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TRANSLATION INVARIANT CLOSED $*$ DERIVATIONS

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If G is a locally compact group and δ is a left invariant closed $*$ derivation in $C_0(G)$, then δ generates a C^* dynamics of $C_0(G)$. If G is a Lie group, $C_c^\infty(G)$ is a core for δ. Similar results are obtained for coset spaces.

1. Introduction. This paper is a study of translation invariant closed $*$ derivations in C_0 of locally compact groups and their coset spaces. Our starting point is a theorem of S. Sakai [8, Proposition 1.17]: *A nonzero translation invariant closed $*$ derivation in $C(S^1)$ has domain $C^1(S^1)$ and is a constant multiple of the derivative.* Our object is to generalize this result first to Lie groups and their homogeneous spaces, and then to locally compact groups and certain coset spaces. Theorems A and B, stated below, are the main results.

DEFINITIONS. Let G be a locally compact Hausdorff space. A linear map δ in $C_0(G)$ is a *$*$ derivation* if its domain $\mathscr{D}(\delta)$ is a dense conjugate-closed subalgebra of $C_0(G)$, $\delta(\bar{f}) = \overline{\delta(f)}$, and $\delta(fg) = f\delta(g) + \delta(f)g$ $(f, g \in \mathscr{D}(\delta))$. The derivation δ is *closed* if its graph is closed. Now let G be a locally compact group and let H be a closed subgroup. Left translations in G/H and in $C_0(G/H)$ are defined by $\ell_s(tH) = stH$ and $\ell_s f = f\circ\ell_{s^{-1}}$ $(s, t \in G,\ f \in C_0(G/H))$. We say that a closed $*$ derivation δ in $C_0(G/H)$ is *G-invariant* or *translation invariant* if $\ell_s\circ\delta\circ\ell_{s^{-1}} = \delta$ for all $s \in G$. A closed $*$ derivation in $C_0(G)$ is *left invariant* if it is invariant under left translations by elements of G.

NOTATION. If F is a class of continuous functions on a locally compact space, F_c will denote the elements of F with compact support and $F_{\text{s.a.}}$ will denote the real valued elements of F.

THEOREM A. *Let G be a Lie group and H a closed subgroup. Suppose that δ is a G-invariant closed $*$ derivation in $C_0(G/H)$. Then*

(i) *$C_c^\infty(G/H) \subseteq \mathscr{D}(\delta)$ and there is a G-invariant vector field X on G/H such that $\delta(f) = X(f)$ for all $f \in C_c^\infty(G/H)$.*

(ii) *$C_c^\infty(G/H)$ is a core for δ.*

(iii) *The C^*dynamics (strongly continuous one-parameter group of $*$ automorphisms) of $C_0(G/H)$ corresponding to the complete vector field X has generator δ.*

The point of this is that δ is not assumed at the outset to have anything to do with the differential structure of G/H; but G-invariance implies that δ arises from a uniquely determined G-invariant vector field. In particular, the differential structure of a Lie group G can be recovered from the left invariant closed $*$ derivations in $C_0(G)$.

Theorem A is proved in § 2.

THEOREM B. *Let G be a locally compact group, and let δ be a left invariant closed $*$ derivation in $C_0(G)$. Then δ is the generator of a C^* dynamics of $C_0(G)$.*

This result is derived from Theorem A, using the structure theorem which gives certain locally compact groups as projective limits of Lie groups. The proof is in § 3. (Theorem B in turn implies the special case of Theorem A where H is the identity subgroup.) Some further generalizations to coset spaces are also presented in § 3.

The Lie algebra of a locally compact group is discussed in § 4.

We now recall some facts about a closed $*$ derivation δ in a commutative C^* algebra $C_0(X)$. [1, 4, 8]

The algebra $\mathscr{D}(\delta)$, with the graph norm $\|\ \|_\delta = \|\ \|_\infty + \|\delta(\ )\|_\infty$ is a Silov algebra with structure space X. $\mathscr{D}(\delta)$ has a C^1 functional calculus and $\delta(f\circ g) = (f'\circ g)\delta(g)$ for $f \in C^1(\boldsymbol{R})$ and $g \in \mathscr{D}(\delta)_{\text{s.a.}}$.[1] The derivation δ is local; that is, if $f, g \in \mathscr{D}(\delta)$ agree near $x \in X$, then $\delta(f)(x) = \delta(g)(x)$. The minimum closed primary ideal in $\mathscr{D}(\delta)$ at $x \in X$ is $\{f \in \mathscr{D}(\delta)\colon f(x) = \delta(f)(x) = 0\}$.

