replaced by linear combinations of elements of L. Since C is a subset of the basis L for R, the elements of C are linearly independent and so C is a basis for I.

6. The algorithm φ. The algorithm φ for determining whether a polynomial r of R is in I consists of using the lemma in §5 to replace in r the *pp* belonging to B and continuing until r is expressible as

$$r = f_1\alpha_1\xi_1 + \cdots + f_m\alpha_m\xi_m\,, \qquad f_h \in F\,, \qquad \alpha_h \in A\,,$$

with each ξ_h a *pp* in the x_t with $t \in T$. Then r is in I if and only if each ξ_h has positive degree in the x_t.

The description of φ implies that a nonzero polynomial r of R is not in I if the *pp* of each term of r is in A. This motivates the presentation of some simple sufficient conditions for a *pp* π to be in A. First, for a given Q, if $\pi = \rho\eta_1\eta_2 \cdots \eta_n$, then π is in A if $\deg \eta_i = 0$ for some i.

Secondly, let a *pp* π have a function $j(i, k)$ and integers g_i as in §2. The condition

$$q_1 + j(2, 1) + \cdots + j(n, 1) \geqq d_1 + q$$
$$(q = q_1 + \cdots + q_n + q_{n+1})$$

is sufficient for g_n to be 0 and hence for π to be in A. Interchanging 1 and some h, $1 < h \leqq n$, as subscripts i leads to

$$q_h + j(1, 1) + \cdots + j(n, 1) \geqq d_h + q + j(h, 1) .$$

If for some fixed h this inequality holds for the pp of all the terms in a nonzero r of R, then r is not in I.

7. Application to differential ideals. Let $y_1, \cdots, y_n$ be independent differential indeterminates over a differential field F of characteristic 0. Let $z = y_1 y_2 \cdots y_n$, and let y_{ij} and z_j be the jth derivatives of y_i and z, respectively.

For any choice of $q_1, q_2, \cdots, q_n, q_{n+1}$ as nonnegative integers with $q = q_1 + \cdots + q_{n+1}$ and $T = \{q, q+1, \cdots\}$, the z_k meet the conditions required of the x_k in §1 and hence it follows that the set A of §5 forms an α-set for the differential ideal

$$I = [z_q] = (z_q, z_{q+1}, \cdots)$$

and hence $\{z_q\}$ is a strong sequence.

8. Combinatorial applications. Let the *signature* of a π in P be the n-tuple $E = [e_1, \cdots, e_n]$ with e_h the degree of π in the factors y_{ij} with $i = h$. A polynomial is *homogeneous* with signature E if it is in the subspace $V[E]$ generated by the π of signature E. A polynomial is *isobaric* of weight w if it is in the subspace V_w generated by the π of weight w. Let $V(w, E) = V[E] \cap V_w$ and let $p(w, E)$ be the dimension of the subspace $V(w, E)$.

Let $S = [s_1, s_2, \cdots, s_n]$, with the s_i nonnegative integers that are not all zero. A strong sequence $\{x_j\} = x_q, x_{q+1}, \cdots$ will be called an *S-sequence* if:

(i) each x_j is homogeneous with signature S and is isobaric with weight j, and

(ii) $\{x_j\}$ has an α-set consisting of homogeneous and isobaric polynomials.

Let A be such an α-set for fixed S-sequence $\{x_j\} = x_q, x_{q+1}, \cdots$ and let

$$n_\alpha(w, E) = n_\alpha(w; e_1, \cdots, e_n)$$

be the number of elements in $A \cap V(w, E)$. We note here that $n_\alpha(w; 0, 0, \cdots, 0)$ equals 1 if $w = 0$ and equals 0 if $w > 0$.

THEOREM 2. *The dimension of the vector space $V(w, E)$ can be expressed as*

$$(8) \qquad p(w, E) = n_\alpha(w, E) + \sum n_\alpha(i; e_1 - ks_1, \cdots, e_n - ks_n)p(j, k)$$

where the sum is taken over all nonnegative integers i, j, k such that $i + j + qk = w$, $k > 0$, and the $e_h - ks_h \geqq 0$ for $1 \leqq h \leqq n$.