LEMMA 1.1. *Let X be a locally compact Hausdorff space and let δ be a closed $*$ derivation in $C_0(X)$. Then $\mathscr{D}(\delta)_c$ is dense in $\mathscr{D}(\delta)$ in the graph norm.*

Proof. Let $X \cup \{\infty\}$ be the one point compactification of X. Define a closed $*$ derivation δ_1 in $C(X \cup \{\infty\})$ "extending" δ by taking $\mathscr{D}(\delta_1) = \mathscr{D}(\delta) \oplus \boldsymbol{C}1$ and setting $\delta_1 = \delta \oplus 0$. Then $\mathscr{D}(\delta) = \{f \in \mathscr{D}(\delta_1)\colon f(\infty) = \delta_1(f)(\infty) = 0\}$. But this is the minimum closed ideal at ∞ in $\mathscr{D}(\delta_1)$, and is therefore the closure in $\mathscr{D}(\delta_1)$ of the ideal of functions vanishing in a neighborhood of ∞. [3, Theorem 36.1]. That is, $\mathscr{D}(\delta)_c$ is dense in $\mathscr{D}(\delta)$. □

DEFINITION. A closed subset $E \subseteq X$ is called a *restriction set* for δ if whenever $f \in \mathscr{D}(\delta)$ and $f|_E = 0$, it follows that $\delta(f)|_E = 0$.

[1] One must require $f(0) = 0$, unless X is compact.

If E is a restriction set, the formula $\delta_E(f|_E) = \delta(f)|_E$ defines a * derivation in $C_0(E)$ with domain $\{f|_E: f \in \mathscr{D}(\delta)\}$.

LEMMA 1.2. *If V is open and closed in X, then V is a restriction set for δ, and δ_V is closed.*

Proof. That V is a restriction set follows from the fact that δ is local. The characteristic function 1_V of V is locally in $\mathscr{D}(\delta)$, since $\mathscr{D}(\delta)$ is Silov regular. (We say that a function g is locally in $\mathscr{D}(\delta)$ if for each $x \in X$ there is an $f \in \mathscr{D}(\delta)$ such that $f = g$ in a neighborhood of x.) If $f \in \mathscr{D}(\delta)_c$, then $f1_V \in \mathscr{D}(\delta)$, because a Silov algebra contains each function of compact support which is locally in the algebra. Given $f \in \mathscr{D}(\delta)$, let $\langle f_n \rangle$ be a sequence in $\mathscr{D}(\delta)_c$ such that $\|f_n - f\|_\delta \to 0$ (Lemma 1.1); then $\|f_n 1_V - f1_V\|_\infty \to 0$, and $\delta(f_n 1_V) = \delta(f_n)1_V \to \delta(f)1_V$ uniformly. Since δ is closed, $f1_V \in \mathscr{D}(\delta)$. It now follows that any $g \in \mathscr{D}(\delta_V)$ can be extended *isometrically* to a function g_1 in $\mathscr{D}(\delta)$ by setting $g_1(x) = 0$ for $x \in X \backslash V$. This implies that δ_V is closed. □

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2. The proof of Theorem A. Let the dimensions of G and H be d and $d - c$ respectively. Let $\pi: G \to G/H$ be the canonical map, and let η be a C^∞ section of π defined in a neighborhood of H in G/H.

For each $g \in \mathscr{D}(\delta)$, the map $s \to \ell_s g$ is continuous from G to $(\mathscr{D}(\delta), \|\ \|_\delta)$. Therefore for $f \in C_c(G)$, $f*g = \int_G f(s)\ell_s(g)ds$ is an element of $\mathscr{D}(\delta)$, and

$$\begin{aligned}\delta(f*g)(tH) &= \int_G f(s)\delta(l_s(g))(tH)ds \\ &= \int_G f(s)l_s(\delta(g))(tH)ds \\ &= f*\delta(g)(tH)\ .\end{aligned}$$

(The integrations are with respect to a fixed left Haar measure on G.) If $f \in C_c^\infty(G)$, then $f*g \in \mathscr{D}(\delta) \cap C^\infty(G/H)$.

We want to produce a co-ordinate system on a neighborhood of H in G/H such that the co-ordinate functions extend to elements of $\mathscr{D}(\delta) \cap C^\infty(G/H)$. In the special case that $H = \{1\}$, this can be done very easily. Let $\{x_i: 1 \leqq i \leqq d\}$ be elements of $C_c^\infty(G)$ which form a local co-ordinate system near 1. Because $\mathscr{D}(\delta)$ is Silov regular, G has a right approximate identity $\langle f_n \rangle$ for convolution, with each f_n an element of $\mathscr{D}(\delta)$. For large n, $\{x_i * f_n: 1 \leqq i \leqq d\}$ is a co-ordinate system in a neighborhood of 1, and each $x_i * f_n$ is an element of $\mathscr{D}(\delta) \cap C_c^\infty(G)$. The proof in the following paragraphs for H arbitrary follows the same basic line, although it is somewhat more convoluted.

We will show that $\{d(f*g)(H): f \in C_c^\infty(G)_{\text{s.a.}},\ g \in \mathscr{D}(\delta)_{\text{s.a.}}\}$ spans the cotangent space $T_H^*(G/H)$. If not, there is a tangent vector $w_1 \in T_H(G/H)$ such that $w_1(f*g) = 0$ for all $f \in C_c^\infty(G)_{\text{s.a.}}$ and $g \in \mathscr{D}(\delta)_{\text{s.a.}}$. Let $w_i (1 \leqq i \leqq c)$ be a basis of $T_H(G/H)$ and let $v_i = d\eta(w_i)$. Choose a basis $\{v_i: c+1 \leqq i \leqq d\}$ of $T_1(H) \subseteqq T_1(G)$. Then $\{v_i: 1 \leqq i \leqq d\}$ is a basis of $T_1(G)$. Let X_i be a *right* invariant vector field on G such that $X_i(1) = v_i$ $(1 \leqq i \leqq d)$. There is a positive constant a such that the map

$$\phi^{-1}: (r_1, \cdots, r_d) \longmapsto \exp(r_1 X_1) \cdots \exp(r_d X_d)$$

is a diffeomorphism of the cube $\{r \in \boldsymbol{R}^d: |r_i| < 4a\ \ (1 \leqq i \leqq d)\}$ onto a neighborhood U of 1 in G, and the map

$$(r_{c+1}, \cdots, r_d) \longmapsto \exp(r_{c+1} X_{c+1}) \cdots \exp(r_d X_d)$$

is a diffeomorphism of the cube $\{r \in \boldsymbol{R}^{d-c}: |r_i| < 4a\ \ (c+1 \leqq i \leqq d)\}$ onto the neighborhood $U \cap H$ of 1 in H. Let $\{x_i: 1 \leqq i \leqq d\}$ be the co-ordinate functions of the co-ordinate system (U, ϕ). Let $e: \boldsymbol{R} \to \boldsymbol{R}$ be a C^∞ function satisfying $0 \leqq e \leqq 1$, $e|_{[-a,a]} = 1$, and $\operatorname{supp}(e) \subseteqq]-2a, 2a[$. Define

$$F(s) = \begin{cases} \prod_{i=1}^{d} e(x_i(s)) & (s \in U) \\ 0 & (s \notin U)\,. \end{cases}$$

Let V be a symmetric neighborhood of 1 in G satisfying

(1) $\phi^{-1}([-2a, 2a]^d) \cdot V \subseteqq \phi^{-1}(]-3a, 3a[^d)$

(2) $\phi^{-1}([-3a, 3a]^d) \cdot V \subseteqq U$, and

(3) $|x_1(s)| < a/2$ for $s \in (H \cap \phi^{-1}([-3a, 3a]^d)) \cdot V$.

(Point (3) is possible since $x_1|_{H \cap U} = 0$.) Let $g \in \mathscr{D}(\delta)$ satisfy $g(H) > 0$, $g \geqq 0$, and $\operatorname{supp}(g) \subseteqq \pi(V)$.

We next observe that $X_1(F)(s) g(\pi(s^{-1})) = 0$ for all $s \in G$. Of course $X_1(F(s)) = 0$ for $s \notin U$. Suppose that for some $s \in U$, $X_1(F)(s) g(\pi(s^{-1})) \neq 0$. Since $X_1(F)(s) = e'(x_1(s)) \prod_{i=2}^d e(x_i(s))$, we must have $|x_i(s)| \leqq 2a$ $(2 \leqq i \leqq d)$, and

$$a \leqq |x_1(s)| \leqq 2a\,. \tag{4}$$

Because $g(\pi(s^{-1})) \neq 0$, $\pi(s^{-1}) \in \pi(V)$. Thus $\exists h \in H$ and $\exists w \in V$ such that $s^{-1}h = w$. Since $s \in \phi^{-1}([-2a, 2a]^d)$, $h = sw$ is an element of $H \cap \phi^{-1}([-3a, 3a]^d)$, according to (1). Therefore $s = hw^{-1}$ is an element of $(H \cap \phi^{-1}([-3a, 3a]^d)) \cdot V$, and (3) implies $|x_1(s)| < a/2$. This contradicts (4).