Proof. Each side of equation (8) is the dimension of the finite dimensional space $V(w, E)$. The left side is the number of elements in the basis consisting of the π in $V(w, E)$ while the right side is the number of $\alpha\xi$, associated with the strong sequence $\{x_j\}$, in $V(w, E)$.

The formula (8) in Theorem 2 enables one to calculate the $n_\alpha(w, E)$ recursively, i.e., for a given $E = [e_1, \cdots, e_n]$ in terms of values $n_\alpha(w', E')$ with signatures $E' = [e'_1, \cdots, e'_n]$ having $e'_h \leqq e_h$ for $1 \leqq h \leqq n$ and $e'_h < e_h$ for some h. We next use this to obtain the following:

THEOREM 3. *The number $n_\alpha(w, E)$ depends only on w, E, q, and S and can be written as $n_\alpha(w, E, q, S)$.*

Proof. For definiteness, let $s_1 > 0$. Then we use induction on e_1. If $e_1 = 0$, $n_\alpha(w, E)$ clearly depends only on w, E, q, and S since $n_\alpha(w, E) = p(w, E)$ in this case. By Theorem 2, $n_\alpha(w, E) = p(w, E) - \sum n_\alpha(i; e_1 - ks_1, \cdots, e_n - ks_n)p(j, k)$. Since $k > 0$ and $s_1 > 0$, $e_1 - ks_1 < e_1$. Now our result follows using the inductive hypothesis on the factors $n_\alpha(i; e_1 - ks_1, \cdots, e_n - ks_n)$.

If one has two S-sequences (with the same S) the easiest way to calculate $n_\alpha(w, E, q, S)$ for one of the sequences may be to calculate it using the α-set for the other sequence (and Theorem 3). Also the identity in (8) can be used to show that a given sequence may not be strong. We next illustrate these two types of applications of Theorems 2 and 3.

First, let $Q = \{q_1, q_2, \cdots, q_{n+1}\}$ be a fixed $(n + 1)$-tuple of nonnegative integers with $q_1 + q_2 + \cdots + q_{n+1} = q = \binom{n}{2}$ and let $T = \{q, q + 1, \cdots\}$. For $t \in T$, let x_t be a linear combination over F, of all products $y_{1j_1} \cdots y_{nj_n}$ with $j_1 + \cdots + j_n = t$ such that those products with $j_i \geqq q_i$ for $1 \leqq i \leqq n$ have nonzero coefficients. By the results in §2–5, the sequence $\{x_j\}$ for the ideal $J = (x_q, x_{q+1}, \cdots)$ is strong, has an α-set A, and is an S-sequence with $S = [1, 1, \cdots, 1]$.

Next, let $q = \binom{n}{2}$ and $W_{n,q+k}$ be the kth derivative of the Wronskian W_n of n independent differential indeterminates $y_1, \cdots, y_n$. Then W_{nj} is homogeneous with signature $S = [1, 1, \cdots, 1]$ and isobaric with weight j. Also the work in Mead's paper [9] shows that $\{W_{nj}\}$ is an S-sequence. Hence we have the following:

THEOREM 4. *$J \cap V(w, E)$ and $[W_n] \cap V(w, E)$ have the same dimension given by either side of the equation:*

$$p(w, E) - n_\alpha(w, E) = \sum n_\alpha(i; e_1 - k, \cdots, e_n - k)p(j, k)$$

where the sum is over all nonnegative integers i, j, k with

$$i + j + k\binom{n}{2} = w$$

and

$$1 \leqq k \leqq m = \min\{e_1, \cdots, e_n\} .$$

In [3], the author stated that Theorem 4 could be proved and used this result to obtain many identities on combinatorial generating functions.