Let g be as above and define

$$f(s) = \begin{cases} F(s)x_1(s) & (s \in U) \\ 0 & (s \notin U) . \end{cases}$$

Then $f \in C_c^\infty(G)$. We will show that $w_1(f*g) \neq 0$. In fact,

$$\begin{aligned} w_1(f*g) &= v_1((f*g)\circ\pi) \\ &= v_1(f*(g\circ\pi)) \\ &= X_1(f*(g\circ\pi))(1) \\ &= X_1(f)*(g\circ\pi)(1) . \end{aligned}$$

The last equality comes from the right invariance of X_1. Continuing,

$$\begin{aligned} w_1(f*g) &= \int_U X_1(f)(s)g(\pi(s^{-1}))ds \\ &= \int_U x_1(s)X_1(F)(s)g(\pi(s^{-1}))ds + \int_U F(s)g(\pi(s^{-1}))ds . \end{aligned}$$

In the last line, the first integrand is zero, as was noted in the previous paragraph. The second integral is positive. Thus $w_1(f*g) \neq 0$, and this contradiction shows that $\{dy(H): y \in C_c^\infty(G/H) \cap \mathscr{D}(\delta)_{\text{s.a.}}\}$ exhausts $T_H^*(G/H)$. Therefore there exist functions $y_i (1 \leqq i \leqq c)$ in $C_c^\infty(G/H) \cap \mathscr{D}(\delta)_{\text{s.a.}}$ and a neighborhood U_0 of H such that $(U_0, \{y_i\})$ is a co-ordinate system.

Let $C^\infty(y_j)$ denote $\{g\circ y_j : g \in C^\infty(\boldsymbol{R})\}$.[2] Because of the C^1 functional calculus in $\mathscr{D}(\delta)$, $C^\infty(y_j) \subseteqq \mathscr{D}(\delta)$ and $\delta(g\circ y_j) = (g'\circ y_j)\delta(y_j)$ $(g \in C^\infty(\boldsymbol{R}))$. For $s \in U_0$, this is $\delta(g\circ y_j)(s) = \partial/\partial y_j(g\circ y_j)(s)\delta(y_j)(s)$. It follows that for f in the algebra A generated by $\{C^\infty(y_j): 1 \leqq j \leqq c\}$, and for $s \in U_0$, $\delta(f)(s) = X(f)(s)$, where X is the C^0 vector field on U_0,

$$X(s) = \sum_{j=1}^{c} \delta(y_j)(s)\frac{\partial}{\partial y_j} .$$

Now let W be a compact neighborhood of H contained in U_0, such that $\{(y_1(w), \cdots, y_c(w)): w \in W\}$ is a cube in $\boldsymbol{R}^c$. If f is a C^∞ function with support in $\operatorname{int}(W)$, then there is a sequence $\langle f_n \rangle$ in A, each f_n also having support in $\operatorname{int}(W)$, such that $f_n(w) \to f(w)$ and $\delta(f_n)(w) = X(f_n)(w) \to X(f)(w)$ uniformly for $w \in W$. Because

[2] One should require $g(0) = 0$.

all functions are supported in W, $\langle f_n\rangle$ and $\langle\delta(f_n)\rangle$ are uniformly convergent on G/H, and since δ is closed, it follows that $f\in\mathscr{D}(\delta)$. By translation invariance of $\mathscr{D}(\delta)$ every C^∞ function is *locally* in $\mathscr{D}(\delta)$ and therefore $C_c^\infty(G/H)\subseteqq\mathscr{D}(\delta)$. There is a vector field X on G/H such that $\delta(f)=X(f)$ $(f\in C_c^\infty(G/H))$, and because of the G-invariance of δ, X is also G-invariant. This proves the first assertion.

Now let $g\in\mathscr{D}(\delta)_c$ and let $\langle f_n\rangle$ be a left approximate identity for convolution in G, with each $f_n\in C_c^\infty(G)$. Then $f_n*g\in C_c^\infty(G/H)$, $f_n*g\to g$ uniformly and $\delta(f_n*g)=f_n*\delta(g)\to\delta(g)$ uniformly. Thus $C_c^\infty(G/H)$ is dense in $\mathscr{D}(\delta)_c$, with respect to $\|\ \|_\delta$. Since $\mathscr{D}(\delta)_c$ is in turn dense in $\mathscr{D}(\delta)$, by Lemma 1.1, $C_c^\infty(G/H)$ is a core for δ.

X, being G-invariant, is a complete vector field on G/H. Let $\{X_t: t\in\boldsymbol{R}\}$ denote the group of diffeomorphisms of G/H generated by X, and let $\alpha_t(f)=f\circ X_t(f\in C_0(G/H),\ t\in\boldsymbol{R})$. Let δ_1 be the infinitesimal generator of the C^* dynamics $\{\alpha_t\}$. Then $C_c^\infty(G/H)\subseteqq\mathscr{D}(\delta_1)$ and $\delta_1(f)=X(f)=\delta(f)$ $(f\in C_c^\infty(G/H))$. Moreover, it follows from the invariance of X that $l_s\circ\alpha_t\circ l_{s^{-1}}=\alpha_t$ $(t\in\boldsymbol{R},\ s\in G)$ and hence that δ_1 is G-invariant. But then $C_c^\infty(G/H)$ is also a core for δ_1, and since δ and δ_1 agree on $C_c^\infty(G/H)$, $\delta=\delta_1$. □

3. Theorem B and generalizations. Before giving the proof of Theorem B, we note that this theorem contains the case of Theorem A where $H=\{1\}$. Let G be a Lie group and δ a left invariant closed $*$ derivation in $C_0(G)$. If we know that δ generates a C^* dynamics $\{\alpha_t\}$, we can easily obtain the remaining conclusions of Theorem A. Let us see that $\mathscr{D}(\delta)\supseteqq C_c^\infty(G)$. Let $\{X_t\}$ be the group of homeomorphisms of G such that $\alpha_t(f)=f\circ X_t$ $(f\in C_0(G),\ t\in\boldsymbol{R})$, and let $\theta_t=X_t(1)$. Because of the left invariance of δ, $\ell_s\circ\alpha_t\circ\ell_{s^{-1}}=\alpha_t$ and $\ell_s\circ X_t\circ\ell_{s^{-1}}=X_t$ $(s\in G,\ t\in\boldsymbol{R})$. It follows that $X_t(s)=s\theta_t$, $\{\theta_t\}$ is a continuous one-parameter subgroup, and $\alpha_t(f)(s)=f(s\theta_t)$ $(s\in G,\ t\in\boldsymbol{R},\ f\in C_0(G))$. Now the fact that $\{\theta_t\}$ is C^∞ implies that $\mathscr{D}(\delta)\supseteqq C_c^\infty(G)$.

Proof of Theorem B. Suppose first that G is the projective limit of Lie groups: Let $\{G_a: a\in\boldsymbol{B}\}$ be an inverse limit system of Lie groups with homomorphisms $\phi_{ab}: G_b\to G_a$ $(b>a)$ such that $G=\varprojlim G_a$. Let $\phi_a: G\to G_a$ be the natural projection; we are supposing that the kernel N_a of ϕ_a is compact. Thus if $K\subseteqq G_a$ is compact, then $\phi_a^{-1}(K)$ is compact in G. Let $\phi_a^0(f)=f\circ\phi_a$ $(f\in C_0(G_a))$; then ϕ_a^0 is a $*$ isomcrphism of $C_0(G_a)$ into $C_0(G)$ which carries $C_c(G_a)$ into $C_c(G)$. Let $A_a=\phi_a^0(C_0(G_a))$, and let $A=\bigcup_{a\in B}A_a$.

Each A_a is both left and right translation invariant, since

$$r_s(f\circ\phi_a)=(r_{\phi_a(s)}f)\circ\phi_a\ ,$$

and

$$\ell_s(f \circ \phi_a) = (\ell_{\phi_a(s)} f) \circ \phi_a \qquad (f \in C_0(G_a)) .$$

A function $f \in C_0(G)$ is in A_a if and only if $\ell_n(f) = f$ for all $n \in N_a$. If $f \in \mathscr{D}(\delta) \cap A_a$, then $\delta(f)$ is also in A_a, since $\ell_n(\delta(f)) = \delta(\ell_n(f)) = \delta(f)$ $(n \in N_a)$. We next observe that $\mathscr{D}(\delta) \cap A_a$ is dense in A_a. Let dn denote normalized Haar measure on the compact group N_a. Given $g_0 \in A_a$ and $\varepsilon > 0$, choose $g_1 \in \mathscr{D}(\delta)$ such that $\|g_0 - g_1\| < \varepsilon$. Define $g_2 = \int_{N_a} \ell_n(g_1) dn$. Then $g_2 \in \mathscr{D}(\delta) \cap A_a$, and

$$g_0 - g_2 = \int_{N_a} \ell_n(g_0 - g_1) dn .$$

Therefore $\|g_0 - g_2\|_\infty \leqq \int_{N_a} \|\ell_n(g_0 - g_1)\| dn < \varepsilon$.