Secondly, Theorem 2 can be used to show that a given sequence $\{x_j\}$ is not strong, and hence to indicate the existence of syzygies. For example, consider the differential ideal $[x] = (x_2, x_3, \cdots)$ where x_{2+j} is the jth derivative of $x = y_0^2 y_2 + 7y_0 y_1^2$. The recursive calculation, using (8), of $n_\alpha(w, E)$ based on the assumption that $\{x_j\}$ is strong leads to the contradiction that the cardinality of $A \cap V(w, E)$ is negative for some w and E. The following is a partial printout from a computer program designed to compute $n_\alpha(w, E, q, S)$ and $p(w, E) - n_\alpha(w, E, q, S)$ [denoted by n_α and n_β resp. in the table] for this example where we have

$$(9) \qquad n = 1 , \quad q = 2 , \quad S = [s_1] = [3] .$$

deg / wgt	1		2		3		4		5		6	
	n_α	n_β	n_α	n_β	n_α	n_β	n_α	n_β	n_α	n_β	n_α	n_β
0	1	0	1	0	1	0	1	0	1	0	1	0
1	1	0	1	0	1	0	1	0	1	0	1	0
2	1	0	2	0	1	1	1	1	1	1	1	1
3	1	0	2	0	2	1	1	2	1	2	1	2
4	1	0	3	0	3	1	2	3	1	4	1	4
5	1	0	3	0	4	1	2	4	1	6	1	6
6	1	0	4	0	6	1	4	5	1	9	1	10
7	1	0	4	0	7	1	5	6	1	12	0	14
8	1	0	5	0	9	1	8	7	2	16	−1	21

The negative entry -1 for n_α in the table for weight 8 and degree 6 shows that no sequence $\{x_j\}$ satisfying properties (9) is strong, and also indicates that any such sequence has some syzygies involving only polynomials with the weight bounded by 8 and the degree bounded by 6.

9. Bibliography. The many fields of mathematics in which tableaux and skew-tableau play an important role are described in the papers of the report [2]. The set of all linear combinations of partitions is shown to be isomorphic to the differential polynomial ring in one indeterminate in [4]. The ordered pairs of generalized tableaux used by Mead in [9] appear in a more general setting in [1]. The different proof of Mead's Theorem 2 in that paper could be eliminated by a reference to D. Knuth's generalization of the Robinson-Schensted insertion into tableau algorithm in [6]. The ordering of power products described in §2 above made possible the generalization, given here, of the results of [5] and [7].

REFERENCES

1. P. Doubilet, G. C. Rota and J. Stein, *On the foundations of combinatorial theory: IX combinatorial methods in invariant theory*, Studies in Applied Mathematics, Vol. LIII, No. 3, September 1974, 185-216.
2. D. Foata, ed., *Combinatoire et Représentation du Groupe Symétrique* (Actes de la Table Ronde du C. N. R. S. tenue a l'Université Louis-Pasteur de Strasbourg, du 26 au 30 avril 1976).
3. R. M. Grassl, *Multisectioned partitions of integers*, Pacific J. Math., **69** (1977), 415-424.
4. A. P. Hillman and R. M. Grassl, *Polynomial ring generated by tableau shapes*, European Journal of Combinatorics, to appear.
5. A. P. Hillman, D. G. Mead, K. B. O'Keefe and E. S. O'Keefe, *Ideals generated by products*, Proc. Amer. Math. Soc., **17** (1966), 717-719.
6. D. E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math., **34** (1970), 709-727.
7. H. Levi, *On the structure of differential polynomials and on their theory of ideals*, Trans. Amer. Math' Soc., **51** (1942), 532-568.
8. D. G. Mead, *Differential ideals*, Proc. Amer. Math. Soc., **6** (1955), 420-432.
9. ———, *Determinantal ideals, identities and the Wronskian*, Pacific J. Math., **42** (1972), 165-175.
10. D. G. Mead and B. D. McLemore, *Ritt's question on the Wronskian*, Pacific J. Math., **35** (1970), 467-472.

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