Define δ_a in $C_0(G_a)$ by $\mathscr{D}(\delta_a) = (\phi_a^0)^{-1}(\mathscr{D}(\delta) \cap A_a)$, $\delta_a = (\phi_a^0)^{-1}\delta\phi_a^0$. Then δ_a is a densely defined * derivation in $C_0(G_a)$, and it is straightforward to check that δ_a is closed and left invariant. Since G_a is a Lie group, δ_a generates a C^* dynamics of $C_0(G_a)$ (Theorem A). Let $\psi_t^a = \phi_a^0 \exp(t\delta_a)(\phi_a^0)^{-1}$ be the corresponding C^* dynamics of A_a. If $b > a$ in $\boldsymbol{A}$, then $N_b \subseteqq N_a$, and $A_a \subseteqq A_b$. We observe that $\psi_t^b | A_a = \psi_t^a$. For $s \in G$ and $t \in \boldsymbol{R}$,

$$\begin{aligned}
\ell_s \psi_t^b \ell_s^{-1} &= \ell_s \phi_b^0 \exp(t\delta_b)(\phi_b^0)^{-1} \ell_s^{-1} \\
&= \phi_b^0 \ell_{\phi_b(s)} \exp(t\delta_b) \ell_{\phi_b^{(s^{-1})}} (\phi_b^0)^{-1} \\
&= \phi_b^0 \exp(t\delta_b)(\phi_b^0)^{-1} \\
&= \psi_t^b .
\end{aligned}$$

For $f \in A_a$ and $n \in N_a$,

$$\psi_t^b(f) = \ell_n \psi_t^b \ell_n^{-1}(f) = \ell_n \psi_t^b(f) .$$

Therefore ψ_t^b maps A_a into A_a. Now both $\{\psi_t^a\}$ and $\{\psi_t^b | A_a\}$ have generator $\delta|_{\mathscr{D}(\delta) \cap A_a}$, so $\psi_t^b|_{A_a} = \psi_t^a$. We define a group of * automorphisms of $A = \bigcup_{a \in B} A_a$ by $\psi_t(f) = \psi_t^a(f)$ if $f \in A_a$; $\{\psi_t\}$ is strongly continuous on A. A is a conjugate-closed subalgebra of $C_0(G)$. If $s \neq 1$, there is a kernel N_a such that $s \notin N_a$, and there is an $f \in A_a$ such that $f(s) \neq f(1)$. So A separates points of G and is dense in $C_0(G)$. Each ψ_t extends uniquely to a * automorphism of $C_0(G)$ and $\{\psi_t\}$ is a C^* dynamics of $C_0(G)$. Let δ_1 be the generator of $\{\psi_t\}$.

We show that $\delta = \delta_1$. First it is evident that $\mathscr{D}(\delta_1) \cap A = \mathscr{D}(\delta) \cap A$ and $\delta_1(f) = \delta(f)$ $(f \in \mathscr{D}(\delta) \cap A)$. Since $\ell_s \psi_t \ell_{s^{-1}} = \psi_t$ $(t \in \boldsymbol{R}$, $s \in G)$, δ_1 is left invariant. Given $g \in \mathscr{D}(\delta)$ and $\varepsilon > 0$, there is a neighborhood U of 1 in G such that $\|\ell_s(g) - g\|_\delta < \varepsilon$ for all $s \in U$, and there is an $a \in \boldsymbol{B}$ such that $N_a \subseteqq U$. Let $g_1 = \int_{N_a} \ell_n(g) dn$ (with

dn denoting normalized Haar measure on N_a). Then $g_1 \in \mathscr{D}(\delta) \cap A_a$, and $\|g - g_1\|_\delta < \varepsilon$. Thus $A \cap \mathscr{D}(\delta)$ is a core for δ. Similarly $A \cap \mathscr{D}(\delta_1) = A \cap \mathscr{D}(\delta)$ is a core for δ_1. Since δ and δ_1 agree on $A \cap \mathscr{D}(\delta)$, $\delta = \delta_1$. This completes the proof in the case that G is the projective limit of Lie groups.

Now let G be any locally compact group, with identity component G_0. Let G_1 be the pre-image in G of a compact-open subgroup of G/G_0 [6, Theorem 2.3]. Thus G_1 is an open (and closed) subgroup, and according to Lemma 1.2, G_1 is a restriction set for δ and δ_{G_1} is closed. It is clear that δ_{G_1} is G_1-invariant. Since G_1/G_0 is compact, G_1 is the projective limit of Lie groups [6, Theorem 4.6]. By the first part of the proof, δ_{G_1} generates a C^* dynamics $\{\alpha_t\}$ of $C_0(G_1)$. There is a continuous one-parameter subgroup $\{\theta_t\}$ of G_0 such that $\alpha_t(f) = r_{\theta_t}(f)$ $(f \in C_0(G_1),\, t \in \boldsymbol{R})$. (See the remarks at the beginning of this section.) We define a C^* dynamics $\{\psi_t\}$ of $C_0(G)$ by the formula

$$\psi_t(f) = r_{\theta_t}(f) \quad (f \in C_0(G)) .$$

If δ_1 is the generator of $\{\psi_t\}$, then G_1 is also a restriction set for δ_1 and $(\delta_1)_{G_1} = \delta_{G_1}$, since $\psi_t(f)|_{G_1} = \alpha_t(f|_{G_1})$ $(f \in C_0(G))$. By translation invariance of δ and δ_1, a function is locally in $\mathscr{D}(\delta)$ if and only if it is locally in $\mathscr{D}(\delta_1)$. Hence $\mathscr{D}(\delta)_c = \mathscr{D}(\delta_1)_c$. Again by translation invariance, δ and δ_1 agree on $\mathscr{D}(\delta)_c$, and Lemma 1.1 implies that $\delta = \delta_1$. □

COROLLARY 3.1. *Let δ be a left invariant closed $*$ derivation in $C_0(G)$, where G is a locally compact group. The identity component of G is a restriction set for δ.*

Proof. This follows immediately from the existence of a one-parameter subgroup $\{\theta_t\}$ of G such that

$$\delta(f)(s) = \frac{d}{dt}\Big|_{t=0} f(s\theta_t) \qquad (s \in G,\, f \in \mathscr{D}(\delta)) .$$

We next consider generalizations of Theorem B to coset spaces.

THEOREM 3.2. *Let G be a locally compact group. Suppose that there is an inverse limit system $\{G_a\}$ of Lie groups such that $G = \varprojlim G_a$ and each projection $G \to G_a$ has compact kernel. Let H be a closed subgroup of G and let δ be a G-invariant closed $*$ derivation in $C_0(G/H)$. Then δ is the generator of a C^* dynamics of $C_0(G/H)$.* □

The first part of the proof of Theorem B can be modified to prove this result. Theorem 3.2 applies in particular if G is compact or connected. We have not obtained the analogous result for arbitrary G, but we do have the result for another special class of groups:

THEOREM 3.3. *Let G be a locally compact group which can be covered by countably many translates of an arbitrary neighborhood of the identity. Let H be a closed subgroup and let δ be a G-invariant closed * derivation in $C_0(G/H)$. Then δ generates a C^* dynamics of $C_0(G/H)$.*

Proof. Let G_1 be an open subgroup of G which is the projection limit of Lie groups. Each G_1-orbit M in G/H is open and closed, and is therefore a restriction set for δ. The derivation δ_M is closed and evidently G_1-invariant.

G_1 inherits the property of being covered by countably many translates of an arbitrary neighborhood of the identity. Because of this, there is a homeomorphism of M onto a coset space of G_1 which respects the G_1 actions. It now follows from Theorem 3.2 that δ_M generates a C^* dynamics $\{\alpha_t^M\}$ of $C_0(M)$.

Define $\alpha_t\colon C_0(G/H) \to C_0(G/H)$ by

$$\alpha_t(f)|_M = \alpha_t^M(f\,|\,M) ,$$

for each G_1-orbit M. Then $\{\alpha_t\}$ is a C^* dynamics of $C_0(G/H)$. Let δ_1 be the generator of $\{\alpha_t\}$. For each G_1-orbit M, $(\delta_1)_M = \delta_M$. It follows that $\mathscr{D}(\delta_1)_c = \mathscr{D}(\delta)_c$ and $\delta_1(f) = \delta(f)$ for $f \in \mathscr{D}(\delta)_c$. Now Lemma 1.1 implies that $\delta = \delta_1$. □

4. The Lie algebra of a locally compact group. The Lie algebra of a connected locally compact group G was defined by Lashof in [5] to be the projective limit of the Lie algebras of Lie groups forming a projective limit system for G. The Lie algebra of an arbitrary locally compact group is defined to be the same as the Lie algebra of its connected component. Bruhat [2] identified the Lie algebra of G with a closed subspace of the dual of the algebra $\mathscr{D}(G)$ of regular functions on G.

For the remainder of this section let G be a connected locally compact group. We show that the set $L(G)$ of left invariant closed * derivations in $C_0(G)$ has a natural Lie algebra structure and can be identified with the Lie algebra of G.

Let $\{G_a\colon a \in \boldsymbol{B}\}$ be an inverse limit system of Lie groups such that $G = \varprojlim G_a$. We adopt the notation of the proof of Theorem B. The algebra $\mathscr{D}(G)$ is defined to be $\bigcup_{a\in B} \phi_a^0(C_c^\infty(G_a))$. $\mathscr{D}(G)$ is independent if the choice of the inverse limit system $\{G_a\}$. [2]

PROPOSITION. (i) *$\mathscr{D}(G)$ is a core for each element of $L(G)$.*
(ii) *Each element of $L(G)$ maps $\mathscr{D}(G)$ into $\mathscr{D}(G)$.*
(iii) *Each $*$ derivation in $C_0(G)$ with domain $\mathscr{D}(G)$ is closable.*

Proof. Let δ be an element of $L(G)$. By Theorem A the derivation δ_a induced by δ in $C_0(G_a)$ has domain containing $C_c^\infty(G_a)$ and maps $C_c^\infty(G_a)$ into itself. Therefore $\mathscr{D}(\delta) \supseteqq \phi_a^0(C_c^\infty(G_a))$ and δ maps $\phi_a^0(C_c^\infty(G_a))$ into itself. This proves (ii). Now let $g \in \mathscr{D}(\delta)$ and $\varepsilon > 0$. By the proof of Theorem B, there is an $a \in \boldsymbol{B}$ and $g_1 \in \mathscr{D}(\delta) \cap A_a$ such that $\|g - g_1\|_\delta < \varepsilon$. Since $C_c^\infty(G_a)$ is a core for δ_a, it follows that there is a $g_2 \in \phi_a^0(C_c^\infty(G_a))$ such that $\|g_1 - g_2\|_\delta < \varepsilon$. Thus $\mathscr{D}(G)$ is a core for δ.

A $*$ derivation δ in $C_0(X)$ is said to be well behaved if it satisfies the following equivalent conditions: (a) $\|f \pm \delta(f)\| \geqq \|f\|$ for all $f \in \mathscr{D}(\delta)_{\text{s.a.}}$, and (b) if $f \in \mathscr{D}(\delta)_{\text{s.a.}}$ attains its maximum at $s \in X$, then $\delta(f)(s) = 0$. A well behaved $*$ derivation is closable [8, Theorem 2.8].

Now let δ be a $*$ derivation in $C_0(G)$ with domain $\mathscr{D}(G)$. For each $a \in \boldsymbol{B}$, define $P_a: C_0(G) \to A_a$ by $P_a(f) = \int_{N_a} l_n(f)dn$. Since P_a is a conditional expectation, $P_a \circ \delta | \phi_a^0(C_c^\infty(G_a)) = \delta_a$ is a $*$ derivation in A_a. The $*$ derivation $(\phi_a^0)^{-1} \circ \delta_a \circ \phi_a^0$ in $C_0(G_a)$ with domain $C_c^\infty(G_a)$ is defined by a continuous vector field and is therefore well behaved (condition (b)). So δ_a is well behaved. Fix $a \in \boldsymbol{B}$ and $f \in \phi_a^0(C_c^\infty(G_a))_{\text{s.a.}}$; for all $b \geqq a$, $\|f \pm P_b(\delta(f))\| \leqq \|f\|$. Since $P_b(\delta(f)) \to \delta(f)$ uniformly, $\|f \pm \delta(f)\| \geqq \|f\|$. Thus δ is well behaved and closable. □

Given this result, we can define each Lie algebra operation on $L(G)$ by restricting the elements of $L(G)$ to $\mathscr{D}(G)$, performing the operation on the restrictions, and closing the resulting left invariant $*$ derivation on $\mathscr{D}(G)$.

It remains to identify $L(G)$ with the Lie algebra of G as defined in [5]. Let $\mathfrak{g}_a$ be the Lie algebra of G_a; $\{\mathfrak{g}_a: a \in \boldsymbol{B}\}$ forms an inverse limit system with homomorphisms $d\phi_{ab}: \mathfrak{g}_b \to \mathfrak{g}_a$ $(b > a)$. Regard $L(G)$ as the set of left invariant $*$ derivations of $\mathscr{D}(G)$ and $\mathfrak{g}_a$ as the set of left invariant $*$ derivations of $C_c^\infty(G_a)$. Then $\delta \to \{\delta_a: a \in \boldsymbol{B}\}$ is a Lie algebra isomorphism of $L(G)$ onto $\varprojlim \mathfrak{g}_a$.

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